Bol. Soc. Paran. Mat. (3s.) **v. 27** 1 (2009): 29–39.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/spm doi:10.5269/bspm.v27i1.9066

Wave Equation with Acoustic/Memory Boundary Conditions

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ABSTRACT: In this paper we prove the existence and uniqueness of global solution to the mixed problem for the wave equation with acoustic boundary conditions on a portion of the boundary and memory type conditions on the rest of it.

Key Words: wave equation, acoustic boundary conditions, memory boundary term.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with smooth boundary Γ and $T > 0$. Suppose Γ is divided into two portion of positive measure $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let ν be the outward unit normal vector on Γ. Moreover, consider the following given functions $F : \Omega \times (0,T) \to \mathbb{R}$; $f, g, h : \overline{\Gamma_1} \to \mathbb{R}$; $\beta : \mathbb{R}^+ \to \mathbb{R}$; $u_0, u_1 : \Omega \to \mathbb{R}$ and $\delta_0 : \Gamma_1 \to \mathbb{R}$. In this work we study the mixed problem for the wave equation with acoustic/memory boundary conditions

$$u'' - \Delta u = F \qquad \text{in } \Omega \times (0,T), \tag{1}$$

$$u + \int_0^t \beta(t-s) \frac{\partial u}{\partial \nu}(s)\, ds = 0 \qquad \text{on } \Gamma_0 \times (0,T), \tag{2}$$

$$\frac{\partial u}{\partial \nu} = \delta' \qquad \text{on } \Gamma_1 \times (0,T), \tag{3}$$

$$u' + f\delta'' + g\delta' + h\delta = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{4}$$

$$u(x,0) = u_0(x), \;\; u'(x,0) = u_1(x)\,, \qquad x \in \Omega, \tag{5}$$

$$\delta(x,0) = \delta_0(x), \;\; \delta'(x,0) = \frac{\partial u_0}{\partial \nu}(x)\,, \qquad x \in \Gamma_1, \tag{6}$$

where $' = \dfrac{\partial}{\partial t}$ and $\Delta = \displaystyle\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator.

2000 *Mathematics Subject Classification:* 35L05, 35L20

Mixed problems for wave equations with homogeneous boundary conditions have been studied for a long time. However, time-dependent boundary conditions seems to be more suitable to model concrete applications, see [2], [10] , [14].

In this direction boundary conditions of memory type, as equation (2), imposed on a portion of the boundary and Dirichlet condition on the rest of the boundary, have been considered, for instance [1], [4], [12] and [13]. Equation (2) means that the portion Γ_0 is clamped in a body with viscoelastic properties. On the other hand, wave equations equipped with time-dependent acoustic boundary conditions have been considered also. For locally reacting boundaries, conditions (3) and (4), were introduced by Beale-Rosencrans [2] and studied in [3], [5], [6], [7], [8] and [11]. In these cases, the solution u of the wave equation (1) is the velocity potential of a fluid undergoing acoustic wave motion and $\delta(x,t)$ is the normal displacement to the boundary at time t with the boundary point x. Similarly, acoustic boundary conditions have been coupled with homogeneous Dirichlet condition on a portion of the boundary, excepted in [7], [9] and [11] where the acoustic boundary condition were imposed in the whole boundary Γ.

The main purpose of this paper is to study the combination of acoustic and memory boundary conditions. We prove the existence and uniqueness of global solution to the problem (1)-(6). Our proof is based on Galerkin's method and compactness arguments. Technical difficulties in studying equation (2) lead us, by using the inverse Volterra's operator, to another equivalent condition in which the normal derivative $\frac{\partial u}{\partial \nu}$ appears explicit. Furthermore, the usual approach to the Galerkin's method meet up with technical problems when estimating approximate solutions $u''_m(0)$. In order to avoid these difficulties we first solve a problem with homogeneous initial data and then we take an appropriated transformation to reduce the study of the nonhomogeneous case to similar one with homogeneous initial data. Finally we observe that when we have homogeneous Dirichlet condition on a portion Γ_0 of the boundary with positive measure, the natural space to be considered is $\{u \in H^1(\Omega)$ such that the trace of $u = 0$ a. e. in $\Gamma_0\}$ and Poincaré's inequality trivially holds in such space. In our case, we do not have homogeneous Dirichlet condition on a portion of the boundary, then we introduce a close subspace W of $H^1(\Omega)$ where the Poincaré inequality is satisfied.

Our paper is organized as follows. In section 2 we introduce the notations and the main result. In section 3 we deal with the perturbed problem and then prove the main result Theorem 2.1.

