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A note on almost δ-semicontinuous functions

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ABSTRACT: In this note, we obtain some improvements of results established on δ-semicontinuous functions in [3] and show that a function $f : (X, \tau) \to (Y, \sigma)$ is almost δ-semicontinuous if and only if $f : (X, \tau_s) \to (Y, \sigma_s)$ is semi-continuous, where τ_s and σ_s are the semiregularizations of τ and σ, respectively.

Key Words: semi-regularization, δ-semicontinuity, almost δ-semicontinuity, semi-continuity.

Contents

1. Introduction

The notions of semi-open sets and semi-continuity in topological spaces were first introduced and investigated by Levine [6]. Since then, many generalizations of these notions are introduced and studied in the literature. In 1997, Park et al. [13] introduced the notion of δ-semiopen sets by using δ-open sets due to Veličko [16]. Recently, δ-semicontinuity in topological spaces has been defined independently by Ekici and Navalagi [3] and Noiri [12]. Furthermore, Ekici [1] intrduced and investigated almost δ-semicontinuous functions.

In this note, we obtain some improvements of results established in [3] and show that a function $f : (X, \tau) \to (Y, \sigma)$ is almost δ-semicontinuous if and only if $f : (X, \tau_s) \to (Y, \sigma_s)$ is semi-continuous, where τ_s and σ_s are the semiregualarizations of τ and σ, respectively. We point out that if we use this fact then several properties of almost δ-semicontinuous functions established in [1] follow from the corresponding known properties of semi-continuity.

2. Preliminaries

Let (X, τ) be a topological space and A be a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

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Definition 2.1. *A subset A of a space X is said to be regular open (resp. regular closed) [15] if $A = int(cl(A))$ (resp. if $A = cl(int(A))$).*

A point $x \in X$ is called a δ-cluster point of A [16] if $A \cap int(cl(B)) \neq \varnothing$ for each open set B containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δ-$cl(A)$. If δ-$cl(A) = A$, then A is said to be δ-closed. The complement of a δ-closed set is said to be δ-open. The set $\{x \in X : x \in G \subset A$ for some regular open set G of $X\}$ is called the δ-interior of A and is denoted by δ-$int(A)$.

The family of all regular open (resp. regular closed) sets of a space X will be denoted by $RO(X)$ (resp. $RC(X)$).

Definition 2.2. *A subset A of a space X is said to be*

(1) semiopen [6] if $A \subset cl(int(A))$,

(2) δ-semiopen [13] if $A \subset cl(\delta\text{-}int(A))$.

The family of all δ-semiopen sets of a space X will be denoted by $\delta SO(X)$.

Remark 2.1. *(1) It is shown in [13] that openness and δ-semiopenness are independent.*

(2) The following diagram holds for the subsets defined above:

$$\begin{array}{ccc} \delta\text{-}open & \Rightarrow & \delta\text{-}semiopen \\ \Downarrow & & \Downarrow \\ open & \Rightarrow & semiopen \end{array}$$

Definition 2.3. *([15]) The collection of all regular open sets in a space (X, τ) forms a base for a topology τ_s. It is called the semiregularization.*

Lemma 2.1. *([12]) A subset A is δ-semiopen in (X, τ) if and only if A is semiopen in (X, τ_s).*

3. δ-semicontinuous functions

In this section, we obtain some improvements of the results established in [3].

Definition 3.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*

(1) almost semicontinuous [7] if $f^{-1}(V) \in SO(X)$ for every $V \in RO(Y)$,

(2) almost δ-semicontinuous [1] if for each $x \in X$ and each $V \in RO(Y)$ containing $f(x)$, there exists $U \in \delta SO(X)$ containing x such that $f(U) \subset V$.

(3) δ-semicontinuous [3], [12] if $f^{-1}(V)$ is δ-semiopen for every open set $V \subset Y$.

(4) super-continuous [8] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(int(cl(U))) \subset V$.

(5) semi-continuous [9] if $f^{-1}(V)$ is semi-open in X for every open set $V \subset Y$.

Remark 3.1. *The following implications are hold for a function* $f : X \to Y$*:*

$$\begin{array}{ccccc} \textit{super continuity} & \Rightarrow & \delta\textit{-semicontinuity} & \Rightarrow & \textit{almost } \delta\textit{-semicontinuity} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \textit{continuity} & \Rightarrow & \textit{semi-continuity} & \Rightarrow & \textit{almost semi-continuity} \end{array}$$

In the above diagram, none of the implications is reversible as shown by the following examples.

