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Limit cycles for Singular Perturbation Problems via Inverse Integrating Factor

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ABSTRACT: In this paper singularly perturbed vector fields X_ε defined in $\mathbb{R}^2$ are discussed. The main results use the solutions of the linear partial differential equation $X_\varepsilon V = \operatorname{div}(X_\varepsilon)V$ to give conditions for the existence of limit cycles converging to a singular orbit with respect to the Hausdorff distance.

Key Words: Limit cycles, vector fields, singular perturbation, inverse integrating factor.

Contents

1. Introduction and statement of the main results

The present work fits within the geometric study of singular perturbation problems expressed by one–parameter families of vector fields $X_\varepsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where

$$X_\varepsilon(x,y) = (f(x,y,\varepsilon), \varepsilon g(x,y,\varepsilon)) \tag{1}$$

with $\varepsilon \geq 0$, $f,g \in C^r$ for $r \geq 1$ or $f,g \in C^\varpi$ for which we want to study the phase portrait, for sufficient small ε, near the set of singular points of X_0 :

$$\Sigma = \{(x,y) \in \mathbb{R}^2 : f(x,y,0) = 0\}.$$

Special emphasis will be given on systems which the solutions of the linear partial differential equation

$$X_\varepsilon V := f\frac{\partial V}{\partial x} + \varepsilon g \frac{\partial V}{\partial y} = \operatorname{div}(X_\varepsilon)\, V$$

are known.

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The system of differential equations associated to X_ε is

$$x' = f(x, y, \varepsilon), \quad y' = \varepsilon g(x, y, \varepsilon) \tag{2}$$

with $x = x(\tau), y = y(\tau)$.

The main trick in geometric singular perturbation (GSP) is to consider the above family in addition to the family

$$\varepsilon \dot{x} = f(x, y, \varepsilon), \quad \dot{y} = g(x, y, \varepsilon) \tag{3}$$

with $x = x(t), y = y(t)$ obtained after the time rescaling $t = \varepsilon\tau$.

System (2) is the *fast system* and (3) is the *slow system*.

Observe that for $\varepsilon > 0$ the phase portrait of the fast and the slow systems coincide, but for $\varepsilon = 0$ the problems are completely different.

We call $\sum$ the *slow manifold* of the singular perturbation problem, and the dynamical system defined by (3) on $\sum$, for $\varepsilon = 0$, is called the *reduced problem*.

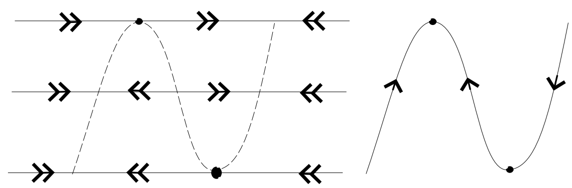

Figure 1: Fast and slow dynamics.

Combining results on the dynamics of these two limiting problems, with $\varepsilon = 0$, one obtains information on the dynamics for small values of ε. In fact, such techniques can be exploited to formally construct approximate solutions on pieces of curves that satisfy some limiting version of the original equation as ε goes to zero.

Let n_1 and n_2 be normally hyperbolic points on $\sum$, see for a definition Section 2. A *singular orbit* consists of three pieces of smooth curves: an orbit of the reduced problem starting at n_1, an orbit of the reduced problem ending at n_2 and a orbit of the fast problem connecting the two previous peaces.

For two compact sets $A, B \subseteq \mathbb{R}^2$ we define the *Hausdorff distance* by

$$D(A, B) = \max_{z_1 \in A, z_2 \in B} \{d(z_1, B), d(z_2, A)\}.$$

The main question in GSP–theory is to exhibit conditions under which a singular orbit can be approached by regular orbits for $\varepsilon \searrow 0$, with respect to the Hausdorff distance. The most interesting question is to decide if X_ε has a limit cycle approaching a singular orbit. In this case, the singular orbit should have a non normally hyperbolic point, that means there is a turning point in the usual terminology, i.e an extreme local of the function defined implicitly by $f(x, y, 0) = 0$. Some papers are in this direction [2,3,4,8,9,10].