2. Notations and Principal Result

The inner product and norm in $L^2(\Omega)$, $L^2(\Gamma)$ and $L^2(\Gamma_i)$, $i = 0, 1$, are denoted, respectively, by

$$(u,v) = \int_\Omega u(x)v(x)\,dx\,; \qquad |u| = \left(\int_\Omega |u(x)|^2\,dx\right)^{\frac{1}{2}}\,;$$
$$(\delta,\theta)_\Gamma = \int_\Gamma \delta(x)\theta(x)\,d\Gamma\,; \qquad |\delta|_\Gamma = \left(\int_\Gamma |\delta(x)|^2\,d\Gamma\right)^{\frac{1}{2}}\,;$$
$$(\delta,\theta)_{\Gamma_i} = \int_{\Gamma_i} \delta(x)\theta(x)\,d\Gamma\,; \qquad |\delta|_{\Gamma_i} = \left(\int_{\Gamma_i} |\delta(x)|^2\,d\Gamma\right)^{\frac{1}{2}}\,.$$

Let $H(\Delta,\Omega) = \{u \in H^1(\Omega); \Delta u \in L^2(\Omega)\}$ be the Hilbert space equipped with the norm $\|u\|_{H(\Delta,\Omega)} = \left(\|u\|^2_{H^1(\Omega)} + |\Delta u|^2\right)^{\frac{1}{2}}$, where $H^1(\Omega)$ is the real Sobolev space of first order. Denoting $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ and $\gamma_1 : H(\Delta,\Omega) \to H^{-\frac{1}{2}}(\Gamma)$ the trace map of order zero and the Neumann trace map on $H(\Delta,\Omega)$, respectively, we have $\gamma_0(u) = u_{|\Gamma}$ and $\gamma_1(u) = \left(\frac{\partial u}{\partial \nu}\right)_{|\Gamma}$ for all $u \in \mathcal{D}(\overline{\Omega})$. Some times to simplify the notation we write u and $\frac{\partial u}{\partial \nu}$ instead of $\gamma_0(u)$ and $\gamma_1(u)$, respectively.

For each point x_0 fixed in Γ let $V_{x_0} = \{u \in C^1(\overline{\Omega})\,\text{such that}\; u(x_0) = 0\}$. The Poincaré inequality holds in V_{x_0}, that is,

$$|u|^2 \le\, D^2\,|\nabla u|^2 \quad \text{for all } u \in V_{x_0},$$

where D is the diameter of Ω. Now we consider $V = \bigcup\limits_{x\in\Gamma} V_x$ and Poincaré's inequality holds in V also, since the constant D is independent of $x \in \Gamma$. By density the Poincaré inequality still holds in the $H^1(\Omega)$ closure of V which we denote by $W := \overline{V}^{H^1(\Omega)}$. The inner product and norm in W are denoted, respectively, by

$$((u,v)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)dx,$$

$$\|u\| = \left(\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^2 dx\right)^{\frac{1}{2}} = \left(\int_\Omega |\nabla u(x)|^2\,dx\right)^{\frac{1}{2}}\,.$$

Poincaré's inequality and the continuity of trace map yield a constant C such that

$$|\gamma_0(u)|^2_\Gamma \le C\,\|u\|^2, \qquad \text{for all } u \in W. \tag{7}$$

As we sad before we shall replace equation (2) to another equivalent one. We write the convolution product operator

$$(\beta * \varphi)(t) = \int_0^t \beta(t-s)\varphi(s)\,ds.$$

Suppose that $\beta : [0,\infty) \to \mathbb{R}$ satisfies

$$\beta \in W^{2,1}(0,\infty)\,,\ \beta(0) \neq 0 \quad \text{and} \quad \eta\,|\beta'|_{L^1(0,\infty)} < 1\,, \quad \text{where } \eta = \frac{1}{\beta(0)}\,, \tag{8}$$

then from Banach's fix point Theorem there exists a unique function $k \in W^{1,1}(0,\infty)$, usually called by resolvent kernel, such that

$$k(t) = -\eta\beta'(t) - \eta(\beta' * k)(t), \ \text{a.e. in } (0,\infty). \tag{9}$$

The Volterra Operator $\psi(\xi) = -\beta(0)\xi - \beta' * \xi$ for all $\xi \in L^\infty(0,T;L^2(\Gamma_0))$ is well defined and its inverse is given by $\psi^{-1}(\zeta) = -\eta(\zeta + k * \zeta)$.

