Decomposition of continuity via \( b \)-open set

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Abstract: We introduce the notions of locally \( b \)-closed, \( b\)-\( t \)-set, \( b\)-\( B \)-set, locally \( b \)-closed continuous, \( b\)-\( t \)-continuous, \( b\)-\( B \)-continuous functions and obtain decomposition of continuity and complete continuity.

Key Words: \( b \)-open set, \( t \)-set, \( B \)-set, locally closed, decomposition of continuity

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1. introduction

Tong [17,18], Ganster-Reilly [5], Hatice [7], Hatir-Noiri [8], Przemski [20], Noiri-Sayed [13] and Erguang-Pengfei [4], gave some decompositions of continuity. Andrijevic [2] introduced a class of generalized open sets in a topological space, the so-called \( b \)-open sets. The class of \( b \)-open sets is contained in the class of semi-preopen sets and contains all semi-open sets and preopen sets. Tong [18] introduced the concept of \( t \)-set and \( B \)-set in topological space. In this paper, we introduce the notions of locally \( b \)-closed sets, \( b\)-\( t \)-set, \( b\)-\( B \)-set, \( b \)-closed continuous, \( b\)-\( t \)-continuous and \( b\)-\( B \)-continuous function, and obtain another decomposition of continuity. All through this paper \((X, \tau)\) and \((Y, \sigma)\) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let \( A \subseteq X \), the closure of \( A \) and the interior of \( A \) will be denoted by \( Cl(A) \) and \( Int(A) \), respectively. \( A \) is regular open if \( A = Int(Cl(A)) \) and \( A \) is regular closed if its complement is regular open; equivalently \( A \) is regular closed if \( A = Cl(Int(A)) \). The complement of a \( b \)-open set is said to be \( b \)-closed. The intersection of all \( b \)-closed sets of \( X \) containing \( A \) is called the \( b \)-closure of \( A \) and is denoted by \( bCl(A) \) of \( A \). The union of all \( b \)-open (resp. \( \alpha \)-open, semi open, preopen) sets of \( X \) contain in \( A \) is called \( b \)-interior (resp. \( \alpha \)-interior, semi-interior, pre-interior) of \( A \) and is denoted by \( bInt(A) \) (resp. \( \alpha Int(A), sInt(A), pInt(A) \)). The family of all \( b \)-open (resp. \( \alpha \)-open, semi-open, preopen, regular open, \( b \)-closed, preclosed) subsets of a space \( X \) is denoted by \( bO(X) \) (resp. \( \alpha O(X), SO(X) \) and \( PO(X), bC(X), PC(X) \) respectively).

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Definition 1.1. A subset $A$ of a space $X$ is said to be:

1. $\alpha$-open \cite{12} if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$;
2. Semi-open \cite{9} if $A \subseteq \text{Cl}(\text{Int}(A))$;
3. preopen \cite{15} if $A \subseteq \text{Int}(\text{Cl}(A))$;
4. $b$-open \cite{1} if $A \subseteq \text{Cl}(\text{Int}(A)) \cap \text{Int}(\text{Cl}(A))$;
5. Semi-preopen \cite{2} if $A \subseteq \text{Cl}(\text{Int}(A))$.

The following result will be useful in the sequel.

Lemma 1.1. \cite{1} If $A$ is a subset of a space $(X, \tau)$, then

1. $s\text{Int}(A) = A \cap \text{Cl}(\text{Int}(A))$;
2. $p\text{Int}(A) = A \cap \text{Int}(\text{Cl}(A))$;
3. $\alpha\text{Int}(A) = A \cap \text{Int}(\text{Cl}(\text{Int}(A)))$;
4. $b\text{Int}(A) = s\text{Int}(A) \cup p\text{Int}(A)$.