In the qualitative theory of differential equations, research on limit cycles is a difficult part. Limit cycles of planar vector fields were defined by Poincaré and at the end of the 1920s van der Pol, Liénard and Andronov proved that a closed trajectory of a self–sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. There are some methods for proving the nonexistence and existence of limit cycles: Bendixon–Dulac, Poincaré–Bendixson, the return map, etc. The main trick used in this paper is to use the criteria introduced in [6] to study the limit cycles of X_ε, for $\varepsilon \searrow 0$.

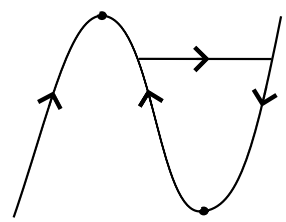

Figure 2: Singular orbit.

The main results of this paper are the following.

Theorem 1.1. *Let X_ε be the vector field (1). Consider $\varepsilon_0 > 0$ and let $V_\varepsilon(x,y) = V(x,y,\varepsilon)$ be a C^1 solution of the linear partial differential equation $X_\varepsilon V = div(X_\varepsilon)V$, defined in an open set $U \subseteq \mathbb{R}^2$, for any $0 \leq \varepsilon < \varepsilon_0$. Let $\Gamma \subset U$ be a singular orbit and Γ_ε be a limit cycle of X_ε in U, for $\varepsilon \in (0, \varepsilon_0)$, with $\Gamma_\varepsilon \longrightarrow \Gamma$, according Hausdorff distance. Then $V_0(\Gamma) = 0$.*

Corollary 1.1A. *Consider X_ε and V like in Theorem 1.1. If the level zero of the function $V(x,y,\varepsilon)$ does not contain a closed curve, for $0 < \varepsilon < \varepsilon_0$, then X_ε does not present a limit cycle.*

We remark that Theorem 1.1 provides a necessary condition in order that a singular orbit Γ can generate, for $\varepsilon > 0$ sufficiently small, a limit cycle. More specifically, if $V(x,y,\varepsilon)$ is a solution of $X_\varepsilon V = \text{div}(X_\varepsilon)V$, defined in the open set U of $\mathbb{R}^2$, then the necessary condition for $\Gamma \subseteq U$ is that $V_0(\Gamma) = 0$.

Theorem 1.2. *Consider X_ε and V like in Theorem 1.1. If*

$$f(x,y,\varepsilon) = f_0(x,y) + f_1(x,y)\varepsilon + f_2(x,y)\varepsilon^2 + ...$$

and

$$g(x,y,\varepsilon) = g_0(x,y) + g_1(x,y)\varepsilon + g_2(x,y)\varepsilon^2 + ...$$

are analytical in their variables, then $V(x,y,\varepsilon)$ is analytical, and

$$V(x,y,\varepsilon) = V_0(x,y) + \varepsilon V_1(x,y) + \varepsilon^2 V_2(x,y) + ...$$

with

$$V_0(x,y) = \varphi(y) f_0(x,y),$$

for some C^1 *function* φ*, and*

$$\sum_{i+j=k} \left(f_i \frac{\partial V_j}{\partial x} - \frac{\partial f_i}{\partial x} V_j \right) = \sum_{i+j=k-1} \left(\frac{\partial g_i}{\partial y} V_j - g_i \frac{\partial V_j}{\partial y} \right).$$

We remark that Theorem 1.2 provides an way to compute an approximation of the solution $V(x,y,\varepsilon)$.

In Section 2 we present basic facts of the GSP–theory and one criteria for the study the existence and nonexistence of limit cycles introduced in [6]. In Section 3 we prove the main result and in Section 4 we present some examples and applications.

2. Basic facts of GSP-theory and inverse integrating factor

2.1. The GSP-theory. The foundation of GSP-theory, which is briefly summarized here, was laid by Fenichel in [5]. We consider only planar problems but remember that in [5] one can check the general case.

Let $X_\varepsilon(x,y) = (f(x,y,\varepsilon), \varepsilon g(x,y,\varepsilon))$ with $(x,y) \in \mathbb{R}^2$ and the slow manifold Σ given implicitly by $f(x,y,0) = 0$.

We say that $p = (x_0, y_0) \in \Sigma$ is *normally hyperbolic* if $\dfrac{\partial f}{\partial x}(p,0) \neq 0$.

We assume that, for every normally hyperbolic $p \in \Sigma$, $\dfrac{\partial f}{\partial x}(p,0)$ has k^s eigenvalues with negative real part and k^u eigenvalues with positive real part.

