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The beginning of the Fučik spectrum for a Steklov Problem

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ABSTRACT: In this paper, we give some properties of the first nonprincipal eigenvalue for an asymmetric Steklov problem with weights, and we study the Fučik spectrum.

Key Words: Steklov problem, Fučik spectrum, First curve, weights.

Contents

1. Introduction

In a previous work [4], we investigated the eigenvalues of the following asymmetric Steklov problem with weights

$$\begin{cases} \triangle_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ν denotes the unit exterior normal, $1 < p < \infty$ and $\triangle_p u = div(|\nabla u|^{p-2}\nabla u)$ indicate the p–Laplacian. $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary, where $N \geq 2$, $m,\ n \in L^q(\partial\Omega)$ with $\frac{N-1}{p-1} < q$ if $1 < p \leq N$ and $q \geq 1$ if $p > N$. We proved the existence of a first nonprincipal positive eigenvalue $c(m,n)$ for (1).

Our purpose in the present paper is to give some properties of $c(m,n)$ and to study the Fučik spectrum of the Steklov problem on $W^{1,p}(\Omega)$. Recall that the latter is defined as the set $\Sigma = \Sigma(m,n)$ of those $(\alpha,\beta) \in \mathbb{R}^2$ such that

$$\begin{cases} \triangle_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \alpha m(x)(u^+)^{p-1} - \beta n(x)(u^-)^{p-1} & \text{on } \partial\Omega, \end{cases} \tag{2}$$

has a nontrivial solution. Let $\lambda_1(m)$ be the principal positive eigenvalue of the following Steklov problem

$$\begin{cases} \triangle_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m(x)|u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{3}$$

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The Fučik spectrum Σ clearly contains the lines $\{0\} \times \mathbb{R}$, $\mathbb{R} \times \{0\}$, $\{\lambda_1(m)\} \times \mathbb{R}$, $\mathbb{R} \times \{\lambda_1(n)\}$ and also possibly the lines $\mathbb{R} \times \{-\lambda_1(-n)\}$ and $\{-\lambda_1(-m)\} \times \mathbb{R}$. It will be convenient to denote by $\Sigma^* = \Sigma^*(m,n)$ the set Σ without these 2, 3 or 4 lines. In this work, we show in particular that if m and n both change sign in $\partial\Omega$, then each of the four quadrants in the (α, β) plane contains a first (nontrivial) curve of Σ. This is probably the main results of our paper.

In the preliminary Section 2, we collect some results relative to the usual Steklov problem (3). We also recall some results concerning $c(m,n)$ the first nonprincipal positive eigenvalue of (1). Several properties of the $c(m,n)$ as a function of the weights m, n are investigated in Section 3: continuity, monotonicity and homogeneity. In Section 4, we apply our results to the study of the Fučik spectrum. We also proved the continuity, the monotonicity of the first (nontrivial) curve of Σ, and we show that the two trivial lines $\{\lambda_1(m)\} \times \mathbb{R}$, $\mathbb{R} \times \{\lambda_1(n)\}$ are isolated in Σ.

2. Preliminaries

Throughout this paper Ω will be a bounded domain in $\mathbb{R}^N$ with a Lipschitz continuous boundary. We assume that m, $n \in L^q(\partial\Omega)$, where q as above. We also assume that

$$m^+ = \max(m, 0) \not\equiv 0 \text{ and } n^+ = \max(n, 0) \not\equiv 0, \tag{4}$$

and $d\sigma$ is the $N-1$ dimensional Hausdorff measure. We start by recall some results relative to the usual Steklov problem (3). Let

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx \right)^{1/p}$$

be the norm of $W^{1,p}(\Omega)$, and for any integer $k \geq 1$ let

$$\frac{1}{\lambda_k(m)} = \sup_{C \in C_k} \min_{u \in C} \frac{\int_{\partial\Omega} m|u|^p d\sigma}{\|u\|^p_{W^{1,p}(\Omega)}},$$

where

$$C_k = \{C \subset W^{1,p}(\Omega); C \text{ is compact, symmetric and } \gamma(C) \geq k\},$$

with γ is the Krasnoselski genus.