Differentiating (2) with respect to t and applying the inverse Volterra's Operator, we obtain

$$\frac{\partial u}{\partial \nu} = -\eta\,(u' + k(0)u - k(t)u_0 + k' * u) \ \text{ on } \Gamma_0 \times (0,T)\,, \tag{10}$$

which is equivalent to (2). Hence we have the following equivalent problem

$$u'' - \Delta u = F \quad \text{in } \Omega \times (0,T), \tag{11}$$

$$\frac{\partial u}{\partial \nu} = -\eta\,(u' + k(0)u - k(t)u_0 + k' * u) \quad \text{on } \Gamma_0 \times (0,T), \tag{12}$$

$$\frac{\partial u}{\partial \nu} = \delta' \quad \text{on } \Gamma_1 \times (0,T), \tag{13}$$

$$u' + f\delta'' + g\delta' + h\delta = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{14}$$

$$u(x,0) = u_0(x), \ u'(x,0) = u_1(x) \quad \text{in } \Omega, \tag{15}$$

$$\delta(x,0) = \delta_0(x), \ \delta'(x,0) = \frac{\partial u_0}{\partial \nu}(x) \quad \text{on } \Gamma_1. \tag{16}$$

In order to state our main result we assume:

$$f, g, h \in C(\overline{\Gamma_1}) \text{ such that } f(x), h(x) > 0 \text{ and } g(x) \geq 0\,, \text{ for all } x \in \overline{\Gamma_1} \tag{17}$$

and

$$k \in W^{1,1}(0,\infty) \cap W^{1,2}(0,\infty)\,. \tag{18}$$

Theorem 2.1 *Suppose (17) and (18) hold. Let $u_0, u_1 \in W \cap H^2(\Omega)$, $\delta_0 \in L^2(\Gamma_1)$, $F, F' \in L^\infty_{loc}(0,\infty;L^2(\Omega))$ with*

$$\frac{\partial u_0}{\partial \nu} = -\eta\, u_1 \ \text{ on } \Gamma_0. \tag{19}$$

Then, for all $T > 0$, there exist a unique pair (u,δ), in the class

$$u, u' \in L^\infty(0,T;W) \text{ such that } u(t) \in H(\Delta,\Omega) \text{ a.e. in } [0,T]\,; \tag{20}$$

$$u'' \in L^\infty(0,T;L^2(\Omega));\ \delta, \delta', \delta'' \in L^\infty(0,T;L^2(\Gamma_1)), \tag{21}$$

which comprise a solution to the problem (11)-(16).

3. Perturbed Problem

Let f, g, h, F, k, u_0, u_1 and δ_0 be as in Theorem 2.1. We define

$$\phi(x,t) = u_0(x) + tu_1(x)\,, \tag{22}$$
$$\mathcal{F}(x,t) = F(x,t) + \Delta\phi(x,t)\,, \tag{23}$$
$$r(x,t) = -\eta[\phi'(x,t) + k(0)\phi(x,t) - k(t)\phi(x,0) + (k' * \phi)(t)] - \frac{\partial\phi}{\partial\nu}(x,t). \tag{24}$$

Whence we consider the following perturbed problem

$$v'' - \Delta v = \mathcal{F} \quad \text{in } \Omega\times(0,T), \tag{25}$$
$$\frac{\partial v}{\partial\nu} = -\eta\,(v' + k(0)v + k' * v) + r \quad \text{on } \Gamma_0\times(0,T), \tag{26}$$
$$\frac{\partial v}{\partial\nu} = \delta' - \frac{\partial\phi}{\partial\nu} \quad \text{on } \Gamma_1\times(0,T), \tag{27}$$
$$v' + f\delta'' + g\delta' + h\delta = -\phi' \quad \text{on } \Gamma_1\times(0,T), \tag{28}$$
$$v(x,0) = v'(x,0) = 0 \quad \text{in } \Omega, \tag{29}$$
$$\delta(x,0) = \delta_0(x),\;\; \delta'(x,0) = \frac{\partial u_0}{\partial\nu}(x) \quad \text{on } \Gamma_1. \tag{30}$$

Theorem 3.1 (Perturbed Problem) *Let $\phi, \mathcal{F}$ and r given by (22)-(24). For all $T>0$, there exist a unique pair (v,δ), in the class*

$$v, v' \in L^\infty(0,T;W) \text{ such that } v(t)\in H(\Delta,\Omega) \text{ a.e. in } [0,T]\,; \tag{31}$$
$$v'' \in L^\infty(0,T;L^2(\Omega));\;\; \delta,\delta',\delta'' \in L^\infty(0,T;L^2(\Gamma_1)), \tag{32}$$

which comprise a solution to the problem (25)-(30).