2. locally $b$-closed sets

Definition 2.1. A subset $A$ of a space $X$ is called:

1. $t$-set \cite{18} if $\text{Int}(A) = \text{Int}(\text{Cl}(A))$.
2. $B$-set \cite{18} if $A = U \cap V$, where $U \in \tau$ and $V$ is a $t$-set.
3. locally closed \cite{3} if $A = U \cap V$, where $U \in \tau$ and $V$ is a closed.
4. locally $b$-closed if $A = U \cap V$, where $U \in \tau$ and $V$ is a $b$-closed.

We recall that a topological space $(X, \tau)$ is said to be extremally disconnected (briefly E.D.) if the closure of every open set of $X$ is open in $X$. We note that a subset $A$ of $X$ is locally closed if and only if $A = U \cap \text{Cl}(A)$ for some open set $U$ (see \cite{3}). The following example shows that the two notions of $b$-open and locally closed are independent.

Example 2.1. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{b, c, d\}, \{b\}, \{a, b\}\}$ with $BO(X, \tau) = \{X, \phi, \{b\}, \{a, b\}, \{b, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$, and the family of all locally closed is $LC(X, \tau) = \{X, \phi, \{b\}, \{a\}, \{a, b\}, \{c, d\}, \{b, c, d\}\}$ it is clear that $\{a\}$ is locally closed but not $b$-open and $\{b, d\}$ is $b$-open and not locally closed.

Theorem 2.1. For a subset $A$ of an extremally disconnected space $(X, \tau)$, the following are equivalent:

1. $A$ is open,
2. $A$ is $b$-open and locally closed.
Example 2.2. \( \text{Definition 2.2.} \) \( \text{independent.} \)

\( f \text{BO} \)

Then \( A = U \cap \text{Cl}(A) \) . Then

\[ A \subseteq U \cap (\text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))) \]

\[ \subseteq [\text{Int}(U \cap \text{Cl}(A))] \cup [U \cap \text{Int}(\text{Cl}(A))] \]

\[ \subseteq [\text{Int}(U \cap \text{Cl}(A))] \cup [U \cap \text{Int}(\text{Cl}(A))] \]

\( = \text{Int}(A) \cup \text{Int}(A) = \text{Int}(A) \)

Therefore \( A \) is open. \( \square \)

**Definition 2.2.** A subset \( A \) of a topological space \( X \) is called \( D(c,b) \)-set if \( \text{Int}(A) = b\text{Int}(A) \).

From the following examples one can deduce that \( b \)-open and \( D(c,b) \)-set are independent.

**Example 2.2.** Let \( X = \{a, b, c, d\} \) and \( \tau = \{X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\} \).

Then \( \text{BO}(X, \tau) = \{\emptyset, X, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}, \{c, d\}, \{b, c\}, \{b, d\}, \{a, b\}, \{a, c\}, \{d\}, \{c\}\} \), it is clear that \( A = \{a\} \) is \( D(c,b) \)-set but not \( b \)-open. Also \( B = \{a, b, d\} \) is \( b \)-open but \( B \) is not \( D(c,b) \).

**Theorem 2.2.** For a subset \( A \) of a space \( (X, \tau) \), the following are equivalent:

1. \( A \) is open,
2. \( A \) is \( b \)-open and \( D(c,b) \)-set.

Proof. \((1) \Rightarrow (2)\) If \( A \) is open then \( A \) is \( b \)-open and \( A = \text{Int}(A) = b\text{Int}(A) \) so \( A \) is \( D(c,b) \)-set.

\((2) \Rightarrow (1)\) The condition \( A \in \text{BO}(X) \) and \( A \in D(c,b) \) imply \( A = b\text{Int}(A) \) and \( \text{Int}(A) = b\text{Int}(A) \) and consequently \( A \) is open. \( \square \)

**Proposition 2.1.** Let \( H \) be a subset of \( (X, \tau) \), \( H \) is locally \( b \)-closed if and only if there exists an open set \( U \subseteq X \) such that \( H = U \cap b\text{Cl}(H) \)