Theorem 2.1. *Let $n \in \Sigma$ be a hyperbolic singular point of the slow flow with j^s–dimensional local stable manifold W^s and a j^u–dimensional local unstable manifold W^u. Then there exists an ε-continuous family n_ε such that $n_0 = n$ and n_ε has a $(j^s + k^s)$– dimensional local stable manifold W^s_ε and a $(j^u + k^u)$– dimensional local unstable manifold W^u_ε.*

For a proof see [5]. The importance of this theorem is that every structure of the slow system which persists under regular perturbation also persists under singular perturbation. The next step is to decide if a singular orbit can be approached by regular orbits.

Theorem 2.2. *If $n, m \in \Sigma$, like in Theorem 2.1, are connected by an orbit of the fast problem then there exists an orbit of X_ε connecting n_ε and m_ε.*

For a proof see [11]. Combining Theorem 2.1 and Theorem 2.2 one can see that if a singular orbit Γ is composed by orbits of the reduced problem on the normally hyperbolic part of the slow manifold and connected by orbits of the fast problem, then there are regular orbits Γ_ε of X_ε, such that $\Gamma_\varepsilon \longrightarrow \Gamma$, for $\varepsilon \searrow 0$, according Hausdorff distance. To analyse the non–normally hyperbolic case there is a new technique introduced by Dumortier and Roussarie in [4] which is based on the blow up techniques. Another approach can be obtained in [3] for the same problems by assuming that the systems are time reversible.

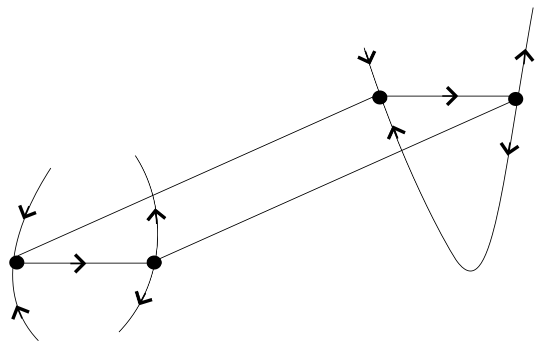

Figure 3: Singular perturbation with normal hyperbolicity.

2.2. THE INVERSE INTEGRATING FACTOR. Let U be the domain of definition of the vector field $X(x,y) = (p(x,y), q(x,y))$ and let W be an open subset of U. A non-zero function V on W that satisfies the linear partial differential equation $XV = \text{div}(X)V$, is called an *inverse integrating factor* of the vector field X.

This function V is important because

(i) $R = 1/V$ defines on $W \setminus \{V = 0\}$ an integrating factor of the differential system associated to the vector field.

(ii) $\{V = 0\}$ contains the limit cycles of the phase portrait of the vector field X. This fact allows to study the limit cycles which bifurcate from periodic orbits of a centre (Hamiltonian or not) and compute their shape. For doing that we develop the function V in power series of the small perturbation parameter. A remarkable fact is that the first term of this expansion coincides with the first non–identically zero Melnikov function.

(iii) There are a great number of examples of vector fields with an inverse integrating function V being an easier function than their first integrals.

3. Proof of the main results

In this section we shall prove the results state in the introduction.

Proof of Theorem 1.1: We suppose that the function $V(x,y,\varepsilon)$ is a solution of the equation $X_\varepsilon V = \text{div}(X_\varepsilon)V$ on the open subset U of $\mathbb{R}^2$, for $\varepsilon \in [0, \varepsilon_0)$. It means that the system

$$x' = f(x,y,\varepsilon)/V(x,y,\varepsilon), \quad y' = \varepsilon g(x,y,\varepsilon)/V(x,y,\varepsilon), \tag{4}$$

on $U \setminus \{V = 0\}$ is Hamiltonian. Note that System (2) and System (4) are topologically equivalent in $U \setminus \{V = 0\}$. Since System 4 is Hamiltonian, it has no limit cycles on $\{V \neq 0\}$. Therefore, if system (2) has a limit cycle in U it must be contained on $\{V = 0\}$. Thus, we have $V_\varepsilon(\Gamma_\varepsilon) = 0$. If there exists $p \in \Gamma$ such that $V_0(p) \neq 0$ then there exists an open subset $W \subseteq \mathbb{R}^2$ such that $p \in W$ and $V_0(q) \neq 0$, for any $q \in W$. The Hausdorff convergence $\Gamma_\varepsilon \longrightarrow \Gamma$ implies that there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that $\Gamma_\varepsilon \bigcap W \neq \emptyset$ for any $0 < \varepsilon < \varepsilon_1$. In this case there exists $q \in \Gamma_\varepsilon$ with $V_0(q) \neq 0$. The continuity of V with respect to ε gives that there exists $0 < \varepsilon_2 \in (0, \varepsilon_1)$ such that for $0 < \varepsilon < \varepsilon_2$ we have $V_\varepsilon(q) \neq 0$, and it is a contradiction. Then $V_0(\Gamma) = 0$. ∎