Proof.: Let $(w_j)_{j\in\mathbb{N}}$, $(z_j)_{j\in\mathbb{N}}$ be orthonormal bases in W and $L^2(\Gamma_1)$, respectively. For each $m\in\mathbb{N}$ let $U_m = \text{span}\{w_1, w_2, \ldots, w_m\}$ and $Z_m = \text{span}\{z_1, z_2, \ldots, z_m\}$. From ODE theory, we can find $0 < T_m \le T$, $v_m : \Omega\times[0,T_m]\to\mathbb{R}$ and $\delta_m : \Gamma_1\times[0,T_m]\to\mathbb{R}$ such that

$$v_m(x,t) = \sum_{j=1}^m \alpha_{jm}(t) w_j(x) \quad \text{and} \quad \delta_m(x,t) = \sum_{j=1}^m \beta_{jm}(t) z_j(x)$$

satisfy the approximate problem

$$\begin{aligned}&(v_m''(t), w_j) + ((v_m(t), w_j)) + (\eta[v_m'(t) + k(0)v_m(t) + (k' * v_m)(t)], \gamma_0(w_j))_{\Gamma_0}\\ &- (r(t), \gamma_0(w_j))_{\Gamma_0} + \left(\frac{\partial\phi(t)}{\partial\nu} - \delta_m'(t), \gamma_0(w_j)\right)_{\Gamma_1} = (\mathcal{F}(t), w_j)\,,\end{aligned} \tag{33}$$
$$-(v_m'(t), z_j)_{\Gamma_1} = (f\delta_m''(t) + g\delta_m'(t) + h\delta_m(t), z_j)_{\Gamma_1} + (\phi'(t), z_j)_{\Gamma_1}\,, \tag{34}$$
$$v_m(0) = v_m'(0) = 0, \tag{35}$$
$$\delta_m(0) = \delta_{0m} = \sum_{i=1}^m (\delta_0, z_i)_{\Gamma_1} z_i \to \delta_0 \text{ em } L^2(\Gamma_1), \quad \delta_m'(0) = \frac{\partial u_0}{\partial\nu}, \tag{36}$$

for $j = 1, \ldots, m$.

Now we need estimates which allow us to extend the solutions u_m, δ_m to the whole interval $[0, T]$ and pass to limit as $m \to \infty$. From (33) and (34) we have the following approximate equations:

$$(v_m''(t), w) + ((v_m(t), w)) + (\eta[v_m'(t) + k(0)v_m(t) + (k' * v_m)(t)] - r(t), \gamma_0(w))_{\Gamma_0} + \\ + \left(\frac{\partial \phi(t)}{\partial \nu} - \delta_m'(t), \gamma_0(w) \right)_{\Gamma_1} = (\mathcal{F}(t), w)\,, \;\; \forall\, w \in U_m, \tag{37}$$

$$-(v_m'(t), z)_{\Gamma_1} = (f\delta_m''(t) + g\delta_m'(t) + h\delta_m(t), z)_{\Gamma_1} + (\phi'(t), z)_{\Gamma_1}\,, \;\; \forall\, z \in Z_m. \tag{38}$$

Estimate I: Taking $w = 2v_m'(t)$ in (37), $z = 2\delta_m'(t)$ in (38) and substituting the second equation into to the first, we obtain

$$\frac{d}{dt}\left(|v_m'(t)|^2 + \|v_m(t)\|^2 + |f^{\frac{1}{2}}\delta_m'(t)|_{\Gamma_1}^2 + |h^{\frac{1}{2}}\delta_m(t)|_{\Gamma_1}^2 \right) + 2\eta|v_m'(t)|_{\Gamma_0}^2 + 2|g^{\frac{1}{2}}\delta_m'(t)|_{\Gamma_1}^2 \\ = -2\eta k(0)\,(v_m(t), v_m'(t))_{\Gamma_0} - 2\eta\,((k' * v_m)(t), v_m'(t))_{\Gamma_0} + 2\,(r(t), v_m'(t))_{\Gamma_0} \\ - 2\left(\frac{\partial \phi(t)}{\partial \nu}, v_m'(t) \right)_{\Gamma_1} - 2\,(\phi'(t), \delta_m'(t))_{\Gamma_1} + 2\,(\mathcal{F}(t), v_m'(t))\,. \tag{39}$$

