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On Convex Hull of Orthogonal Scalar Spectral Functions of a Carleman Operator

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ABSTRACT: In this paper we describe the closed convex hull of orthogonal resolvents of an abstract symmetric operator of defect indices $(1,1)$, then we study the convex hull of orthogonal spectral functions of a Carleman operator in the Hilbert space $L^2(X,\mu)$.

Key Words: defect indices, integral operator, spectral theory.

Contents

1. Introduction

Let H be a (separable) Hilbert space and let A be a symmetric operator in H with defect indices $(1,1)$. It is well known that the set $W(A)$ of all generalized spectral functions of A is both convex and closed (in some natural topology). Consider the following problem:

Describe a convex hull $W_0(A)$ of orthogonal spectral functions of A.

This problem has been solved by I.M. Glazman [7] for jacobi matrices corresponding to Hamburger moment problem. To explain his result let us recall that according to the Krein-Naimark formula for generalized resolvents of A the set $W(A)$ is described as follows: $E_t \in W(A)$ if and only if

$$\int \frac{d(E_t f, f)}{t-\lambda} = \frac{D_0(\lambda)\varphi(\lambda) + D_1(\lambda)}{C_0(\lambda)\varphi(\lambda) + C_1(\lambda)}, \quad \varphi \in N, \tag{1}$$

where D_i,C_i $(i = 1,2)$ are entries of the resolvent matrix of A and f is a "scale" vector and N stands for the Nevanlinna class of functions holomorphic in $\mathbb{C}_+$ with non-negative imaginary parts [9].

Glazman [7] proved that $E_t \in W_0(A)$ if and only if φ in (1) admits a representation

$$\varphi(\lambda) = \varphi^*(C_1(\lambda)/C_0(\lambda)), \quad \varphi^* \in N. \tag{2}$$

Though Glazman proved this result for Jacobi matrices, it remains valid for any symmetric operator A with defect indices $(1,1)$. This result may be found in

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[2,5,14]. However, the orthogonal resolvents and the orthogonal spectral functions, which are those of the selfadjoint extensions in the same space, play a central role. We start by describing the closed convex hull of orthogonal resolvents of an abstract symmetric operator of defect indices $(1,1)$, then we study the convex hull of orthogonal spectral functions of a Carleman operator in the Hilbert space $L^2(X,\mu)$. Let us notice that the same problem for the differential operator of second order on $(0,\infty)$ was considered in [8].

2. Resolvents of a symmetric operator of defect indices (1.1)

We begin by introducing the following sets:

Φ the set of the analytical functions $\varphi(z)$ on the unit disc $K=\{z\in\mathbb{C}:|z|<1\}$ such that $|\varphi(z)|\leq 1$, $z\in K$;

$\mathfrak{M}$ the set of all functions $\varphi(z)\in\Phi$ admitting the representation

$$\varphi(z)=\frac{\int_0^{2\pi}\frac{e^{it}}{(1-ze^{it})}dS(t)}{\int_0^{2\pi}\frac{1}{(1-ze^{it})}dS(t)} \tag{3}$$

where $S(t)$ is a monotonic nondecreasing function with total variation equal to one, i.e. $\int_0^{2\pi}dS(t)=1$;

$\mathfrak{M}_0$ the set of all functions $\varphi(z)\in\mathfrak{M}$ with $S(t)$ a step function with a finite number of jumps.

Lemma 2.1. *1. $\mathfrak{M}$ is closed under pointwise convergence and $\mathfrak{M}_0$ is dense in $\mathfrak{M}$.*

2. $\mathfrak{M}=\Phi$.

Proof:

1. We assume that for all $z\in K$

$$\lim_{n\to\infty}\varphi_n(z)=\varphi(z),$$

with

$$\varphi_n(z)=\frac{\int_0^{2\pi}\frac{e^{it}}{(1-ze^{it})}dS_n(t)}{\int_0^{2\pi}\frac{1}{(1-ze^{it})}dS_n(t)}\in\mathfrak{M}.$$

According to a theorem by Helly [10], there exists a nondecreasing function $S(t)$ with total variation equal to 1 and a subsequence n_k such that

$$\lim_{n_k\to\infty}S_{n_k}(t)=S(t)$$

in each point of continuity of $S(t)$. Thus we have

$$\lim_{n_k\to\infty}\varphi_{n_k}(z)=\varphi(z)=\frac{\int_0^{2\pi}\frac{e^{it}}{(1-ze^{it})}dS(t)}{\int_0^{2\pi}\frac{1}{(1-ze^{it})}dS(t)}\in\mathfrak{M},$$

and the ensity of $\mathfrak{M}_0$ in $\mathfrak{M}$ is then straightforward.