Proof of Corollary 1.1A: Since the set $\{V = 0\}$ contains the limit cycles of X_ε in U and it has no closed curve, X_ε can not have limit cycles. ∎

Proof of Theorem 1.2: We deal with planar systems of the form (2) where $f(x, y, \varepsilon)$ and $g(x, y, \varepsilon)$ depend analytically on their variables in an open subset U. Assume that ε is a small parameter. We look for an analytic solution

$$V(x, y, \varepsilon) = \sum_{k=0}^{\infty} V_k(x, y)\varepsilon^k,$$

of the linear partial differential equation

$$f\frac{\partial V}{\partial x} + \varepsilon g\frac{\partial V}{\partial y} = \left(\frac{\partial f}{\partial x} + \varepsilon\frac{\partial g}{\partial y}\right)V. \tag{5}$$

It is known that V is analytic in the variables x, y, ε (see for instance [7]). From equation (5) we deduce the zero–order equation with respect to ε

$$f_0\frac{\partial V_0}{\partial x} = V_0\frac{\partial f_0}{\partial x}. \tag{6}$$

At k–th order with respect to ε we obtain

$$\sum_{i+j=k}\left(f_i\frac{\partial V_j}{\partial x} - \frac{\partial f_i}{\partial x}V_j\right) = \sum_{i+j=k-1}\left(\frac{\partial g_i}{\partial y}V_j - g_i\frac{\partial V_j}{\partial y}\right). \tag{7}$$

For any value of k, the homogeneous partial differential equation for V_k is the same. So, the way to solve (7) is recursive. Since equation (6) becomes $\dfrac{\partial}{\partial x}\left(\dfrac{V_0}{f}\right) = 0$, we have

$$V_0(x, y) = \varphi(y)f(x, y)$$

for some C^1 function φ depending of the variable y.

Besides, if $f_i = g_i = 0$ for $i \geqslant 1$ then (7) implies

$$f\frac{\partial V_k}{\partial x} - V_k\frac{\partial f}{\partial x} = V_{k-1}\frac{\partial g}{\partial y} - g\frac{\partial V_{k-1}}{\partial y}$$

or equivalently

$$f^2\frac{\partial}{\partial x}\left(\frac{V_k}{f}\right) = -g^2\frac{\partial}{\partial y}\left(\frac{V_{k-1}}{g}\right).$$

∎

4. Examples and applications

In the following examples we compute the inverse integrating factor of some vector fields singularly perturbed using the partial differential equations states in Theorem 1.2.

Example 1. Let $X_\varepsilon(x,y) = (y^2 - x^2, \varepsilon x^2)$. We have $f(x,y) = y^2 - x^2$, $V_0(x,y) = yf(x,y), V_1(x,y) = -x^3$, and $V_k(x,y) = 0$, for $k \geq 2$. Thus $V(x,y,\varepsilon) = y^3 - yx^2 - x^3\varepsilon$. Using Corollary 1.1A we conclude that X_ε does not present limit cycles because the levels of V do not contain closed curves.

Example 2. Let $X_\varepsilon(x,y) = (y - x^2, \varepsilon x)$. We have $f(x,y) = y - x^2$, $V_0(x,y) = -2f(x,y), V_1(x,y) = 1$, and $V_k(x,y) = 0$, for $k \geq 2$. Thus $V(x,y,\varepsilon) = -2y+2x^2+\varepsilon$. Using Corollary 1.1A we conclude that X_ε does not present limit cycles because the levels of V do not contain closed curves.