We observe that

$$|2\eta k(0)\,(v_m(t), v_m'(t))_{\Gamma_0}| \le \frac{\eta}{3}|v_m'(t)|_{\Gamma_0}^2 + 3\eta k(0)^2 C\|v_m(t)\|^2; \tag{40}$$

$$|2\eta\,((k' * v_m)(t), v_m'(t))_{\Gamma_0}| \le \frac{\eta}{3}|v_m'(t)|_{\Gamma_0}^2 + 3\eta|k'|_{L^1(0,\infty)}\int_0^t |k'(t-s)||v_m(s)|_{\Gamma_0}^2\,ds; \tag{41}$$

$$|2\,(r(t), v_m'(t))_{\Gamma_0}| \le \frac{\eta}{3}|v_m'(t)|_{\Gamma_0}^2 + \frac{3}{\eta}|r(t)|_{\Gamma_0}^2, \tag{42}$$

Using (40)-(42) in (39) and integrating from 0 to $t \le T_m$, we get

$$|v_m'(t)|^2 + \|v_m(t)\|^2 + |f^{\frac{1}{2}}\delta_m'(t)|_{\Gamma_1}^2 + |h^{\frac{1}{2}}\delta_m(t)|_{\Gamma_1}^2 + \eta\int_0^t |v_m'(\xi)|_{\Gamma_0}^2 d\xi \\ + 2\int_0^t |g^{\frac{1}{2}}\delta_m'(\xi)|_{\Gamma_1}^2 d\xi \le \left| f^{\frac{1}{2}}\frac{\partial u_0}{\partial \nu} \right|_{\Gamma_1}^2 + |h^{\frac{1}{2}}\delta_m(0)|_{\Gamma_1}^2 + 3\eta k(0)^2 C\int_0^t \|v_m(\xi)\|^2 d\xi \\ + 3\eta|k'|_{L^1(0,\infty)}\int_0^t\int_0^\xi |k'(\xi - s)||v_m(s)|_{\Gamma_0}^2\,ds d\xi \\ + \int_0^t \left[\frac{3}{\eta}|r(\xi)|_{\Gamma_0}^2 - 2\left(\frac{\partial \phi(\xi)}{\partial \nu}, v_m'(\xi) \right)_{\Gamma_1} - 2\,(\phi'(\xi), \delta_m'(\xi))_{\Gamma_1} + 2\,(\mathcal{F}(\xi), v_m'(\xi)) \right] d\xi. \tag{43}$$

We note that

$$3\eta|k'|_{L^1(0,\infty)}\int_0^t\int_0^\xi |k'(\xi-s)||v_m(s)|_{\Gamma_0}^2\,dsd\xi \le 3\eta|k'|_{L^1(0,\infty)}^2 C\int_0^t \|v_m(\xi)\|^2\,d\xi; \tag{44}$$

$$\int_0^t \frac{3}{\eta}|r(\xi)|_{\Gamma_0}^2 d\xi \le C_1 + C_2(T+T^3); \tag{45}$$

$$2\int_0^t \left(\frac{\partial\phi(\xi)}{\partial\nu}, v_m'(\xi)\right)_{\Gamma_1} d\xi = 2\int_{\Gamma_1}\int_0^t -\frac{\partial\phi'}{\partial\nu}(x,\xi)v_m(x,\xi)d\xi + \frac{\partial\phi}{\partial\nu}(x,\xi)v_m(x,\xi)|_0^t d\Gamma$$

$$\le C_3 + C_4(T+T^2) + C\int_0^t \|v_m(\xi)\|^2 d\xi + \frac{1}{2}\|v_m(t)\|^2\,. \tag{46}$$

Coming back with these estimates to (43) and applying Gronwall's inequality, we have that there exist a constant, $L_1 = L_1(T) > 0$, independent of the m and $t \in [0, T_m]$, such that

$$|v_m'(t)|^2 + \|v_m(t)\|^2 + |\delta_m'(t)|_{\Gamma_1}^2 + |\delta_m(t)|_{\Gamma_1}^2 \le L_1, \quad \forall\, t \in [0, T_m]. \tag{47}$$

From this estimate we can extend the solution of the approximate problem to the whole interval $[0, T]$ and (47) holds for all $t \in [0, T]$.
Estimate II: Taking $w = v_m''(t)$ in the approximate equation (37) and putting $t = 0$, we come to

$$|v''(0)|^2 = \left(-\eta u_1 - \frac{\partial u_0}{\partial\nu}, v_m''(0)\right)_{\Gamma_0} + \left(F(0) + \Delta u_0, v_m''(0)\right). \tag{48}$$

Now, taking $z = \delta_m''(t)$ in (38) and putting $t = 0$ we get

$$0 = \left(f\delta_m''(0) + g\frac{\partial u_0}{\partial\nu} + h\delta_m(0), \delta_m''(0)\right)_{\Gamma_1} + \left(u_1, \delta_m''(0)\right)_{\Gamma_1}.$$