2. Now, we introduce two other sets :

Φ^+ the set of all functions $f(\lambda)$ analytic in the upper half-plane $\Pi^+ = \{\lambda \in \mathbb{C} : \Im\lambda > 0\}$ with positive imaginary part, i.e. $\Im f(\lambda) \geq 0$, and $\mathfrak{M}^+$ the set of functions $\tau(\lambda) \in \Phi^+$ having the following representation

$$\tau(\lambda) = -\left(\lambda + \frac{1}{F(\lambda)}\right),$$

with

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t-\lambda},$$

$\mu(t)$ being a nondecreasing function with total variation equal to to one.

It is easy to see that the homographic transformation

$$L(\lambda) = \frac{\lambda - i}{\lambda + i} = z \tag{4}$$

establishes a bijection between the sets Φ and Φ^+. Indeed if, $f(\lambda) \in \Phi^+$, then

$$\varphi(z) = L\left(f\left(L^{-1}(z)\right)\right) = L\left(f\left(\frac{1}{i}\frac{1+z}{1-z}\right)\right) = \frac{f\left(\frac{1}{i}\frac{1+z}{1-z}\right) - i}{f\left(\frac{1}{i}\frac{1+z}{1-z}\right) + i} \in \Phi$$

and, reciprocally, if $\varphi(z) \in \Phi$, then

$$f(\lambda) = L^{-1}(\varphi(L(\lambda))) \in \Phi^+.$$

We can also see that (4) establishes the same bijection enters $\mathfrak{M}$ and $\mathfrak{M}^+$. Indeed, we have for $\tau(\lambda) \in \mathfrak{M}^+$

$$-\left(\lambda + \frac{1}{F(\lambda)}\right) = -\left(\lambda + \frac{1}{\int_{-\infty}^{+\infty} \frac{d\mu(t)}{t-\lambda}}\right) = \frac{\int_{-\infty}^{+\infty} \frac{t\,d\mu(t)}{t-\lambda}}{\int_{-\infty}^{+\infty} \frac{d\mu(t)}{t-\lambda}}. \tag{5}$$

And making in the formula (5) the following change of variables

$$\frac{\lambda - i}{\lambda + i} = z;\ \ \lambda = i\frac{1+z}{1-z};\ \ e^{i\alpha} = \frac{t-i}{t+i};\ \ t = i\frac{1+e^{i\alpha}}{1-e^{i\alpha}} = -\cot\frac{\alpha}{2},$$

we will have

$$-\left(\lambda + \frac{1}{F(\lambda)}\right) = -\frac{i\int_0^{2\pi} \frac{1+e^{i\alpha}}{e^{i\alpha}-z} dS(\alpha)}{\int_0^{2\pi} \frac{1-e^{i\alpha}}{e^{i\alpha}-z} dS(\alpha)} = f(z),$$

where $S(\alpha) = \mu\left(-\cot\frac{\alpha}{2}\right)$. Thus

$$\frac{f(z) - i}{f(z) + i} = \frac{i\int_0^{2\pi} \frac{1}{e^{i\alpha}-z} dS(\alpha)}{\int_0^{2\pi} \frac{e^{i\alpha}}{e^{i\alpha}-z} dS(\alpha)} \in \mathfrak{M}.$$

But we know [8] that $\mathfrak{M}^+$ is dense in Φ^+, consequently $\mathfrak{M}$ is dense in Φ for pointwise convergence. As $\mathfrak{M}$ is closed for this convergence then $\mathfrak{M} = \Phi$.

□

Let A be a closed symmetric operator defined in a Hilbert space H with the scalar product $(.,.)$. Its domain of definition $D(A)$ is assumed to be dense in H. For $\lambda \in \mathbb{C},\ \{\Im\lambda \neq 0$ we set

$$\mathfrak{M}_\lambda = (A - \lambda I)\, D(A),\ \mathfrak{N}_\lambda = H \ominus \mathfrak{M}_\lambda;$$

where I is the identity operator in H. We recall that $D(A)$ and the defect subspaces $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\overline{\lambda}}$ are linearly independent.