Example 3. Let $X_\varepsilon(x,y) = (-y + x^2, \varepsilon x)$. We have $f(x,y) = -y + x^2$, $V_0(x,y) = -f(x,y), V_1(x,y) = 1/2$, and $V_k(x,y) = 0$, for $k \geq 2$. Thus $V(x,y,\varepsilon) = y - x^2 + (1/2)\varepsilon$. Using Corollary 1.1A we conclude that X_ε does not present limit cycles because the levels of V do not contain closed curves. It is interesting to observe that the singularity $(0,0,\varepsilon)$ is a centre, because the system is invariant by the symmetry $(x,y,t) \longmapsto (-x,y,-t)$, and its eigenvalues are $\pm\sqrt{\varepsilon}i$.

Now we prove a proposition which will be used in next example.

Proposition 4.1. *Let $V(x,y)$ be a C^1-function defined in some open subset $U \subseteq \mathbb{R}^2$. If $\lambda \in \mathbb{R}$ and $g(x,y) \in C^1(U)$ then $V(x,y)$ is an inverse integrating factor of the vector field*

$$X(x,y) = (-\lambda\partial V/\partial y - V\partial g/\partial y, \lambda\partial V/\partial x + V\partial g/\partial x).$$

Proof: We have that

$$\begin{aligned}(XV)(x,y) &= \frac{\partial V}{\partial x}\left(-\lambda\frac{\partial V}{\partial y} - V\frac{\partial g}{\partial y}\right) + \frac{\partial V}{\partial y}\left(\lambda\frac{\partial V}{\partial x} + V\frac{\partial g}{\partial x}\right) = \\ &= \left(\frac{\partial V}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial g}{\partial y}\right)V = \operatorname{div}(X)V(x,y),\end{aligned}$$

for any $(x,y) \in U$. ∎

Example 4. Let X_ε be the vector field given by

$$X_\varepsilon(x,y) = (-2\varepsilon y - (x^2 + y^2 - 1), \varepsilon(2x + x^2 + y^2 - 1)).$$

The slow manifold is given by $\Sigma = \{(x,y) : x^2 + y^2 = 1\}$ and the function $V_\varepsilon(x,y) = V_0(x,y) = x^2 + y^2 - 1$ is an inverse integrating factor of X_ε. In fact we can apply Proposition 4.1 with $\lambda = \varepsilon$ and $g(x, y) = \varepsilon x + y$. The slow system associated to the vector field is

$$\varepsilon\dot{x} = -2\varepsilon y - (x^2 + y^2 - 1), \quad \dot{y} = 2x + (x^2 + y^2 - 1),$$

and the reduced problem is

$$x^2 + y^2 - 1 = 0, \quad \dot{y} = 2x.$$

The fast and slow dynamics are illustrated in Figure 4. According Theorem 1.2 the only singular orbit which can be approached by limit cycles is the slow manifold $x^2 + y^2 - 1 = 0$.

The curve $x^2 + y^2 - 1 = 0$ is an invariant of the vector field because $X_\varepsilon V_0 = 0$ if $V_0 = 0$. Moreover, the system

$$-2\varepsilon y - (x^2 + y^2 - 1) = 0, \quad 2x + (x^2 + y^2 - 1) = 0, \quad x^2 + y^2 - 1 = 0$$

has solution only if $\varepsilon = 0$. Thus X_ε does not present critical points on $V_0 = 0$ for $\varepsilon > 0$. The periodic orbit of X_ε corresponding to $V_0 = 0$ is

$$x(t) = cos(2\varepsilon t), \quad y(t) = sin(2\varepsilon t).$$

A direct calculation shows that

$$\int_0^{2\pi} div(X_\varepsilon)(x(t), y(t))dt = 1 - \frac{sin(4\varepsilon\pi)}{\varepsilon} - cos(4\varepsilon\pi).$$

Therefore the closed orbit defined by $V_0 = 0$ is a limit cycle of X_ε for $\varepsilon \searrow 0$.

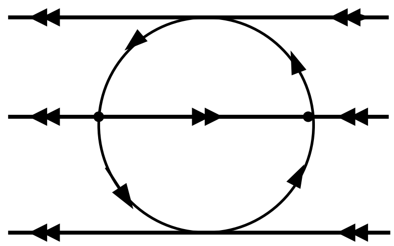

Figure 4: Fast and slow dynamics for $\varepsilon = 0$.

In our last example, we consider a vector field which do not have a polynomial inverse integrating factor.