This inequality, (48), the assumptions on $u_0, u_1, \delta_0, F, f, g, h$ and (19) yield a constant C_5 such that

$$|v_m''(0)| + |\delta_m''(0)|_{\Gamma_1} \le C_5. \tag{49}$$

Differentiating (37) and (38) with respect to t and taking $w = 2v_m''(t)$ and $z = 2\delta_m''(t)$ we find

$$\frac{d}{dt}\left(|v_m''(t)|^2 + \|v_m'(t)\|^2 + |f^{\frac12}\delta_m''(t)|_{\Gamma_1}^2 + |h^{\frac12}\delta_m'(t)|_{\Gamma_1}^2\right) + 2\eta|v_m''(t)|_{\Gamma_0}^2 + 2|g^{\frac12}\delta_m''(t)|_{\Gamma_1}^2$$

$$= -2\eta k(0)\left(v_m'(t), v_m''(t)\right)_{\Gamma_0} - 2\eta\left(\frac{d}{dt}(k' * v_m)(t), v_m''(t)\right)_{\Gamma_0}$$

$$+ 2\left(r'(t), v_m''(t)\right)_{\Gamma_0} - 2\left(\frac{\partial\phi'(t)}{\partial\nu}, v_m''(t)\right)_{\Gamma_1} + 2\left(\mathcal{F}'(t), v_m''(t)\right). \tag{50}$$

Next we estimate each term in the right hand side of (50). We note that

$$|2\eta k(0)\left(v_m'(t), v_m''(t)\right)_{\Gamma_0}| \leq \frac{\eta}{3}|v_m''(t)|_{\Gamma_0}^2 + 3\eta k(0)^2 C\|v_m'(t)\|^2; \tag{51}$$

$$|2\left(r'(t), v_m''(t)\right)_{\Gamma_0}| \leq \frac{\eta}{3}|v_m''(t)|_{\Gamma_0}^2 + \frac{3}{\eta}|r'(t)|_{\Gamma_0}^2. \tag{52}$$

Since

$$\frac{d}{dt}(k' * v_m)(t) = k'(t)v_m(0) + \int_0^t k'(\tau)v_m'(t-\tau)d\tau = (k' * v_m')(t),$$

analogously to the (41) we have

$$\left|2\eta\left(\frac{d}{dt}(k' * v_m)(t), v_m''(t)\right)_{\Gamma_0}\right| \leq \frac{\eta}{3}|v_m''(t)|_{\Gamma_0}^2$$

$$+ 3\eta|k'|_{L^1(0,\infty)}\int_0^t |k'(t-s)||v_m'(s)|_{\Gamma_0}^2\, ds. \tag{53}$$

Using (51)-(53) in (50), integrating from 0 to $t \leq T$ and noting (49), we obtain

$$|v_m''(t)|^2 + \|v_m'(t)\|^2 + |f^{\frac{1}{2}}\delta_m''(t)|_{\Gamma_1}^2 + |h^{\frac{1}{2}}\delta_m'(t)|_{\Gamma_1}^2 + \eta\int_0^t |v_m''(\xi)|_{\Gamma_0}^2 d\xi$$

$$+2\int_0^t |g^{\frac{1}{2}}\delta_m''(\xi)|_{\Gamma_1}^2 d\xi \leq C_6 + 3\eta k(0)^2 C\int_0^t \|v_m'(\xi)\|^2 d\xi$$

$$+3\eta|k'|_{L^1(0,\infty)}\int_0^t\int_0^\xi |k'(\xi-s)||v_m'(s)|_{\Gamma_0}^2\, ds d\xi$$

$$+\int_0^t \left[\frac{3}{\eta}|r'(\xi)|_{\Gamma_0}^2 - 2\left(\frac{\partial\phi'(\xi)}{\partial\nu}, v_m''(\xi)\right)_{\Gamma_1} + 2\left(\mathcal{F}'(\xi), v_m''(\xi)\right)\right] d\xi. \tag{54}$$

We have

$$3\eta|k'|_{L^1(0,\infty)}\int_0^t\int_0^\xi |k'(\xi-s)||v_m'(s)|_{\Gamma_0}^2\, ds d\xi \leq 3\eta|k'|_{L^1(0,\infty)}^2 C\int_0^t \|v_m'(\xi)\|^2\, d\xi; \tag{55}$$

$$\int_0^t \frac{3}{\eta}|r'(\xi)|_{\Gamma_0}^2 d\xi \leq C_7 + C_8 T; \tag{56}$$

and proceeding as in (46), we obtain

$$2\int_0^t \left(\frac{\partial\phi'(\xi)}{\partial\nu}, v_m''(\xi)\right)_{\Gamma_1} d\xi \leq C_9 + \frac{1}{2}\|v_m'(t)\|^2. \tag{57}$$

Substituting (55)-(57) in (54), we can apply Gronwall's inequality to get a constant $L_2 = L_2(T) > 0$, independent of the m and $t \in [0,T]$, such that

$$|v_m''(t)|^2 + \|v_m'(t)\|^2 + |\delta_m''(t)|_{\Gamma_1}^2 + |\delta_m'(t)|_{\Gamma_1}^2 \leq L_2, \quad \forall\, t \in [0,T]. \tag{58}$$

which is the second estimate.