In the sequel we suppose the operator A with defect indices $(1,1)$, i.e.

$$\dim \mathfrak{N}_\lambda = \dim \mathfrak{N}_{\overline{\lambda}} = 1.$$

We now state the formula of generalized resolvents $R_\omega(\lambda)$ of the operator A obtained in [2] and [4]

$$R_\omega(\lambda) f = (A_\omega - \lambda I)^{-1} f = \overset{\circ}{R}(\lambda) f + \frac{1-\omega(\lambda)}{\omega(\lambda) C(\lambda) - 1} \cdot \frac{\left(f, \varphi_{\overline{\lambda}}\right)}{(\lambda + i)(\varphi_\lambda, \varphi_i)} \varphi_\lambda, \quad (6)$$

where $\overset{\circ}{R}(\lambda) = \left(\overset{\circ}{A} - \lambda I\right)^{-1}$ is the resolvent of the selfadjoint extension $\overset{\circ}{A}$ of the operator A, $\omega(\lambda)$ an analytical function in Π^+ such that $|\omega(\lambda)| \leq 1,\ \Im(\lambda) > 0, \varphi_i \in \mathfrak{N}_{-i},\ \varphi_\lambda = \overset{\circ}{U}_{\lambda i}\varphi_i = \left(\overset{\circ}{A} - iI\right)\left(\overset{\circ}{A} - \lambda I\right)^{-1} \varphi_i \in \mathfrak{N}_{\overline{\lambda}}$ [9] and

$$C(\lambda) = \frac{\lambda - i}{\lambda + i} . \frac{(\varphi_\lambda, \varphi_{-i})}{(\varphi_\lambda, \varphi_i)}$$

the characteristic function of A checking for $\lambda, \Im\lambda > 0 : |C(\lambda)| < 1$.

The generalized resolvent defined by (6) is a resolvent of a selfadjoint extension of A if and only if $\omega(\lambda) = \varkappa = const,\ |\varkappa| = 1$.We will call it orthogonal (or canonical) resolvent.

Every generalized resolvent $R_\omega(\lambda)$ is connected with the generalized spectral function E_t^ω by the relation

$$R_\omega(\lambda) = \int_{-\infty}^{+\infty} \frac{1}{t - \lambda} dE_t^\omega \quad (7)$$

As the set of the generalized spectral functions is convex [2], then the set of generalized resolvent is also.

Let $\mathfrak{R}_0$ be the convex set of the orthogonal resolvents $R_{\varkappa_j}(\lambda)\ (j = 1, 2, ..., n)$, corresponding to the constants $\varkappa_j, |\varkappa_j| = 1,\ (j = 1, 2, ..., n)$, i.e.

$$\mathfrak{R}_0 = \left\{ R_\omega(\lambda) = \sum_{j=1}^n \alpha_j R_{\varkappa_J}(\lambda),\ \alpha_j > 0,\ \sum_{j=1}^n \alpha_j = 1 \right\}.$$

Thus we have for an $R_\omega(\lambda) \in \mathfrak{R}_0$ and $f \in H$

$$\begin{aligned} R_\omega(\lambda) f &= \overset{\circ}{R}(\lambda) f - \sum_{j=1}^{n} \alpha_j \frac{1-\varkappa_j}{1-\varkappa_j C(\lambda)} h(\lambda) \left(f, \varphi_{\overline{\lambda}}\right) \varphi_\lambda \\ &= \overset{\circ}{R}(\lambda) f - \frac{1-\omega(\lambda)}{1-\omega(\lambda) C(\lambda)} h(\lambda) \left(f, \varphi_{\overline{\lambda}}\right) \varphi_\lambda, \end{aligned}$$

with

$$h(\lambda) = [(\lambda+1)(\varphi_\lambda, \varphi_i)]^{-1}.$$

As

$$\frac{1-\omega(\lambda)}{1-\omega(\lambda) C(\lambda)} = \sum_{j=1}^{n} \varkappa_j \frac{1-\varkappa_j}{1-\varkappa_j C(\lambda)},$$

we obtain

$$\omega(\lambda) = \frac{\sum_{j=1}^{n} \frac{\alpha_j\ \varkappa_j}{1-\varkappa_j C(\lambda)}}{\sum_{j=1}^{n} \varkappa_J \frac{\alpha_J}{1-\varkappa_j C(\lambda)}} = \frac{\int_0^{2\pi} \frac{e^{it}}{1-C(\lambda)e^{it}} dS(t)}{\int_0^{2\pi} \frac{1}{1-C(\lambda)e^{it}} dS(t)} = \varphi(C(\lambda)),$$

where $\varphi(z) \in \mathfrak{M}_0$.