Proof. Since \( H \) is an \( b \)-closed subset, then \( H = U \cap F \), where \( U \) is open and \( F \) is \( b \)-closed. So \( H \subseteq U \) and \( H \subseteq F \) then \( H \subseteq b\text{Cl}(H) \subseteq b\text{Cl}(F) = F \). Therefore \( H \subseteq U \cap b\text{Cl}(H) \subseteq U \cap b\text{Cl}(F) = U \cap F = H \). Hence \( A = U \cap b\text{Cl}(A) \). Conversely since \( b\text{Cl}(H) \) is \( b \)-closed and \( H = U \cap b\text{Cl}(H) \), then \( H \) is locally \( b \)-closed. \( \square \)

**Proposition 2.2.** \( [1] \).

1. The union of any family of \( b \)-open sets is \( b \)-open.
2. The intersection of an open set and a \( b \)-open set is a \( b \)-open set.
**Proposition 2.3.** Let $A$ be a subset of a topological space $X$ if $A$ is locally $b$-closed, then

1. $bCl(A) - A$ is $b$-closed set.
2. $[A \cup (X - bCl(A))]$ is $b$-open.
3. $A \subseteq bInt(A \cup (X - bCl(A))$.

**Proof.**

1. If $A$ is an locally $b$-closed, there exist an $U$ is open such that $A = U \cap bCl(A)$. Now

$$bCl(A) - A = bCl(A) - [U \cap bCl(A)]$$

$$= bCl(A) \cap [X - (U \cap bCl(A))]$$

$$= bCl(A) \cap [(X - U) \cup (X - bCl(A))]$$

$$= [bCl(A) \cap (X - U)] \cup [bCl(A) \cap (X - bCl(A))]$$

$$= bCl(A) \cap (X - U)$$

which is $b$-closed by Proposition 2.2.

2. Since $bCl(A) - A$ is $b$-closed, then $[X - (bCl(A) - A)]$ is $b$-open and

$$[X - (bCl(A) - A)] = X - ((bCl(A) \cap (X - A)) = [A \cup (X - bCl(A))]$$

3. It is clear that $A \subseteq [A \cup (X - bCl(A))] = bInt[A \cup (X - bCl(A))]$.

As a consequence of Proposition 2.2, we have the following

**Corollary 2.2A.** The intersection of a locally $b$-closed set and locally closed set is locally $b$-closed.

Let $A, B \subseteq X$. Then $A$ and $B$ are said to be separated if $A \cap Cl(B) = \phi$ and $B \cap Cl(A) = \phi$.

**Theorem 2.3.** Suppose $(X, \tau)$ is closed under finite unions of $b$-closed sets. Let $A$ and $B$ be locally $b$-closed. If $A$ and $B$ are separated, then $A \cup B$ is locally $b$-closed.

**Proof.** Since $A$ and $B$ are locally $b$-closed, $A = G \cap bCl(A)$ and $B = H \cap bCl(B)$, where $G$ and $H$ are open in $X$. Put $U = G \cap (X \setminus Cl(A))$ and $V = H \cap (X \setminus Cl(A))$. Then $U \cap bCl(A) = (G \cap (X \setminus Cl(B))) \cap bCl(A) = A \cap (X \setminus Cl(B)) = A$, since $A \subseteq X \setminus Cl(B)$. Similarly, $V \cap bCl(B) = B$. And $U \cap bCl(B) \subseteq U \cap Cl(B) = \phi$ and $V \cap bCl(A) \subseteq V \cap Cl(A) = \phi$. Since, $U$ and $V$ are open.

$$(U \cup V) \cap bCl(A \cup B) = (U \cup V) \cap (bCl(A) \cup bCl(B))$$

$$= (U \cap bCl(A)) \cup (U \cap bCl(B)) \cup (V \cap bCl(A)) \cup (V \cap bCl(B))$$

$$= A \cup B$$

Hence $A \cup B$ is locally $b$-closed. \qed
Proposition 2.4. [18] A subset $A$ in a topological space $X$ is open if and only if it is pre-open set and a $B$-set.

