Bol. Soc. Paran. Mat. (3s.) **v. 27** 1 (2009): 59–63.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/spm doi:10.5269/bspm.v27i1.9068

A Note on The Convexity of Chebyshev Sets

T.D. Narang and Sangeeta

ABSTRACT: Perhaps one of the major unsolved problem in Approximation Theory is : Whether or not every Chebyshev subset of a Hilbert space must be convex. Many partial answers to this problem are available in the literature. R.R. Phelps [Proc. Amer. Math. Soc. 8 (1957), 790-797] showed that a Chebyshev set in an inner product space (or in a strictly convex normed linear space) is convex if the associated metric projection is non-expansive. We extend this result to metric spaces.

Key Words: Convex set, Cheybshev set, Convex space, Strongly convex space, Metric projection , Non-expansive map.

Contents

1. Introduction

It is well known (see e.g. [4], p. 35) that a closed convex subset of a Hilbert space is Chebyshev. One of the major unsolved problem in Approximation Theory is its converse i.e. whether every Chebyshev subset of a Hilbert space is convex. Surveys giving various partial answers of this problem were given by Vlasov (1973), Narang (1977), Deutsch (1993) and by Balaganski and Vlasov (1996). Metric projections have been very helpful in giving some partial answers of this problem. Phelps [9] showed that in an inner product space (in a strictly convex normed linear space), a Chebyshev set is convex if the associated metric projection is non-expansive. Here we extend this result to metric spaces.

Before proceeding to our main result, we recall few definitions.

2. Definitions and Notations

Definition 2.1 *Let (X, d) be a metric space and $x, y, z \in X$. We say that z is* ***between*** *x and y if $d(x, z) + d(z, y) = d(x, y)$. For any two points $x, y \in X$, the set*

$$\{z \in X : d(x, z) + d(z, y) = d(x, y)\}$$

is called the ***metric segment*** *and is denoted by $[x, y]$. The set $[x, y, -[= \{z \in X : d(x, y) + d(y, z) = d(x, z)\}$ denotes the half ray starting from x and passing through*

2000 *Mathematics Subject Classification:* 41A65

y. Correspondingly, $]-,x,y]$ is the half ray starting from y and passing through x, $]-x,y,-[$ is the line passing through x and y.

Definition 2.2 *A metric space (X,d) is said to be **convex** [10] if for every x, y in X and for every t, $0 \leq t \leq 1$ there exists at least one point z such that $d(x,z) = (1-t)\ d(x,y)$ and $d(z,y) = t\ d(x,y)$.*

Definition 2.3 *The space X is said to be **strongly convex** [10] or an M-space [7] if such a z exists and is unique for each pair x, y of X.*

Thus for strongly convex metric spaces each t, $0 \leq t \leq 1$, determines a unique point of the segment $[x,y]$.

Definition 2.4 *A metric space (X,d) is called **externally convex** [6] if for all distinct points x, y such that $d(x,y) = \lambda$, and $r > \lambda$ there exists a unique z of X such that $d(x,y) + d(y,z) = d(x,z) = r$.*

Definition 2.5 *A strongly convex metric space (X,d) is said to be **strictly convex** [8] if for every pair x, y of X and $r > 0$.*

$$d(x,p) \leq r, d(y,p) \leq r \text{ imply } d(z,p) < r$$

unless $x = y$, where p is arbitrary but fixed point of X and z is any point in the open metric segment $]x,y[$.

Therefore, in a strictly convex metric space if x and y are any two points on the boundary of a sphere then $]x,y[$ lies strictly inside the sphere.

Remark 2.1: A convex metric space need not be externally convex and an externally convex space need not be convex(see [6]). The unit disc in R^2 with Euclidean distance is a strictly convex M-space and is not a normed linear space. For more examples of convex metric spaces, externally convex metric spaces and M-spaces one may refer to [6] and [11].

Definition 2.6 *A subset K of a metric space (X,d) is said to be **convex** (see [8]) if for every $x,y \in K$, any point between x and y is also in K i.e. for each x, y in K, the metric segment $[x,y]$ lies in K.*

Definition 2.7 *Let S be a subset of a metric space (X,d) and $x \in X$. An element $s_o \in S$ satisfying $d(x,s_o) \equiv \inf\{d(x,y) : y \in S\} \equiv d(x,S)$, is called a **best approximation** to x in S. The set $P_S(x) = \{s_o \in S : d(x,s_o) = d(x,S)\}$ is called the set of best approximants to x in S. The set S is said to be **proximinal** if $P_S(x) \neq \phi$ for each $x \in X$ and is called **Chebyshev** if $P_S(x)$ is exactly singleton for each $x \in X$. The set-valued map $f : X \to 2^S \equiv$ collection of subsets of S, taking each $x \in X$ to the set $P_S(x)$ is called the **nearest point map** or the **metric projection**.*

It is clear that f exists iff S is proximinal and that f is single-valued iff S is Chebyshev.

Definition 2.8 *We say that f shrinks distances or f is **non-expansive** if*

$$d(f(x), f(y)) \leq d(x, y) \text{ for all } x, y \in X.$$

Definition 2.9 *We say that X has **Property (P)** [9] if the nearest point map f shrinks distances whenever it exists for a closed convex set $S \subseteq X$.*

It is well known (see [9]) that every inner product space has Property (P). Infact, Property (P) characterizes inner product spaces of dimension ≥ 3 (Theorem 5.2 [9]).