From (47) and (58) we can find subsequences, still denoted by $(v_m)_{m\in\mathbb{N}}$, $(\delta_m)_{m\in\mathbb{N}}$ and functions $v\,,\delta$ such that

$$\begin{array}{ll} v_m \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T;W)\,, & \delta_m \overset{*}{\rightharpoonup} \delta \text{ in } L^\infty(0,T;L^2(\Gamma_1))\,, \\ v'_m \overset{*}{\rightharpoonup} v' \text{ in } L^\infty(0,T;W)\,, & \delta'_m \overset{*}{\rightharpoonup} \delta' \text{ in } L^\infty(0,T;L^2(\Gamma_1))\,, \\ v''_m \overset{*}{\rightharpoonup} v'' \text{ in } L^\infty(0,T;L^2(\Omega)), & \delta''_m \overset{*}{\rightharpoonup} \delta'' \text{ in } L^\infty(0,T;L^2(\Gamma_1)). \end{array} \tag{59}$$

Using compactness arguments and the convergences (59) we can pass to limit, as $m\to\infty$, in the approximate equations (37), (38). Whence we have

$$\begin{aligned} &(v''(t),w) + ((v(t),w)) + (\eta[v'(t)+k(0)v(t)+(k'*v)(t)] - r(t),\gamma_0(w))_{\Gamma_0} \\ &\qquad + \left(\frac{\partial\phi(t)}{\partial\nu} - \delta'(t),\gamma_0(w)\right)_{\Gamma_1} = (\mathcal{F}(t),w)\,,\ \ \forall\, w\in W\,, \end{aligned} \tag{60}$$

$$-\,(v'(t),z)_{\Gamma_1} = (f\delta''(t)+g\delta'(t)+h\delta(t),z)_{\Gamma_1} + (\phi'(t),z)_{\Gamma_1}\,,\ \ \forall\, z\in L^2(\Gamma_1). \tag{61}$$

The last equation proves (28). Taking $w\in\mathcal{D}(\Omega)$ in (60) we obtain

$$v''(t) - \Delta v(t) = \mathcal{F}(t) \quad \text{in } \mathcal{D}'(\Omega)\,,\ \text{a.e. in } \ (0,T)\,, \tag{62}$$

and since $\mathcal{F}(t), v''(t)\in L^2(\Omega)$, we can see that $\Delta v(t)\in L^2(\Omega)$, and equation (62) holds a.e. in $\Omega\times(0,T)$, which proves (25).

Now we shall interpret the sense in which v and δ satisfy (26) and (27). Multiplying (25) by $w\in W$, integrating over Ω and using the Green's formula we find

$$\int_\Omega v''w\,dx + \int_\Omega \nabla v\nabla w\,dx - \langle\gamma_1(v(t)),\gamma_0(w)\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)} = \int_\Omega \mathcal{F}w\,dx.$$

This and (60) yield

$$\begin{aligned} &\langle\gamma_1(v(t)),\gamma_0(w)\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)} = -\,(\eta[v'(t)+k(0)v(t)+(k'*v)(t)],\gamma_0(w))_{\Gamma_0} \\ &+ (r(t),\gamma_0(w))_{\Gamma_0} - \left(\frac{\partial\phi(t)}{\partial\nu} - \delta'(t),\gamma_0(w)\right)_{\Gamma_1}\,, \quad \forall\, w\in W\,,\ \text{a.e. in } [0,T]\,, \end{aligned} \tag{63}$$

which proves (26) and (27).