Proposition 2.5. [10] A subset $A$ in a topological space $X$ is $\alpha$-set if and only if it is pre-open set and semi-open.

Lemma 2.1. [14] For a subset $V$ of a topological space $Y$, we have $pCl(V) = Cl(V)$ for every $V \in SO(Y)$.

In the topological space $(X, \tau)$ in [7] the author defined, $A_5 = B(X) = \{U \cap F|U \in \tau \text{ and } Int(Cl(F)) \subseteq F\}$. It is easy to see that every element in $A_5$ is $B$-set.

Proposition 2.6. [7] Let $H$ be a subset of $(X, \tau)$, $H \in A_5$ if and only if there exists an open set $U$ such that $H = U \cap sCl(H)$.

Lemma 2.2. A subset $A$ in a topological space $X$ is $B$-set if it is locally $b$-closed and semi-open.

Proof. Let $A$ be locally $b$-closed and semi-open. Then by Proposition 2.1, there exists an open set $U$ such that

$$A = U \cap bCl(A) = U \cap [sCl(A) \cap pCl(A)] = U \cap [sCl(A) \cap Cl(A)]$$

by Lemma 2.1 we have

$$= U \cap sCl(A)$$

Hence by Proposition 2.6 $A \in A_5$, so $A$ is $B$-set. \qed

From the following examples one can deduce that $\alpha$-sets and locally $b$-closed sets are independent.

Example 2.3. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{c\}, \{d\}, \{a, c, d\}, \{b, c, d\}\}$. It is clearly that $\{a, d\}$ is locally $b$-closed but not $\alpha$-open, since $\{a, d\} = \{a, c, d\} \cap \{a, b, d, \}$ and $\{a, d\} \not\subseteq Int(Cl(Int(\{a, d\}))) = \{d\}$.

Example 2.4. Let $X = \{a, b, c, d\}$, and $\tau = \{X, \emptyset, \{b, c, d\}, \{b\}, \{a, b\}\}$ with Then the family of all locally closed set is $LC(X, \tau) = \{X, \emptyset, \{b\}, \{a\}, \{a, b\}, \{c, d\}, \{b, c, d\}\}$. It is clearly $\{c, b\}$ is $\alpha$-open but not locally $b$-closed since $\{c, b\} \neq \{\text{open} \cap \{b\text{-closed}\}}$ and $\{c, b\} \subseteq Int(Cl(Int(\{c, b\}))) = X$.

Theorem 2.4. For a subset $A$ of a space $(X, \tau)$, the following are equivalent:

1. $A$ is open,

2. $A$ is $\alpha$-set and locally $b$-closed.

Proof. It is immediate from Proposition 2.4, Proposition 2.5 and Lemma 2.2. \qed
Definition 2.3. Let $X$ be a space. A subset $A$ of $X$ is called a generalized $b$-closed set (simply; $gb$-closed set) if $bCl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open. The complement of a generalized $b$-closed set is called generalized $b$-open (simply; $gb$-open).

Theorem 2.5. For a subset $A$ of a space $(X, \tau)$, the following are equivalent:

1. $A$ is $b$-closed,
2. $A$ is $gb$-closed and locally $b$-closed.

Proof. Let $A$ be $b$-closed so $A = A \cap X$, then $A$ is locally $b$-closed. And if $A \subseteq U$ then $bCl(A) = A \subseteq U$ and $A$ is $gb$-closed. Conversely if $A$ is locally $b$-closed, then there exist open set $U$ such that $A = U \cap bCl(A)$, since $A \subseteq U$ and $A$ is $gb$-closed then $bCl(A) \subseteq U$. Therefore $bCl(A) \subseteq U \cap bCl(A) = A$. Hence $A$ is $b$-closed. \qed

3. $b$-t-sets

In this section, we introduce the following notions.

