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A note on iterative solutions for a nonlinear fourth order ode *

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ABSTRACT: This work is concerned with the existence of iterative solutions for a class of fourth order differential equations with nonlinear boundary conditions modeling beams on nonlinear elastic foundations. Some numerical simulations are also considered.

Key Words: Beam equation, nonlinear boundary, numerical solutions.

Contents

1. Introduction

In this work we are concerned with the boundary value problem

$$u^{(iv)}(t) = f(t,u,u'), \quad 0 < t < L \tag{1}$$

$$u(0) = 0, \quad u(L) = 0 \tag{2}$$

$$u''(0) = g(u'(0)), \quad u''(L) = h(u'(L)), \tag{3}$$

which models bending equilibrium of elastic beams on nonlinear supports. Following Ginsberg [7] or Grossinho and Tersian [8], u represents the configuration of an elastic beam of length L, subject to a force f exerted by the foundation. Both ends are attached to fixed torsional springs represented by the functions g and h.

Our objective is to show the existence of iterative solutions under local conditions on the functions f, g, h. Some numerical simulations are also presented. We refer the reader to [2,3,4,5,8,9] for other related works.

2. Iterative Solutions

Our existence result is the following.

Theorem 2.1 *Suppose that f, g, h are continuous functions and there exist constants $A, B, C > 0$ such that*

$$|f(t,u,v)| \leq A, \quad \forall\,(t,u,v) \in [0,T] \times [-\tfrac{L}{2}R, \tfrac{L}{2}R] \times [-R,R], \tag{4}$$

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$$|g(z)| \leq B, \quad \forall\, z \in [-R, R], \tag{5}$$

and

$$|h(z)| \leq C, \quad \forall\, z \in [-R, R]. \tag{6}$$

Then if

$$\frac{L^3}{16}A + \frac{L}{2}(B + C) \leq R, \tag{7}$$

problem (1)-(3) *has at least a solution.*

Theorem 2.2 *Suppose the assumptions of Theorem 2.1 hold. Suppose further that there exist constants* $\lambda_f, \lambda_g, \lambda_h > 0$ *such that*

$$|f(t, u, u') - f(t, v, v')| \leq \lambda_f \max\{|u - v|, |u' - v'|\}, \tag{8}$$

for all $(t, u, u'), (t, v, v') \in [0, L] \times [-\frac{L}{2}R\,,\ \frac{L}{2}R] \times [-R, R]$,

$$|g(u) - g(v)| \leq \lambda_g |u - v|, \quad \forall\, u, v \in [-R, R], \tag{9}$$

and

$$|h(u) - h(v)| \leq \lambda_h |u - v|, \quad \forall\, u, v \in [-R, R]. \tag{10}$$

Then if

$$\frac{L^3}{16}\max\{\tfrac{L}{2}, 1\}\lambda_f + \frac{L}{2}(\lambda_g + \lambda_h) < 1, \tag{11}$$

problem (1)-(3) *has an iterative solution* u *with* $\|u'\|_\infty \leq R$.

The proofs rely on fixed point theorems. We begin by rewriting problem (1)-(3) into a second order system. If $v = u''$ then we have

$$\begin{cases} u'' = v, \quad 0 < t < L \\ u(0) = 0, \quad u(L) = 0 \end{cases} \tag{12}$$

and

$$\begin{cases} v''(t) = f(t, u, u') \\ v(0) = g(u'(0)), \quad v(L) = h(u'(L)). \end{cases} \tag{13}$$

The Green's function associated to the second order problem (12) is precisely

$$G(x, t) = \begin{cases} \frac{x(L-t)}{L}, & \text{if } \ x \leq t \leq L \\ \frac{t(L-x)}{L}, & \text{if } \ t \leq x \leq L, \end{cases}$$

and gives

$$u(x) = \int_0^L -G(x, t)v(t)dt.$$

Analogously, from (13) we have

$$v(t) = \int_0^L -G(t, s)f(s, u(s), u'(s))ds + \frac{L - t}{L}g(u'(0)) + \frac{t}{L}h(u'(L)).$$