Let us denote by $\mathfrak{R} = \overline{\mathfrak{R}_0}$ the closed convex hull of orthogonal resolvents in strong topology.

Theorem 2.1. *The closed convex hull* $\mathfrak{R}$ *is characterized by the formula (6), with* $\omega(\lambda) = \varphi(C(\lambda))$ *and* $\varphi(z)$ *is an arbitrary function from* $\mathfrak{M}$.

Proof: If $R_\omega(\lambda) \in \mathfrak{R}_0$, then $\omega(\lambda) = \varphi(C(\lambda))$, $\varphi(z) \in \mathfrak{M}_0$. We assume that $R_\omega(\lambda) \in \mathfrak{R}$ and $R_\omega(\lambda) \notin \mathfrak{R}_0$. Therefore, there is a sequence $R_{\omega_n}(\lambda)$ convergent to $R_\omega(\lambda)$,as $n \to \infty$, $R_{\omega_n}(\lambda) \in \mathfrak{R}_0$ for strong topology, i.e.

$$R_{\omega_n}(\lambda) f = \overset{\circ}{R}(\lambda) f - \frac{1-\omega_n(\lambda)}{1-\omega_n(\lambda) C(\lambda)} h(\lambda) \left(f, \varphi_{\overline{\lambda}}\right) \varphi_\lambda,$$

with $\omega_n(\lambda) = \varphi_n C(\lambda), \varphi_n(z) \in \mathfrak{M}_0$.

As

$$\lim_{n\to\infty} R_{\omega_n}(\lambda) = R_\omega(\lambda)$$

it follows that

$$\lim_{n\to\infty} [1-\omega_n(\lambda)] [1-\omega_n(\lambda) C(\lambda)]^{-1} = [1-\omega(\lambda)] [1-\omega(\lambda) C(\lambda)]^{-1}$$

for all λ, $\Im\lambda > 0$. Finally, according to the lemma1

$$\lim_{n\to\infty} \omega_n(\lambda) = \lim_{n\to\infty} \varphi_n(C(\lambda)) = \omega(\lambda) = \varphi(C(\lambda)),$$

$\varphi(z) \in \mathfrak{M}$. □

Let $E_t^{\varkappa_j}$, $j = 1, 2, ..., n$ be the orthogonal spectral functions connected with the orthogonal resolvents $R_{\varkappa_J}(\lambda)$ by (7). We denote by $\mathfrak{E}_0$ the convex hull of orthogonal spectral functions

$$\mathfrak{E}_0 = \left\{ E_t^{\omega} = \sum_{j=1}^{n} \alpha_j E_t^{\varkappa_j},\ \alpha_j > 0,\ \sum_{j=1}^{n} \alpha_j = 1 \right\}$$

and $\mathfrak{E} = \overline{\mathfrak{E}_0}$ for strong topology. It is obvious that each $E_t^{\omega} \in \mathfrak{E}$ defines by (7) a generalized resolvent $R_{\omega}(\lambda) \in \mathfrak{R}$ and, reciprocally, each generalized resolvent $R_{\omega}(\lambda) \in \mathfrak{R}$ defines by (7) a generalized spectral function $E_t^{\omega} \in \mathfrak{E}$.

3. Orthogonal scalar spectral functions of a Carleman operator

Let X be an arbitrary set, μ a σ-finite measure on X, $L_2(X, \mu)$ the Hilbert space of square integrable functions defined on X, $\{\psi_p(x)\}_{p=1}^{\infty}$ an orthonormal sequence in $L_2(X, \mu)$, $\{\gamma_p\}_{p=1}^{\infty}$ a sequence of real numbers such that $\sum\limits_{p=1}^{\infty} \gamma_p \psi_p(x) = 0$ for almost all $x \in X$, $K(x, y)$ a Carleman kernel

$$K(x, y) = \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)},$$

where $\{a_p\}_{p=1}^{\infty}$ is a sequence of real numbers.