Definition 3.1. A subset $A$ of a space $X$ is said to be:

1. $b$-t-set if $Int(A) = Int(bCl(A));$
2. $b$-$B$-set if $A = U \cap V$, where $U \in \tau$ and $V$ is a $b$-t-set;
3. $b$-semiopen if $A \subseteq Cl(bInt(A));$
4. $b$-preopen if $A \subseteq Int(bCl(A)).$

Proposition 3.1. For subsets $A$ and $B$ of a space $(X, \tau)$, the following properties hold:

1. $A$ is a $b$-t-set if and only if it is $b$-semiclosed.
2. If $A$ is $b$-closed, then it is a $b$-t-set.
3. If $A$ and $B$ are $b$-t-sets, then $A \cap B$ is a $b$-t-set.

Proof. (1) Let $A$ be $b$-t-set. Then $Int(A) = Int(bCl(A)).$ Therefore $Int(bCl(A)) \subseteq Int(A) \subseteq A$ and $A$ is $b$-semiclosed. Conversely if $A$ is $b$-semiclosed, then $Int(bCl(A)) \subseteq A$ thus $Int(bCl(A)) \subseteq Int(A).$ Also $A \subseteq bCl(A)$ and $Int(A) \subseteq Int(bCl(A)).$ Hence $Int(A) = Int(bCl(A)).$

(2) Let $A$ be $b$-closed, then $A = bCl(A)$, and $Int(A) = Int(bCl(A))$ therefore $A$ is $b$-t-set.

(3) Let $A$ and $B$ be $b$-t-set. Then we have

$$Int(A \cap B) \subseteq Int(bCl(A \cap B)) \subseteq (Int(bCl(A)) \cap (bCl(B))) = Int(bCl(A) \cap Int(bClB)) = Int(A) \cap Int(B)$$

( since $A$ and $B$ are $b$-t-set) $= Int(A \cap B)$
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Then $\text{Int}(A \cap B) = \text{Int}(b\text{Cl}(A \cap B))$ hence $A \cap B$ is $b$-$t$-set.

The converses of the statements in Proposition 3.1 (2) are false as the following example shows.

Example 3.1. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Then $\{b, c\}$ is $b$-$t$-set but it is not $b$-closed.

Proposition 3.2. For a subset $A$ of a space $(X, \tau)$, the following properties hold:

1. If $A$ is $t$-set then it is $b$-$t$-set;
2. If $A$ is $b$-$t$-set then it is $b$-$B$-set;
3. If $A$ is $B$-set then it is $b$-$B$-set.

Theorem 3.1. For a subset $A$ of a space $(X, \tau)$, the following are equivalent:

1. $A$ is open,
2. $A$ is $b$-preopen and a $b$-$B$-set.

Proof. (1) $\Rightarrow$ (2) Let $A$ be open. Then $A \subseteq b\text{Cl}(A), A = \text{Int}(A) \subseteq \text{Int}(b\text{Cl}(A))$ and $A$ is $b$-preopens. Also $A = A \cap X$ hence $A$ is $b$-$B$-set.

(2) $\Rightarrow$ (1) Since $A$ is $b$-$B$-set, we have $A = U \cap V$, where $U$ is open set and $\text{Int}(V) = \text{Int}(b\text{Cl}(V))$. By the hypothesis, $A$ is also $b$-preopen, and we have

$$A \subseteq \text{Int}(b\text{Cl}(A)) = \text{Int}(b\text{Cl}(U \cap V)) \subseteq \text{Int}(b\text{Cl}(U) \cap b\text{Cl}(V)) = \text{Int}(b\text{Cl}(U)) \cap \text{Int}(b\text{Cl}(V)) = \text{Int}(b\text{Cl}(U)) \cap \text{Int}(V)$$

Hence

$$A = U \cap V = (U \cap V) \cap U \subseteq (\text{Int}(b\text{Cl}(U)) \cap \text{Int}(V)) \cap U = (\text{Int}(b\text{Cl}(U)) \cap U) \cap \text{Int}(V) = U \cap \text{Int}(V)$$

Therefore $A = (U \cap V) = (U \cap \text{Int}(V))$, and $A$ is open.

From the following examples one can deduce that $b$-preopen sets and a $b$-$B$-sets are independent.