Then, combining the above identities we get

$$
\begin{aligned}
u(x) = & \int_0^L \int_0^L G(x,t)G(t,s)f(s,u(s),u'(s))dsdt \\
& - \int_0^L G(x,t)\left[\frac{(L-t)}{L}g(u'(0)) + \frac{t}{L}h(u'(L))\right]dt. \qquad (14)
\end{aligned}
$$

We can see that u is a solution of (1)-(3) if and only if it is a solution of (14). Next we apply fixed point arguments to solve (14). In view of (2) we apply fixed point theorems on the Banach space

$$E = \{u \in C^1([0,L]) \mid u(0) = u(L) = 0\}.$$

Because $u(0) = u(L) = 0$, we see that

$$\|u\|_\infty \leq \frac{L}{2}\|u'\|_\infty, \quad \forall\, u \in E,$$

and, in particular, the usual norm $\|u\|_{C^1} = \max\{\|u\|_\infty, \|u'\|_\infty\}$ is equivalent to

$$\|u\|_E = \|u'\|_\infty, \qquad (15)$$

which will be adopted here. Then we note that $\|u\|_E \leq R$ implies $|u'(x)| \leq R$ and $|u(x)| \leq \frac{L}{2}R$, for all $x \in [0,L]$.

Proof of Theorem 2.1 Let us define the operator $T : E \to E$ with $(Tu)(x)$ equal to the right hand side of (14). Then fixed points of T are solutions of problem (1)-(3). Next we show that T maps the closed ball $B[0,R]$ of E into itself. Indeed, noting that

$$\int_0^L |G(x,t)|\,dt \leq \frac{L^2}{8} \quad \text{and} \quad \int_0^L |G_x(x,t)|\,dt \leq \frac{L}{2},$$

we have from

$$
\begin{aligned}
(Tu)'(x) = & \int_0^L G_x(x,t)\left[\int_0^L G(t,s)f(s,u(s),u'(s))ds\right]dt \\
& - \int_0^L G_x(x,t)\left[\frac{(L-t)}{L}g(u'(0)) + \frac{t}{L}h(u'(L))\right]dt,
\end{aligned}
$$

that for $u \in B[0,R]$ and using (4)-(7),

$$
\begin{aligned}
\|(Tu)'\|_\infty & \leq \frac{L^3}{16}\max|f(t,u,u')| + \frac{L}{2}(|g(u'(L))| + |h(u'(L))|) \\
& \leq \frac{L^3}{16}A + \frac{L}{2}(B+C) \leq R.
\end{aligned}
$$

Therefore with respect to the norm (15), $T(B[0,R]) \subset B[0,R]$. To conclude the proof we note that T is completely continuous on $B[0,R]$ (by Arzela-Ascoli theorem) and therefore it has a fixed point by the Schauder's fixed point theorem (e.g. [1]). □

Proof of Theorem 2.2 Let $u,v \in B[0,R]$. Then as before, but using (8)-(10),

$$\begin{aligned}
\|(Tu-Tv)'\|_\infty &\le \frac{L^3}{16}\max|f(t,u,u')-f(t,v,v')| \\
&\qquad + \frac{L}{2}|g(u'(L))-g(v'(L))| + \frac{L}{2}|h(u'(L))-h(v'(L))| \\
&\le \frac{L^3}{16}\lambda_f \max\{|u-v|,|u'-v'|\} + \frac{L}{2}(\lambda_g+\lambda_h)|u'-v'| \\
&\le \frac{L^3}{16}\lambda_f \max\{\tfrac{L}{2},1\}\|u'-v'\|_\infty + \frac{L}{2}(\lambda_g+\lambda_h)\|u'-v'\|_\infty .
\end{aligned}$$