We assume that for almost all $x \in X$

$$\sum_{p=0}^{\infty} |a_p \psi_p(x)|^2 < \infty,\ \sum_{p=0}^{\infty} |\psi_p(x)|^2 < \infty$$

and

$$\sum_{p=0}^{\infty} \left| \frac{\gamma_p}{a_p - \lambda} \right|^2 < \infty.$$

With these conditions, the symmetric operator $A = (A^*)^*$ admits the defect indices $(1, 1)$ [3],

$$\begin{array}{l} A^* f(x) = \sum\limits_{p=0}^{\infty} a_p (f, \psi_p) \psi_p(x), \\ D(A^*) = \left\{ f \in L^2(X, \mu) : \sum\limits_{p=0}^{\infty} a_p (f, \psi_p) \psi_p(x) \in L^2(X, \mu) \right\} \end{array}.$$

and moreover, we have

$$\left\{ \begin{array}{l} \varphi_{\lambda}(x) = \sum\limits_{p=0}^{\infty} \frac{\gamma_p}{a_p - \lambda} \psi_p(x) \in \mathfrak{N}_{\overline{\lambda}},\ \lambda \in \mathbb{C},\ \lambda \neq a_k,\ k = 1, 2, ... \\ \varphi_{a_k}(x) = \psi_k(x). \end{array} \right.$$

We denote by $\mathfrak{L}_\psi$ the sub-space of $L_2(X,\mu)$ generated by the sequence $\{\Psi_p(x)\}_{p=0}^{\infty}$. As $\mathfrak{L}_\psi$ is reduced by A, we consider A on $\mathfrak{L}_\psi$. Then, we have for all $f \in \mathfrak{L}_\psi$ [4]

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} \frac{(f,\varphi_\sigma)\varphi_\sigma(x)}{(\sigma^2+1)\left|\left(\varphi_\sigma,\mathring{\varphi}_i\right)\right|} d\rho(\sigma) \quad \text{for almost all } x \in X, \\ \|f\|^2 &= \int_{-\infty}^{+\infty} \frac{|(f,\varphi_\sigma)|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\mathring{\varphi}_i\right)\right|} d\rho(\sigma), \end{aligned} \tag{8}$$

with

$$\mathring{\varphi}_i = \frac{\varphi_i}{\|\varphi_i\|},$$

and

$$\rho(\sigma) = \frac{1}{\pi}\lim_{\tau\to+0}\int_0^\sigma \Re\frac{1+\omega(t+i\tau)C(t+i\tau)}{1-\omega(t+i\tau)C(t+i\tau)}dt, \tag{9}$$

$\omega(\lambda)$ is an analytical function on Π^+ with $|\omega(\lambda)| \le 1, (\Im\lambda > 0)$. We call the function $\rho(\sigma)$ scalar spectral function of the operator A.

Let us designate by $\mathfrak{P}$, the set of the functions defined by (9) with $\omega(\lambda)$ analytic in Π^+ and $|\omega(\lambda)| \le 1$. As the set of the spectral functions of a symmetric operator is convex thus $\mathfrak{P}$ is too . Subsequently we will assume that $\rho(\sigma) \in \mathfrak{P}$ is normalized by continuity on the left, i.e. $\rho(\sigma) = \rho(\sigma - 0)$. In $\mathfrak{P}$, we consider pointwise convergence, i.e one says that $\rho_n(t) \to \rho(t)$, $\rho_n(t) \in \mathfrak{P}$, $n \to \infty$ if $\lim_{n\to\infty}\rho_n(t) = \rho(t)$ for each continuity point of $\rho(t)$.

Theorem 3.1. *The set $\mathfrak{P}$ is closed for pointwise convergence.*

Proof: Let's $\rho_n(t) \in \mathfrak{P}$ and $\rho_n(\sigma) \longrightarrow \rho(\sigma)$, $n \to \infty$. It is suffice to establish the possibility of the passage to the limit under the integral sign :

$$\|f\|^2 = \int_{-\infty}^{+\infty} \frac{\|(f,\varphi_\sigma)\|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\mathring{\varphi}_i\right)\right|^2} d\rho(\sigma), \quad f \in \mathfrak{L}_\psi$$

as n tends to ∞.