Example 3.2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. It is clearly that $\{c, b\}$ is $b$-$B$-set but it is not $b$-preopen, since $\{c, b\} = \{b, c, d\} \cap \{c, b\}, \{c, b\}$ is $b$-$t$-set and $\{c, b\} \not\subseteq \text{Int}(b\text{Cl}(\{c, b\})) = \{c\}$.
Example 3.3. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{b, c, d\}, \{b\}, \{a, b\}\}$. It is clearly that $\{c, b\}$ is $b$-preopen but it is not $b$-$B$-set, since $\{c, b\} \subseteq \text{Int}(b\text{Cl}(\{c, b\})) = X$, since $\{c, b\}$ is not $b$-$t$-set and the open set containing $\{c, b\}$ is $X$ or $\{b, c, d\}$, therefore $\{c, b\}$ is not $b$-$B$-set.

Lemma 3.1. Let $A$ be an open subset of a space $X$. Then $b\text{Cl}(A) = \text{Int}(\text{Cl}(A))$ and $\text{Int}(b\text{Cl}(A)) = \text{Int}(\text{Cl}(A))$.

Proof. Let $A$ be open set, then
\[
\begin{align*}
b\text{Cl}(A) &= s\text{Cl}(A) \cap p\text{Cl}(A) \\
&= A \cup (\text{Int}(\text{Cl}(A)) \cap \text{Cl}(\text{Int}(A))) \\
&= A \cup (\text{Int}(\text{Cl}(A)) \cap \text{Cl}(A)) \\
&= A \cup \text{Int}(\text{Cl}(A)) \\
&= \text{Int}(\text{Cl}(A)).
\end{align*}
\]

\[\square\]

Proposition 3.3. For a subset $A$ of space $(X, \tau)$, the following are equivalent:

1. $A$ is regular open,
2. $A = \text{Int}(b\text{Cl}(A))$,
3. $A$ is $b$-preopen and a $b$-$t$-set.

Proof. (1) $\Rightarrow$ (2) Let $A$ be regular open. Since $b\text{Cl} \subseteq \text{Cl}(A)$. Therefore, we have $\text{Int}(b\text{Cl}(A)) \subseteq \text{Int}(\text{Cl}(A)) = A$, since $A$ is open, $A \subseteq \text{Int}(b\text{Cl}(A))$ hence $A = \text{Int}(b\text{Cl}(A))$.

(2) $\Rightarrow$ (3) This is obvious

(3) $\Rightarrow$ (1) Let $A$ be $b$-preopen and a $b$-$t$-set. Then $A \subseteq \text{Int}(b\text{Cl}(A)) = \text{Int}(A) \subseteq A$ and $A$ is open. by Lemma 3.1, $A = \text{Int}(b\text{Cl}(A)) = \text{Int}(\text{Cl}(A))$, hence $A$ is regular open. \[\square\]

Definition 3.2. A subset $A$ of a topological space $X$ is called $sb$-generalized closed if $s(b\text{Cl}(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is $b$-preopen.

Theorem 3.2. For a subset $A$ of topological space $X$, the following properties are equivalent:

1. $A$ is regular open;
2. $A$ is $b$-preopen and $sb$-generalized.

Proof. (1) $\Rightarrow$ (2) Let $A$ be regular open. Then $A$ is $b$-open. $A \subseteq \text{Int}(b\text{Cl}(A))$. Moreover, by Lemma 3.1 $s(b\text{Cl}(A)) = A \cup \text{Int}(b\text{Cl}(A)) = \text{Int}(b\text{Cl}(A)) = \text{Int}(\text{Cl}(A)) = A$. Therefore, $A$ is $sb$-generalized closed.