Therefore

$$\|Tu-Tv\|_E \le \left(\frac{L^3}{16}\max\{\tfrac{L}{2},1\}\lambda_f + \frac{L}{2}(\lambda_g+\lambda_h)\right)\|u-v\|_E .$$

From (11) we see that T is a contraction on $B[0,R]$ and then it has a fixed point from the Banach's fixed point theorem (e.g. [1]). □

3. Numerical Simulations

From Theorem 2.2 we obtain the iterative formulae $u^{k+1} = Tu^k$, were

$$\begin{aligned}
u^{k+1}(x) &= \int_0^L\int_0^L G(x,t)G(t,s)f(s,u^k(s),u^{k\prime}(s))dsdt \\
&\quad - \int_0^L G(x,t)\left[\frac{(L-t)}{L}g(u^{k\prime}(0)) + \frac{t}{L}h(u^{k\prime}(L))\right]dt, \qquad (16)
\end{aligned}$$

which converges to a solution of (1)-(3) for any initial approximation $u^0 \in B[0,R]$.

We show two numerical simulations to illustrate the use of (16). In both examples, $L=1$, $u^0=0$ and mesh size is 0.1. The integrals are approximated by trapezoidal method.

Example 1 First example we take

$$f(x,u,v) = x^5 - x^4 - x^3 + 121x - 24 - u,$$

$$g(v) = 0 \quad \text{and} \quad h(v) = -2v.$$

The exact solution in $[0,1]$ is

$$u(x) = x^5 - x^4 - x^3 + x.$$

After 10 iterations we get maximum error

$$E = \|u - u^{10}\|_\infty = .303411 \times 10^{-2}.$$

Other values are shown in the Table 1.

Table 1: Errors for Example 1 using mesh size $\Delta = 0.1$.

Iteration	E^k
1	.135393e-0
2	.811294e-1
3	.437432e-1
10	.303411e-2
20	.208102e-2
30	.207709e-2

Example 2 In this example we take

$$f(x,u,v) = 4\pi^4 \sin(\pi x)\cos(\pi x) - \frac{1}{16}\sin^2(\pi x)\cos^2(\pi x) + u^2,$$

$$g(v) = h(v) = \frac{v}{2} - \frac{\pi}{8}.$$

Then the exact solution in $[0,1]$ is

$$u(x) = \frac{1}{4}\sin(\pi x)\cos(\pi x).$$

After 4 iterations we get maximum error

$$E = \|u - u^4\|_\infty = .812751 \times 10^{-2}.$$

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References

1. R. P. Agarwal, D. O'Regan and M. Meehan, Fixed Point Theory and Appplications, Cambridge University Press, 2001.

2. E. Alves, M. L. Pelicer and T. F. Ma, *Monotone positive solutions for a fourth order equation with nonlinear boundary conditions*, Nonlinear Anal. 71 (2009) 3834-3841.

3. P. Amster and M. C. Mariani, *A fixed point operator for a nonlinear boundary value problem.* J. Math. Anal. Appl. 266 (2002) 160-168.

4. D. Andrade and T. F. Ma, *Numerical solutions for a nonlocal equation with reflection of the argument*, Neural Parallel Sci. Comput. 10 (2002), 227-234.

5. A. Cabada and F. M. Minhós, *Fully nonlinear fourth-order equations with functional boundary conditions*, J. Math. Anal. Appl. 340 (2008), 239-251.

6. D. Franco, *Fourth order problems with nonlinear boundary conditions*, J. Comput. Appl. Math. 174 (2005), 315-327.

7. J. H. Ginsberg, Mechanical and Structural Vibrations, John Wiley and Sons, New York, 2001.

8. M. R. Grossinho and St. A. Tersian, *The dual variational principle and equilibria for a beam resting on a discontinuous nonlinear elastic foundation*, Nonlinear Anal. 41 (2000), 417-431.

9. T. F. Ma and J. Silva, *Iterative solutions for a beam equation with nonlinear boundary conditions of third order*, Appl. Math. Comput. 159 (2004), 11-18.

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