However, for all $f \in D(A)$ and for all $b > 0$

$$\begin{aligned}
\int_b^{+\infty} \frac{|(f,\varphi_\sigma)|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\overset{\circ}{\varphi}_i\right)\right|^2} d\rho_n(\sigma) &= \int_b^{+\infty} \frac{1}{\sigma^2} \frac{|(f,\sigma\varphi_\sigma)|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\overset{\circ}{\varphi}_i\right)\right|^2} \rho_n(\sigma) \\
&\leq \frac{1}{b^2} \int_b^{+\infty} \frac{|(f,A^*\varphi_\sigma)|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\overset{\circ}{\varphi}_i\right)\right|^2} d\rho_n(\sigma) \\
&= \frac{1}{b^2} \int_b^{+\infty} \frac{|(Af,\varphi_\sigma)|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\overset{\circ}{\varphi}_i\right)\right|^2} \rho_b(\sigma) \\
&\leq \frac{1}{b^2} \int_b^{+\infty} \frac{|(Af,\varphi_\sigma)|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\overset{\circ}{\varphi}_i\right)\right|^2} d\rho_n(\sigma) \\
&= \frac{1}{b^2} \|Af\| \to 0, \text{ as } b \to +\infty.
\end{aligned}$$

In the same way, we show that :

$$\int_{-\infty}^{-a} \frac{|(f,\varphi_\sigma)|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\overset{\circ}{\varphi}_i\right)\right|^2} d\rho_n(\sigma) \to 0 \text{ as } a \to +\infty.$$

By applying the modified theorem of Helly [10], we have for all $f \in D(A)$

$$\|f\|^2 = \int_{-\infty}^{+\infty} \frac{\|(f,\varphi_\sigma)\|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\overset{\circ}{\varphi}_i\right)\right|^2} d\rho_n(\sigma) \to \int_{-\infty}^{+\infty} \frac{\|(f,\varphi_\sigma)\|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\overset{\circ}{\varphi}_i\right)\right|^2} d\rho_n(\sigma),$$

as $n \to \infty$,or

$$\|f\|^2 = \int_{-\infty}^{+\infty} \frac{\|(f,\varphi_\sigma)\|^2}{(\sigma^2+1)\left|\left(\varphi_\sigma,\overset{\circ}{\varphi}_i\right)\right|^2} d\rho(\sigma)$$

As $\overline{D(A)} = \mathfrak{L}_\psi$, this equality is true for all $f \in \mathfrak{L}_\psi$. □

We will call $\rho_\varkappa(\sigma) = \rho(\sigma) \in \mathfrak{P}$ orthogonal scalar spectral function if it corresponds to a constant $\varkappa, |\varkappa| = 1$.

Let's $\rho_{\varkappa_k}, |\varkappa_k| = 1, 2, ..., n$ be orthogonal scalar spectral functions corresponding to the constants $\varkappa_1, \varkappa_2, ...\varkappa_n$.

We denote by $\mathfrak{G}_0$ the convex hull of these functions :

$$\mathfrak{G}_0 = \left\{ \rho(\sigma) = \sum_{k=1}^n \alpha_k \rho_{\varkappa_k},\ \alpha_k > 0,\ \sum_{k=1}^n \alpha_k = 1 \right\}$$

and $\mathfrak{G} = \overline{\mathfrak{G}_0}$ for the cenvergence in each point of continuity.

For any function $\rho(\sigma) \in \mathfrak{G}_0$ we have :

$$\rho(\sigma) = \frac{1}{\pi} \lim_{\tau\to\infty} \int_0^\sigma \Re\left[\sum_{k=1}^{n} \alpha_k \frac{1+\varkappa_k C(t+i\tau)}{1-\varkappa_k C(t+i\tau)}\right] dt = \frac{1}{\pi} \lim_{\tau\to\infty} \int_0^\sigma \Re\left[\frac{1+\omega(t+i\tau) C(t+i\tau)}{1-\omega(t+i\tau) C(t+i\tau)}\right] dt \tag{10}$$

where $\omega(\lambda)$ is the analytical function corresponding to $\rho(\sigma) \in \mathfrak{P}$. From where we finds easily :

$$\begin{aligned} \omega(\lambda) &= \frac{\sum_{k=1}^{n} \frac{\varkappa_k x_k}{1-x_k C(\lambda)}}{\sum_{j=1}^{n} \frac{1}{1-x_k C(\lambda)}} \\ &= \frac{\int_{-\infty}^{+\infty} \frac{e^{it}}{1-C(\lambda)e^{it}} dS(t)}{\int_{-\infty}^{+\infty} \frac{1}{1-C(\lambda)e^{it}} dS(t)} = \varphi(C(\lambda)), \end{aligned} \tag{11}$$

with $\varphi(z) \in \mathfrak{M}_0$.