Proposition 2.1 *Assume $m \in L^q(\partial\Omega)$ and $m^+ \not\equiv 0$ in $\partial\Omega$. Then $\lambda_k(m)$ is a sequence of eigenvalues of problem 3 such that $\lambda_k(m) \to +\infty$ as $k \to +\infty$.*

Proposition 2.1 is proved in [5] by applying a general result from infinite dimensional Ljusternik-Schnirelman theory (see [6]). In [5], the authors proved the simplicity, isolation and monotonicity with respect to the weight of the first eigenvalue $\lambda_1(m)$ of the Steklov eigenvalue problem 6.

The lemma below guarantees that in a mountain pass situation, any minimizing path contains a critical point at the mountain pass level.

Lemma 2.1 *(see [1] and [2])*
Let E be a real Banach space and let $M := \{u \in E; g(u) = 1\}$, where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g. Let $f \in C^1(E, \mathbb{R})$ and consider the restriction $\tilde{f}$ of f to M.
Let u, $v \in M$ with $u \neq v$ and assume that

$$H := \{h \in C([0, 1], M); h(0) = u \text{ and } h(1) = v\}$$

is nonempty and that

$$c := \inf_{h \in H} \max_{w \in h([0,1])} f(w) > \max\{f(u), f(v)\}.$$

Suppose that $h \in H$ is such that $\max_{u \in h([0,1])} \tilde{f}(u) = c$. Then there exists $u \in h([0, 1])$ with $\tilde{f}(u) = c$ and which is a critical point of $\tilde{f}$.

Let us conclude this section with some results concerning $c(m, n)$ the first non-principal positive eigenvalue of (1). Let A and $B_{m,n} : W^{1,p}(\Omega) \to \mathbb{R}$, defined by $A(u) = \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}$ and $B_{m,n}(u) = \frac{1}{p}\int_{\partial\Omega}[m(u^+)^p + n(u^-)^p]d\sigma$. At this point let us introduce the set $M_{m,n} := \{u \in W^{1,p}(\Omega); B_{m,n}(u) = 1\}$. The condition $m^+ \not\equiv 0$ implies that $M_{m,n} \neq$. Moreover the set $M_{m,n}$ is a C^1 manifold in $W^{1,p}(\Omega)$. $\tilde{A}$ denotes the restriction of A to the manifold $M_{m,n}$. In [4] we showed the following theorem concerning the first nonprincipal positive eigenvalue $c(m, n)$ for (1), where

$$c(m, n) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[0,1]} \tilde{A}(u) \text{ and} \tag{5}$$

$$\Gamma = \{\gamma \in C([0, 1], M_{m,n}) : \gamma(0) = -\varphi_n \text{ and } \gamma(1) = \varphi_m\},$$

where φ_m denotes the normalized positive first eigenvalue of $\lambda_1(m)$.

Theorem 2.1 *$c(m, n)$ is an eigenvalue of (1) which satisfies*

$$\max\{\lambda_1(m), \lambda_1(n)\} < c(m, n).$$

Moreover there is no eigenvalue of (1) between $\max\{\lambda_1(m), \lambda_1(n)\}$ and $c(m, n)$.

3. Some properties of the first non trivial eigenvalue

In the following proposition suppose that m_k, n_k, m, n satisfy our preliminaries conditions.