Example 5. The vector field $X_{\varepsilon,a}(x, y) = (y - x^3/3 + x, \varepsilon(a - x))$ was considered in [1]. It is an example of *the canard phenomenon*. It is known that for $|a| < 1$ the vector field has a unique limit cycle, which is stable, and for $|a| \geq 1$ does not present limit cycles and it has a singular point $(a, a^3/3 - a)$ which is the ϖ-limit of any orbit. For $|a| = 1$ occurs an Andronov–Hopf bifurcation. When $|a| \nearrow 1$ the amplitude of the limit cycles tends to zero.

Proposition 4.2. *The vector field $X_{\varepsilon,a}$ associated to Example 5 has no polynomial inverse integrating factor.*

Proof: We assume that V is a polynomial solution of degree n of $X_{\varepsilon,a}V = \text{div}X_{\varepsilon,a}V$. We consider $V = \sum_{i=0}^{n} P_i(y)x^i$ a polynomial in variable x with coefficients polynomials in y. The degree of the polynomial $P_i(y)$ is at most $n-i$. We want to determine V such that $\xi(V) = X_{\varepsilon,a}V - \text{div}(X_{\varepsilon,a})V = 0$, for all $x, y \in \mathbb{R}^2$. We denote $P_i = P_i(y)$ for $i = 0 \dots n$. Thus

$$\begin{aligned}
\xi(V) \quad &= \sum_{i=0}^{n+2} a_i(y)x^i = P_1 y + \varepsilon a P_0' - P_0 + (2P_2 y + \varepsilon a P_1' - \varepsilon P_0')x \\
&+ (3P_3 y + P_2 + \varepsilon a P_2' - \varepsilon P_1' + P_0)x^2 \\
&+ \sum_{i=3}^{n-1} \left[(i+1)P_{i+1}y + \varepsilon a P_i' - \varepsilon P_{i-1}' + (i-1)P_i + \frac{5-i}{3}P_{i-2} \right] x^i \\
&+ \left[\varepsilon a P_n' - \varepsilon P_{n-1}' + \frac{5-n}{3}P_{n-2} + (n-1)P_n \right] x^n \\
&+ \left[-\varepsilon P_n' + \frac{4-n}{3}P_{n-1} \right] x^{n+1} + \frac{3-n}{3}P_n x^{n+2}.
\end{aligned}$$

Making $\xi(V) = 0$ for all $x, y \in \mathbb{R}^2$, the coefficients of $x^{n+2}, x^{n+1}, \dots, x^4$, for $n > 5$, give $P_n = P_{n-1} = P_{n-2} = \cdots = P_4 = 0$. So, to conclude the proof we need to show that $P_0 = P_1 = P_2 = P_3 = 0$. Solving $a_i(y) = 0$ for $i = 0, 1, ..., 5$, in function of the polynomial P_3 we get the following ordinary differential equation

$$3\varepsilon^2 a(a+1)P_3''' + 3\varepsilon^2(a - a^3 + 9y)P_3'' + \varepsilon(9ay - 2a^2 + 4)P_3' + 4yP_3 = 0.$$

Substituting the polynomial $P_3 = \sum_{i=0}^{n-3} b_i y^i$ in the last equation and collecting in the variable y, is straightforward that $P_3 = 0$. Then it is easy to see that $P_2 = P_1 = P_0 = 0$. ■

For each $0 \leqslant y_0 < 2/3$ we denote Γ_{y_0} the oval singular orbit in the quadrant $x \leqslant 0, y \geqslant 0$ contained in $\{y = y_0\} \bigcup \{y = x^3/3 - x\}$. According [1] and [4], there exist a C^∞ functions $a_{y_0}(\varepsilon)$, for ε sufficiently small, such that $a_{y_0}(0) = -1$ and such that there exists a limit cycle $\Gamma_{\varepsilon, a_{y_0}(\varepsilon)}$ of $X_{\varepsilon, a_{y_0}(\varepsilon)}$ with $\Gamma_{\varepsilon, a_{y_0}(\varepsilon)} \longrightarrow \Gamma_{y_0}$ when $\varepsilon \searrow 0$. These functions are given implicitly, using perturbations methods and blow up method applied to the variables x and y, and to the parameters a and ε.