Uniqueness: Let (v_1,δ_1) and (v_2,δ_2) be solutions to (25)-(30). Define $\vartheta = v_1 - v_2$ and $\theta = \delta_1-\delta_2$, then

$$\vartheta'' - \Delta\vartheta = 0 \quad \text{a.e. in } \Omega\times(0,T), \tag{64}$$

$$\vartheta' + f\theta'' + g\theta' + h\theta = 0 \quad \text{a.e. on } \Gamma_1\times(0,T), \tag{65}$$

$$\begin{aligned} &\langle\gamma_1(\vartheta(t)),\gamma_0(w)\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)} = -\,(\eta[\vartheta'(t)+k(0)\vartheta(t)+(k'*\vartheta)(t)],\gamma_0(w))_{\Gamma_0} \\ &+ (\theta'(t),\gamma_0(w))_{\Gamma_1}\,, \quad \forall\, w\in W\,,\ \text{a.e. in } [0,T], \end{aligned} \tag{66}$$

$$\vartheta(x,0) = \vartheta'(x,0) = 0 \quad \text{in } \Omega, \tag{67}$$

$$\theta(x,0) = \theta'(x,0) = 0 \quad \text{on } \Gamma_1. \tag{68}$$

Multiplying (64) by ϑ', (65) by θ', integrating over Ω and Γ_1, respectively, and observing (66) we obtain

$$\frac{d}{dt}\left(|\vartheta'(t)|^2 + \|\vartheta(t)\|^2 + |f^{\frac{1}{2}}\theta'(t)|^2_{\Gamma_1} + |h^{\frac{1}{2}}\theta(t)|^2_{\Gamma_1} + \eta k(0)|\vartheta(t)|^2_{\Gamma_0}\right) + 2\eta|\vartheta'(t)|^2_{\Gamma_0}$$

$$+ 2|g^{\frac{1}{2}}\theta'(t)|^2_{\Gamma_1} = -2\eta\left((k' * \vartheta)(t), \vartheta'(t)\right)_{\Gamma_0}. \tag{69}$$

Note that

$$\left|2\eta\left((k' * \vartheta)(t), \vartheta'(t)\right)_{\Gamma_0}\right| \leq \eta|k'|_{L^1(0,\infty)}\int_0^t |k'(t-s)||\vartheta(s)|^2_{\Gamma_0}ds + \eta|\vartheta'(t)|^2_{\Gamma_0}.$$

Using this estimate in (69), integrating from 0 to $t \leq T$ and proceeding as in (44) we can see that there exists a constant $C_{10} > 0$ such that

$$|\vartheta'(t)|^2 + \|\vartheta(t)\|^2 + |\theta'(t)|^2_{\Gamma_1} + |\theta(t)|^2_{\Gamma_1} \leq C_{10}\int_0^t \|\vartheta(\xi)\|^2 d\xi,$$

which yields $\vartheta = 0$ a.e. in $\Omega \times [0,T]$ and $\theta = 0$ a.e. in $\Gamma_1 \times [0,T]$. This complete the proof of uniqueness.

□

Remark 3.1 If we have regularity on function v, for instance $v \in L^\infty(0,T; W \cap H^2(\Omega))$, we can see that (26) and (27) hold a.e. in $\Gamma_0 \times (0,T)$ and $\Gamma_1 \times (0,T)$, respectively. To verify this assertion, let

$$H = \{(\gamma_0(\psi))_{|\Gamma_1} \text{ such that } \psi \in W \text{ with } (\gamma_0(\psi))_{|\Gamma_0} = 0\}$$

Thus H is dense in $L^2(\Gamma_1)$. We can rewrite (63) as

$$\left(\frac{\partial v}{\partial \nu}(t), \gamma_0(w)\right)_{\Gamma_0} + \left(\frac{\partial v}{\partial \nu}(t), \gamma_0(w)\right)_{\Gamma_1} =$$

$$-\left(\eta[v'(t) + k(0)v(t) + (k' * v)(t)] - r(t), \gamma_0(w)\right)_{\Gamma_0} - \left(\frac{\partial \phi(t)}{\partial \nu} - \delta'(t), \gamma_0(w)\right)_{\Gamma_1},$$

$\forall\, w \in W$. Hence,

$$\left(\frac{\partial v}{\partial \nu}(t) + \frac{\partial \phi(t)}{\partial \nu} - \delta'(t), z\right)_{\Gamma_1} = 0, \quad \forall\, z \in H,$$

which gives

$$\frac{\partial v}{\partial \nu} = \delta' - \frac{\partial \phi}{\partial \nu}, \qquad \text{a.e. in } \Gamma_1 \times (0,T).$$

Analogously, we can prove that

$$\frac{\partial v}{\partial \nu} = -\eta[v' + k(0)v + (k' * v)] + r, \qquad \text{a.e. in } \Gamma_0 \times (0,T).$$

Remark 3.2 Let (v, δ) be the solution to the perturbed problem given by theorem (3.1) and ϕ the function defined in (22). Then we can easily verify that (u, δ), where

$$u(x,t) = v(x,t) + \phi(x,t),$$

is the solution to (11)-(16) and theorem (2.1) is proved.

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