Proposition 3.1 *If $(m_k, n_k) \to (m, n)$ in $L^q(\partial\Omega) \times L^q(\partial\Omega)$ then $c(m_k, n_k) \to c(m, n)$.*

Lemma 3.1 *(see [4]) Let $v_k \in W^{1,p}(\Omega)$ with $v_k \geq 0$, $v_k \not\equiv 0$ and $|v_k > 0| \to 0$. Let n_k be bounded in $L^q(\partial\Omega)$. Then $\int_{\partial\Omega} n_k v_k^p d\sigma / \|v_k\|^p_{W^{1,p}(\Omega)} dx \to 0$.*

Proof: [Proof of Proposition 3.1.] We first prove the upper semicontinuity. Let $\varepsilon > 0$ and take $\gamma \in \Gamma$ such that $\max_{t\in[0,1]} A(\gamma(t)) < c(m,n) + \varepsilon$. Since $B_{m,n}(\gamma(t))$ is continuous in its 3 arguments (m,n,t), we deduce that, for k sufficiently large $\max_{t\in[0,1]} A(\gamma(t)/B_{m_k,n_k}(\gamma(t))^{1/p}) < c(m,n)+\varepsilon$, consequently we have $\limsup c(m_k,n_k) \leq c(m,n) + \varepsilon$. Since ε is arbitrary, the upper semicontinuity follows. To prove the lower semicontinuity, suppose by contadiction that, for a subsequence, $c(m_k,n_k) \to c_0$ with $c_0 < c(m,n)$. Let $u_k \in M_{m_k,n_k}$ be a solution of (1), for $\lambda = c(m_k,n_k)$ and for the weights m_k, n_k. As $\|u_k\|_{W^{1,p}(\Omega)}$ remains bounded, for a subsequence $u_k \to u_0$ weakly in $W^{1,p}(\Omega)$; moreover $u_0 \in M_{m,n}$ and u_0 is a solution of (1) for $\lambda = c_0$ and for the weights m, n. Since $c_0 < c(m,n)$ then $c_0 = \lambda_1(m)$ and $u_0 = \varphi_m$ or $c_0 = \lambda_1(n)$ and $u_0 = -\varphi_n$. Consider the first case (similar argument in the other case). In that case $|u_k^-| \to 0$. It then follows from Lemma 3.1 that

$$\int_{\partial\Omega} n(u_k^-)^p d\sigma / \|u\|^p_{W^{1,p}(\Omega)} \to 0. \tag{6}$$

But multiplying by u_k^- the equation satisfied by u_k, one gets that the expression in (6) is equal to $1/c(m_k,n_k)$, which goes to $1/c_0 \neq 0$, a contradiction. □

Proposition 3.2 *If $m \leq \hat{m}$ and $n \leq \hat{n}$, then $c(m,n) \geq c(\hat{m},\hat{n})$.*

Proof: If γ is a path admissible in formula (5) for $c(m,n)$, then $\frac{1}{p}\int_{\partial\Omega}(\hat{m}(\gamma(t)^+)^p + \hat{n}(\gamma(t)^-)^p)d\sigma \geq 1$ and consequently $\hat{\gamma}(t) := \gamma(t)/B_{\hat{m},\hat{n}}(\gamma(t))^{1/p}$ is well–defined and is a path admissible in formula (5) for $c(\hat{m},\hat{n})$. Moreover $A(\hat{\gamma}(t)) \leq A(\gamma(t))$, and the conclusion follows. □

To conclude this section, let us observe that definition (5) clearly implies that $c(m,n)$ is homogeneous of degree -1:

$$c(sm, sn) = c(m,n)/s \text{ for } s > 0. \tag{7}$$

Some sort of separate sub-homogeneity also holds, which will be useful later:

Proposition 3.3 *If $0 < s < \hat{s}$, then $c(\hat{s}m,n) < c(sm,n)$ and $c(m,\hat{s}n) < c(m,sn)$.*

Proof: We will deal with the first inequality (similar argument for the second one). Let u be an eigenfunction in $M_{sm,n}$ associated to $c(sm,n)$ and let γ be the path in $M_{sm,n}$ from φ_{sm} to $-\varphi_n$ constructed from u as in the proof of Proposition 31 of [1]. The path $\hat{\gamma}(t) := (\frac{s}{\hat{s}})^{1/p}\gamma(t)^+ - \gamma(t)^-$ is then admissible in the definition (5) of $c(\hat{s}m,n)$ and we have