3. Convexity of Chebyshev sets

The following lemma, which is easy to prove, will be used in showing that the shrinking property of the metric projection implies the convexity of a Cheybyshev set in externally convex M-spaces :

Lemma 3.1 *For any $z \in \,]-, x, y, -[$ and any closed subset E of $]-, x, y, -[$, we have $d(z, E)$ is attained.*

Theorem 3.1 *If (X, d) is an externally convex M-space and S is a Chebyshev subset of X, then S is convex if f is non-expansive.*

Proof: Suppose S is not convex. Then there exist $x_1, y_1 \in S$, $x_1 \neq y_1$ such that some $u \in]x_1, y_1[$ lies in $X \setminus S$. Let $K_1 = S \cap [x_1, u]$ and $K_2 = S \cap [u, y_1]$. Then both K_1, K_2 are closed subsets of S. Since $u \notin K_1 \cup K_2$, $d(u, K_1) = r_1 > 0$ and $d(u, K_2) = r_2 > 0$. By the lemma, both K_1 and K_2 are proximinal. Hence, there exists $x \in K_1$, $y \in K_2$ such that $d(u, K_1) = d(u, x)$, $d(u, K_2) = d(u, y)$ and the metric segment $]x, y[$ lies in $X \setminus S$. Let z be the mid point of x and y i.e.

$$d(x, z) = d(z, y) = \frac{1}{2} d(x, y). \tag{1}$$

Firstly, we claim that $f(z) \neq x$ and $f(z) \neq y$. If $f(z) = x$ then

$$d(f(y), f(z)) = d(y, x) > \frac{1}{2} d(y, x) = d(y, z),$$

a contradiction to the fact that f is nonexpansive. Same contradiction results if $f(z) = y$. Therefore $f(z)$ is neither equal to x nor equal to y.

Now we claim that

$$d(x, y) < d(x, f(z)) + d(f(z), y). \tag{2}$$

This will be true if we prove that

$$d(x, y) \neq d(x, f(z)) + d(f(z), y).$$

Suppose $d(x, y) = d(x, f(z)) + d(f(z), y)$. Then $f(z) \in]x, y[\subseteq X \setminus S$, a contradiction as $f(z)$ being nearest point to z is in S. Therefore (2) is true.

Now one of $d(x, f(z))$, $d(y, f(z))$ is greater than $\frac{1}{2}d(x, y)$. For otherwise

$$d(x,y) < d(x,f(z)) + d(f(z),y)$$
$$\leq \frac{1}{2}d(x,y) + \frac{1}{2}d(x,y) = d(x,y),$$

a contradiction. Suppose $d(x, f(z)) > \frac{1}{2}d(x, y)$. Then

$$d(f(x), f(z)) = d(x, f(z)) > \frac{1}{2}d(x,y) = d(x,z) \text{ by (1)}$$

i.e. f is not non-expansive. Therefore our supposition that S is not convex is wrong. Same contradiction results if we suppose $d(y, f(z)) > \frac{1}{2}d(x, y)$. Hence S is convex. □

Corollary 3.1A *In a strictly convex metric space which is also externally convex, a Chebyshev set is convex if the nearest point map is non-expansive.*

Since a strictly convex normed linear space is an externally convex M-space, the above theorem extends the following result of Phelps [9]:

Suppose that E is a strictly convex normed linear space and S is a Chebyshev subset of E then S is convex if f shrinks distances.

Since every inner product space is a strictly convex normed linear space ([2] - p. 24), we have:

If S is a Chebyshev set in an inner product space such that the nearest point map is non-expansive then S is convex (see [4] - p. 302).

The following problem (which has a solution in inner product spaces - see [4] - p. 302) is open :

If S is convex Cheybshev set in an externally convex M-space then whether the nearest point map is non-expansive?

Acknowledgments

The authors are thankful to the referee for very valuable comments and suggestions leading to an improvement of the paper. The authors also sincerely acknowledge the help received from Prof. Roshdi Khalil for the revision of the paper

References

1. V. S. Balaganski and L.P. Vlasov, *The problem of convexity of Chebyshev sets*, Russian Math. Surveys, 51 (1996), 1127-1190.
2. E. W. Cheney, *Introduction to Approximation Theory*, McGraw Hill, New York, 1966.
3. F. Deutsch, *The Convexity of Chebyshev sets in Hilbert space*, in Topics in Polynomials of One and Several Variables and their Applications (edited by Th. M. Rassias, H.M. Srivastava and A. Yanushauskas), World Scientific, 1993, 143-150.
4. Frank Deutsch, *Best Approximation in Inner Product Spaces*, Springer, New York (2001).
5. Roshdi Khalil, *Chebyshev sets and strictly convex metric spaces*, Tamkang J. Math. 17(1986), 9-12.

6. Roshdi Khalil, *Best approximation in metric spaces*. Proc. Amer. Math. Soc. 103(1988), 579-586.

7. T. D. Narang, *Convexity of Chebyshev sets*, Nieuw Archief Voor Wiskunde, 25(1977), 377-402.

8. T. D. Narang, *Best approximation and strict convexity of metric spaces*, Arch. Math. 42(1981), 87-90.

9. R. R. Phelps, *Convex sets and nearest points*, Proc. Amer. Math. Soc., 8 (1957), 790-797.

10. Dale Rolfsen, *Geometric Methods in Topological Spaces*, Topology Conference, Arizona State Univ., (1967).

11. W. Takahashi, *A convexity in metric space and nonexpansive mappings I*, Kodai Math. Sem. Rep., 22(1970), 142-149.

12. L. P. Vlasov, *Approximative properties of sets in normed linear spaces*, Russian Math. Surveys 28(1973), 1-66.

T.D. Narang
Department of Mathematics,
Guru Nanak Dev University,
Amritsar - 143005,
India.
E-mail: tdnarang1948@yahoo.co.in

Sangeeta
Department of Mathematics,
Amardeep Singh Shergill Memorial College,
Mukandpur - 144507
India.
seetzz_20@yahoo.co.in