Theorem 1.1 implies that if V^{y_0} is a solution of $X_{\varepsilon,a}V^{y_0} = \text{div}\,(X_{\varepsilon,a})\,V^{y_0}$ then $V^{y_0}(\Gamma_{\varepsilon, a_{y_0}(\varepsilon)}) = 0$, and Theorem 1.2 gives a way to compute an approximation of

V^{y_0}:

$$V^{y_0}(x,y,\varepsilon,a) = V_0^{y_0}(x,y,a) + V_1^{y_0}(x,y,a)\varepsilon + ...$$

with

$$V_0^{y_0}(x,y,a) = F_0(y)\left(y - \frac{x^3}{3} + x\right)$$

and $V_1^{y_0}$ satisfying

$$(y - x^3/3 + x)^2 \frac{\partial}{\partial x}\left(\frac{V_1^{y_0}}{y - x^3/3 + x}\right) = -(a-x)^2 \frac{\partial}{\partial y}\left(\frac{V_0^{y_0}}{a-x}\right). \tag{8}$$

Using Theorem 1.1, we get $F_0(y_0) = 0$, because $V_0(\Gamma_{y_0}) = 0$.

In the sequel, we assume for simplicity that $y_0 = 0$. It is known there exists a limit cycle near Γ_0. So, we want to find $V^0(x,y,\varepsilon,a)$ such that in the neighborhood of Γ_0, $|V_0^0(x,y,a) + \varepsilon V_1^0(x,y,a)|$ is close to zero.

Computing $V_1^0(x,y,a)$ from (8) we obtain

$$\begin{aligned} V_1^0(x,y,a) = \ & \left(-3x + x^3 - 3y\right) F_1(y) + \\ & + \frac{3\left(-2x - 6y + 3x^2 y + a\left(-4 + 2x^2 - 3xy\right)\right) F_0(y)}{4 - 9y^2} + \\ & + \left(\frac{-3x + x^3 - 3y}{4 - 9y^2}\right) \mathrm{RS}\left(3y + 3\xi - \xi^3, H(x,y,\xi)\right), \end{aligned}$$

with

$$H(x,y,\xi) = \frac{\log(x-\xi)}{\xi^2 - 1}\left(\eta_1 F_0(y) + \eta_2 F_0'(y)\right)$$

where $RS(f,H)$ represents the sum of $H(x,y,\xi)$ for all ξ that satisfy the polynomial equation $f(\xi) = 0$, and η_1, η_2 are given by

$$\begin{aligned} \eta_1 = \ & (-4 - 6ay) + (2a + 3y)\xi \\ \eta_2 = \ & -(\xi_i - a) - (4 - 9y^2). \end{aligned}$$

Analysing $V_1^0(x,y,a)$ near $y = 0$ for $x \in (-\sqrt{3}, 0)$, we conclude that $F_1(0)$ must be small, and $F_0'(0) = 0$. Moreover, for $x_0 \in (-\sqrt{3}, 0)$, $\lim_{x \to x_0} V_1^0(x, x^3/3 - x, a) = 0$.

In short, for $F_0(y)$ and $F_1(y)$ satisfying $F_0(0) = F_0'(0) = 0$ and $F_1(0)$ sufficiently small, the function $V_0^0(x,y,a) + \varepsilon V_1^0(x,y,a)$ satisfies that in the neighborhood of Γ_0, $|V_0^0(x,y,a) + \varepsilon V_1^0(x,y,a)|$ is close to zero. For a numerical approach we consider $F_0(y) = y^2$ and $F_1(y) \equiv 0$.

In Figure 5 we observe that there exists an oval on the level 0 of the function $V_0^0 + \varepsilon V_1^0$, in the quadrant $x < 0, y > 0$. This oval approaches a limit cycle of $X_{\varepsilon,a}$ for $\varepsilon = 1/20$ and $a = -0.8$. This limit cycle is stable.

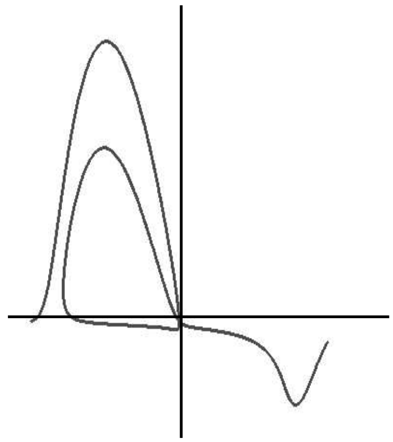

Figure 5: $V_0^0 + \frac{1}{20}V_1^0 = 0$ for $a = -0.8$ and $F_1 \equiv 0$.

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