$$\begin{aligned} A(\hat{\gamma}(t)) &= \frac{s}{p\hat{s}}\|\gamma(t)^+\|^p_{W^{1,p}(\Omega)} + \frac{1}{p}\|\gamma(t)^-\|^p_{W^{1,p}(\Omega)} \\ &\leq \frac{1}{p}\|\gamma(t)\|^p_{W^{1,p}(\Omega)} = A(\gamma(t)), \end{aligned}$$

with strict inequality if $\gamma(t)^+ \not\equiv 0$. So the path $\hat{\gamma}$ goes in $M_{\hat{s}m,n}$ from $\varphi_{\hat{s}m}$ to $-\varphi_n$ and remains at levels $< c(sm,n)$ except at the point $v := -u^-/B_{sm,n}(-u^-)^{1/p}$ where the level is $c(sm,n)$. It follows that $c(\hat{s}m,n) < c(sm,n)$. Assume now by contradiction that $c(\hat{s}m,n) = c(sm,n)$. We can apply Lemma 2.1 to the path $\hat{\gamma}$ in the manifold $M_{\hat{s}m,n}$ to conclude that v must be a critical point of the restriction of A in $M_{\hat{s}m,n}$ at level $c(\hat{s}m,n)$. But this is impossible since v does not change sign. □

4. Fučik spectrum

Assume that the weights m an n satisfy our preliminaries conditions. The Fučik spectrum is thus defined as the set $\Sigma = \Sigma(m,n)$ of those $(\alpha,\beta) \in \mathbb{R}^2$ such that (2) has a nontrivial solution. Σ clearly contains the lines $\{0\}\times\mathbb{R}$, $\mathbb{R}\times\{0\}$, $\{\lambda_1(m)\}\times\mathbb{R}$, $\mathbb{R}\times\{\lambda_1(n)\}$ and also possibly the lines $\mathbb{R}\times\{-\lambda_1(-n)\}$ and $\{-\lambda_1(-m)\}\times\mathbb{R}$. It will be convenient to denote by $\Sigma^* = \Sigma^*(m,n)$ the set Σ without these 2, 3 or 4 lines.

We will start by looking at the part of Σ^* which lies in $\mathbb{R}^+\times\mathbb{R}^+$. The case of the other quadrants will be considered briefly at the end of the section. From the properties of $\lambda_1(n), \lambda_1(m)$ follows that if $(\alpha,\beta) \in \Sigma^* \cap (\mathbb{R}^+\times\mathbb{R}^+)$, then $\alpha > \lambda_1(m)$ and $\beta > \lambda_1(n)$.

Theorem 4.1 *For any $s > 0$, the line $\beta = s\alpha$ in the (α,β) plane intersects $\Sigma^* \cap (\mathbb{R}^+\times\mathbb{R}^+)$. Moreover the first point in this intersection is given by $\alpha(s) = c(m,sn)$, $\beta(s) = s\alpha(s)$, where $c(.,.)$ is defined in (3).*

Proof: It is easily adapted from Theorem 4.1 of [3]. □

Letting $s > 0$ varying, we get in this way a first curve $C := \{(\alpha(s),\beta(s)) : s > 0\}$ in $\Sigma^* \cap (\mathbb{R}^+\times\mathbb{R}^+)$. Here are some properties of this curve.

Proposition 4.1 *The functions $\alpha(s)$ and $\beta(s)$ in Theorem 4.1 are continuous. Moreover $\alpha(s)$ is strictly decreasing and $\beta(s)$ is strictly increasing. One also has that $\alpha(s) \to$ as $s \to 0$ and $\beta(s) \to +\infty$ as $s \to +\infty$.*

Proof: It is easily adapted from Proposition 4.2 of [3]. □

Lemma 4.1 *The lines $\mathbb{R}\times\{\lambda_1(n)\}$ and $\{\lambda_1(m)\}\times\mathbb{R}$ are isolated in $\Sigma^*\cap(\mathbb{R}^+\times\mathbb{R}^+)$.*