(2) $\Rightarrow$ (1) Let $A$ be $b$-preopen and $sb$-generalized closed. Then we have $s(b\text{Cl}(A)) \subseteq A$ and hence $A$ is $b$-semiclosed. Therefore $\text{Int}(b(\text{Cl}(A))) \subseteq A$. Also $A$ is $b$-preopen, $A \subseteq \text{Int}(b(\text{Cl}(A)))$ then $A = \text{Int}(b(\text{Cl}(A)))$. Therefore by Proposition 3.3 $A$ is regular open. \[\square\]
4. Decompositions of continuity

In this section, we provide some theorems concerning the decomposition of continuity via the notion of locally $b$-closed set.

**Definition 4.1.** A function $f : X \to Y$ is called $b$-continuous \([19]\)(resp. $\alpha$-continuous \([16]\), semi continuous \([9]\), $B$-continuous \([18]\) locally closed continuous \([6]\), $D(c, b)$-continuous, locally $b$-closed continuous) if $f^{-1}(V)$ is $b$-open (resp. $\alpha$-open, semi open, $B$-set, locally closed, $D(c, b)$-set, locally $b$-closed) in $X$ for each open set $V$ of $Y$.

**Theorem 4.1.** Let $f : X \to Y$ be a function. Then

1. If $X$ is extremally disconnected, $f$ is continuous if and only if $f$ is $b$-continuous and locally closed-continuous.

2. $f$ is continuous if and only if $f$ is $b$-continuous and $D(c, b)$-continuous.

3. $f$ is continuous if and only if $f$ is $\alpha$-continuous and locally $b$-closed-continuous.

**Proof.**

1. It follows from Theorem 2.1.

2. It follows from Theorem 2.2.

3. It follows from Theorem 2.4.

\[\square\]

From the following example we can see that $b$-continuous functions and $D(c, b)$-continuous functions are independent.

**Example 4.1.** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{c\}, \{d\}, \{a, c, d\}, \{b, c, d\}\}$.

Define a function $f : X \to X$ such that $f(a) = c$, $f(b) = a$, and $f(c) = d = f(d)$. Then $f$ is $D(c, d)$-continuous, but it is not $b$-continuous, since $f^{-1}(\{c\}) = \{a\}$ is $D(c, d)$-set, but it is not a $b$-open. And a function $f : X \to X$ such that $f(a) = c$, $f(b) = a$, $f(c) = b$, $f(d) = d$. Then $f$ is $b$-continuous, but it is not $D(c, d)$-continuous, since $f^{-1}(\{c, d\}) = \{a, b, d\}$ is $b$-open but it is not $D(c, d)$-set.

From the following example we can see that $b$-continuous functions and locally closed-continuous functions are independent.

**Example 4.2.** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{b, c, d\}, \{b\}, \{a, b\}\}$.

Define a function $f : X \to X$ such that $f(a) = b$, $f(b) = c$, $f(c) = c$ and $f(d) = d$. Then $f$ is locally closed continuous, but it is not $b$-continuous, since $f^{-1}(\{b\}) = \{a\}$ is locally closed but it is not $b$-open. And a function $f : X \to X$ such that $f(a) = d$, $f(b) = b$, $f(c) = a$, $f(d) = c$, then $f$ is $b$-continuous, but it is not locally closed-continuous, since $f^{-1}(\{b, c, d\}) = \{a, b, d\}$ which is $b$-open not locally closed.

From the following examples we can see that $\alpha$-continuous functions and locally $b$-closed-continuous functions are independent.
Example 4.3. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $Y = \{a, b, c\}$, with the topology $\sigma = \{\phi, Y, \{a\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = b, f(b) = c, f(c) = a$ and $f(d) = a$. Then $f$ is contra $gb$-continuous, but it is not locally $b$-closed-continuous, since $f^{-1}(\{a\}) = \{c, d\}$ which is $gb$-closed not locally $b$-closed.

Example 4.4. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{b, c, d\}, \{b\}, \{a, b\}\}$. Define a function $f : X \to X$ such that $f(a) = c, f(b) = b, f(c) = a$ and $f(d) = d$ then $f$ is $\alpha$-continuous but it is not locally $b$-closed-continuous, since $f^{-1}(\{a, b\}) = \{c, b\}$ which is $\alpha$-open not locally $b$-closed.