Theorem 3.2. *The closed convex hull $\mathfrak{G}$ of orthogonal scalar spectral functions, is described by (9) where $\omega(\lambda)$ is of the form $\omega(\lambda) = \varphi(C(\lambda))$ and $\varphi(z)$ is an arbitrary function of $\mathfrak{M}$.*

Proof: Let $\rho(t) \in \mathfrak{G}_0$, then according to (11) $\omega(\lambda)$ which corresponds to $\rho(t)$ has the form $\omega(\lambda) = \varphi(C(\lambda))$ with $\varphi(z) \in \mathfrak{M}_0$.

We suppose, now, that $\rho(t) \in \mathfrak{G}$ and $\rho(t) \notin G_0$, there is then a sequence $\rho_n(t) \in \mathfrak{G}_0$ which converges to $\rho(t)$ in each point of continuity of $\rho(t)$. Each function $\rho_n(t) \in \mathfrak{G}_0$ corresponds to a function $\omega_n(\lambda) = \varphi_n(C(\lambda))$ with $\varphi_n(z) \in \mathfrak{M}_0$. As

$$\Re \frac{1+\varphi_n(C(\lambda))\, C(t+i\tau)}{1-\varphi_n(C(\lambda))\, C(t+i\tau)} > 0 \quad (\Im\lambda > 0),$$

by applying the inversion formula of Stieltjes [1], we will have :

$$i \frac{1+\varphi_n(C(\lambda))\, C(t+i\tau)}{1-\varphi_n(C(\lambda))\, C(t+i\tau)} = \int_{-\infty}^{+\infty} \frac{1}{t-\lambda} d\rho_n(t) \tag{12}$$

And while making tend n to ∞ , it is followed from there :

$$i \frac{1+\varphi(C(\lambda))\, C(t+i\tau)}{1-\varphi(C(\lambda))\, C(t+i\tau)} = \int_{-\infty}^{+\infty} \frac{1}{t-\lambda} d\rho(t)$$

As $\varphi_n(z) \in \mathfrak{M}_0$ and $\overline{\mathfrak{M}}_0 = \mathfrak{M}$ then $\varphi(z) \in \mathfrak{M}$.

Reciprocally, let $\omega(\lambda) = \varphi(C(\lambda))$ with $\varphi(z) \in \mathfrak{M}$. The function $\omega(\lambda)$ corresponds to a spectral function $\rho(t)$. Let us show that $\rho(t) \in \mathfrak{G}$. Let us choose a sequence of functions $\varphi(z) \in \mathfrak{M}$ convergent in each point z $(|z| < 1)$ to $\varphi(z)$ and $\varphi_n(z)$ is represented by (3) where $S_n(t)$ have only a finished number of jumps. Each function $\varphi_n(C(\lambda))$ corresponds to a spectral function $\rho_n(t) \in \mathfrak{G}_0$. Owing to the fact that $\mathfrak{G} = \overline{\mathfrak{G}_0}$, there is thus a sub-sequence $\rho_{n_k}(t)$ which converges in each point of continuity towards a function $\widetilde{\rho}(t)$. As

$$i\frac{1+\varphi(C(\lambda))\,C(t+i\tau)}{1-\varphi(C(\lambda))\,C(t+i\tau)} = \int_{-\infty}^{+\infty}\frac{1}{t-\lambda}d\rho(t)$$

and $\rho_{n_k}(t)$ and $\rho_{n_k}(C(\lambda))$ are connected by (12), while making tend $n_k \to \infty$ we will have :

$$\rho_{n_k}(t) \to \widetilde{\rho}(t) = \rho(t),$$

thus $\rho(t) \in \mathfrak{G}$. □

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