Proof: Assume by contradiction the existence of a sequence $(\alpha_k,\beta_k) \in \Sigma^*\cap(\mathbb{R}^+\times\mathbb{R}^+)$ such that $\alpha_k \to \alpha_0$ and $\beta_k \to \beta_0$ with $\alpha_0 \in \mathbb{R}$ and say $\beta_0 = \lambda_1(n)$. Let u_k be an eigenfunction corresponding to (α_k,β_k). Put $v_k = \frac{u_k}{||u_k||}$. Note that v_k changes sign, for a subsequence, $v_k \to v_0$ weakly in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and strongly in $L^{\frac{pq}{q-1}}(\partial\Omega)$ with

$$\int_\Omega |\nabla v_0^-|^p dx + \int_\Omega |v_0^-|^p dx = \lambda_1(n) \int_{\partial\Omega} n(x)(v_0^-)^p d\sigma. \qquad (8)$$

Consequently either (i) $v_0^- \equiv 0$ or (ii) v_0^- is an eigenfunction associated to $\lambda_1(n)$. In case (i), $v_0 \geq 0$, $v_0 \neq 0$ and so $v_0 > 0$ in Ω, which implies $|v_k^- > 0| \to 0$.

It then follows from Lemma 3.1 that

$$\int_{\partial\Omega} \beta_k n(x)(v_k^-)^p / \|v_k^-\|^p_{W^{1,p}(\Omega)} \to 0, \tag{9}$$

which is impossible since by the equation satisfied by v_k, the expression in (9) is equal to 1. In case (ii), $v_0 < 0$ in Ω, which implies $|v_k^+ > 0| \to 0$. An argument as above applied to v_k^+ then leads to a contradiction. □

To conclude this section we consider the distribution Σ^* in the other quadrants of $\mathbb{R} \times \mathbb{R}$. From now on we do not assume below that the weights m, n satisfy the condition (4).

Proposition 4.2 *$\Sigma^*(m,n)$ intersects $\mathbb{R}^+ \times \mathbb{R}^+$ (resp. $\mathbb{R}^- \times \mathbb{R}^-$, $\mathbb{R}^+ \times \mathbb{R}^-$, $\mathbb{R}^- \times \mathbb{R}^+$) if and only if m^+ and $n^+ \not\equiv 0$ in $\partial\Omega$ (rep. m^- and $n^- \not\equiv 0$ in $\partial\Omega$, m^+ and $n^- \not\equiv 0$ in $\partial\Omega$, m^- and $n^+ \not\equiv 0$ in $\partial\Omega$).*

Proof: The necessary conditions follow from the fact that, if $(\alpha, \beta) \in \Sigma^*$, then, for u a corresponding solution of (2),

$$0 < \|u^+\|^p = \alpha \int_{\partial\Omega} m|u^+|^p d\sigma \text{ and } 0 < \|u^-\|^p = \beta \int_{\partial\Omega} m|u^-|^p d\sigma.$$

To prove the sufficient conditions, let us consider for instance $\mathbb{R}^- \times \mathbb{R}^-$ (similar arguments in the other quadrants). We have that $(\alpha, \beta) \in \Sigma^*(m,n) \cap \mathbb{R}^- \times \mathbb{R}^-$ if and only if $(-\alpha, -\beta) \in \Sigma^*(-m,-n) \cap \mathbb{R}^- \times \mathbb{R}^-$. The assumption m^- and $n^- \not\equiv 0$ in $\partial\Omega$ means that the weights $-m$, $-n$ satisfy $(-m)^+$ and $(-n)^+ \not\equiv 0$ in $\partial\Omega$, i.e. satisfy the condition (4). Consequently Theorem 4.1 implies that $\Sigma^*(-m,-n) \cap \mathbb{R}^+ \times \mathbb{R}^+$ is nonempty and consequently $\Sigma^*(m,n) \cap \mathbb{R}^- \times \mathbb{R}^-$ is nonempty. □

Corollary 4.1A *If m and n both change sign in $\partial\Omega$, each of the four quadrants in the (α, β) plane contains a first curve of σ^*.*

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