Definition 4.2. A function $f : X \to Y$ is called contra $b$-continuous [19](resp. contra $gb$-continuous ) if $f^{-1}(V)$ is $b$-closed (resp. $gb$-closed ) in $X$ for each open set $V$ of $Y$.

Theorem 4.2. Let $f : X \to Y$ be a function . Then $f$ is contra $b$-continuous if and only if it is locally $b$-closed-continuous and contra $gb$-continuous.

Proof. It follows from Theorem 2.5

From the following examples we can see that locally $b$-closed-continuous and contra $gb$-continuous are independent.

Example 4.5. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $Y = \{a, b, c\}$, with the topology $\sigma = \{\phi, Y, \{a\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = b, f(b) = c, f(c) = a$ and $f(d) = a$. Then $f$ is contra $gb$-continuous, but it is not locally $b$-closed-continuous, since $f^{-1}(\{a\}) = \{c, d\}$ which is $gb$-closed not locally $b$-closed.

Example 4.6. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{b, c, d\}, \{b\}, \{a, b\}\}$. Let $Y = \{a, b, c\}$, with the topology $\sigma = \{\phi, Y, \{a\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = b, f(b) = a, f(c) = a$ and $f(d) = c$. Then $f$ is locally $b$-closed-continuous, but it is not contra $gb$-continuous, since $f^{-1}(\{a\}) = \{c, b\}$ which is locally $b$-closed not $gb$-closed.

Theorem 4.3. Let $f : X \to Y$ be a function . Then $f$ is $B$-continuous if and only if it is locally $b$-closed-continuous and semi-continuous.

Proof. It follows from Lemma 2.2

Definition 4.3. A function $f : X \to Y$ is called $b$-pre continuous (resp. $b$-$B$-continuous, $b$-$t$-continuous contra $sb$-continuous) if $f^{-1}(V)$ is $b$-preopen (resp. $b$-$B$-set, $b$-$t$-set, $sb$-generalized closed) in $X$ for each open set $V$ of $Y$.

Theorem 4.4. Let $f : X \to Y$ be a function . Then $f$ is continuous if and only if it is $b$-pre-continuous and $b$-$B$-continuous.
Proof. The proof is obvious from Theorem 3.1.

From the following example we can see that $b$-pre continuous functions and $b$-$B$-continuous functions are independent.

Example 4.7. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{b, c, d\}, \{b\}, \{a, b\}\}$. Let $Y = \{a, b, c, \}$, with the topology $\sigma = \{\phi, Y, \{a\}\}$. Then the function $f : (X, \tau) \rightarrow (Y, \sigma)$ is defined by $f(a) = b$, $f(b) = a$, $f(c) = a$ and $f(d) = c$. Then $f$ is $b$-pre-continuous, but it is not $b$-$B$-continuous, since $f^{-1}(\{a\}) = \{c, b\}$ which is $b$-peropen not $b$-$B$-set.

Example 4.8. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{c\}, \{d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $Y = \{a, b, c, \}$, with the topology $\sigma = \{\phi, Y, \{a\}\}$. Then the function $f : (X, \tau) \rightarrow (Y, \sigma)$ is defined by $f(a) = b$, $f(b) = a$, $f(c) = a$ and $f(d) = c$. Then $f$ is $b$-$B$-continuous, but it is not $b$-pre continuous, since $f^{-1}(\{a\}) = \{c, b\}$ which is $b$-$B$-set not $b$-peropen.

Theorem 4.5. Let $f : X \rightarrow Y$ be a function. Then $f$ is completely continuous if and only if it is $b$-pre continuous and $b$-$t$-continuous.

Proof. It follows from Theorem 3.3.

Theorem 4.6. Let $f : X \rightarrow Y$ be a function. Then $f$ is completely continuous if and only if it is $b$-pre continuous and contra $sb$-continuous.

Proof. It follows from Theorem 3.2.

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References


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