Example 3.1. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. *Let* $f : X \to X$ *be a function defined by* $f(a) = a$, $f(b) = d$, $f(c) = c$, $f(d) = d$. *Then,* f *is continuous but not almost* δ*-semicontinuous.*

Example 3.2. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. *Let* $f : X \to X$ *be a function defined by* $f(a) = a$, $f(b) = c$, $f(c) = a$, $f(d) = d$. *Then,* f *is* δ*-semicontinuous but it is not continuous.*

Example 3.3. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. *Let* $f : X \to X$ *be a function defined by* $f(a) = b$, $f(b) = b$, $f(c) = c$, $f(d) = a$. *Then,* f *is almost* δ*-semicontinuous but it is not semi-continuous.*

Theorem 3.1. *([12]) The following properties hold for a function* $f : (X, \tau) \to (Y, \sigma)$*:*

(1) f *is super-continuous if and only if* $f : (X, \tau_s) \to (Y, \sigma)$ *is continuous.*

(2) f *is* δ*-semicontinuous if and only if* $f : (X, \tau_s) \to (Y, \sigma)$ *is semi-continuous.*

Theorem 3.2. *For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following are equivalent:*

(1) $f : (X, \tau) \to (Y, \sigma)$ *is almost* δ*-semicontinuous,*

(2) $f : (X, \tau_s) \to (Y, \sigma)$ *is almost semi-continuous,*

(3) $f : (X, \tau) \to (Y, \sigma_s)$ *is* δ*-semicontinuous,*

(4) $f : (X, \tau_s) \to (Y, \sigma_s)$ *is semi-continuous.*

Proof. (1) $\Rightarrow$ (2) : Let $f : (X, \tau) \to (Y, \sigma)$ be almost δ-semicontinuous and $A \in RO(Y)$. Then, by (1) $f^{-1}(A)$ is δ-semiopen in X. By Lemma 5, $f^{-1}(A)$ is semi-open in (X, τ_s). Thus, f is almost semi-continuous.

(2) $\Rightarrow$ (3) : Let $V \in \sigma_s$. Then there exist regular open sets U_i $(i \in I)$ such that $V = \cup U_i$. Since $f : (X, \tau_s) \to (Y, \sigma)$ is almost semi-continuous, $f^{-1}(U_i)$ is semi-open in (X, τ_s) for each $i \in I$. By Lemma 5, $f^{-1}(U_i)$ is δ-semiopen in (X, τ) for each $i \in I$. Thus, $f^{-1}(V)$ is δ-semiopen in (X, τ) and hence $f : (X, \tau) \to (Y, \sigma_s)$ is δ-semicontinuous.

(3) $\Rightarrow$ (4) : The proof is similar to (1) $\Rightarrow$ (2).

(4) $\Rightarrow$ (1) : Let $f : (X, \tau_s) \to (Y, \sigma_s)$ be semi-continuous and $A \in RO(Y)$. We have $A \in \sigma_s$. By (4), $f^{-1}(A)$ is semi-open in (X, τ_s). By Lemma 5, $f^{-1}(A)$ is δ-semiopen in (X, τ). Thus, it follows from Theorem 11 of [1] that $f : (X, \tau) \to (Y, \sigma)$ is almost δ-semicontinuous. □

Definition 3.2. *A space X is called δ-semiconnected [3] (resp. semi-connected [14]) if X cannot be expressed by the disjoint union of two nonempty δ-semiopen (resp. semi-open) sets.*

It is shown in Theorem 3.3 of [14] that every semi-connected space is connected but the converse is not true.

Lemma 3.1. *For a topological space (X, τ), the following properties are equivalent:*
(1) $cl(V) = X$ for every nonempty open set V of X,
(2) $U \cap V \neq \varnothing$ for any nonempty semi-open sets U and V of X,
(3) (X, τ) is semi-connected,
(4) (X, τ) is δ-semiconnected.

Proof. This follows from Theorem 3.2 of [11] and Theorem 6.3 of [12]. □

Theorem 3.3. *If $f : (X, \tau) \to (Y, \sigma)$ is an almost semi-continuous surjection and (X, τ) is δ-semiconnected, then (Y, σ) is δ-semiconnected.*

Proof. Suppose that (Y, σ) is not a δ-semiconnected space. There exist nonempty disjoint δ-semiopen sets A and B such that $Y = A \cup B$. Since A and B are nonempty δ-semiopen, they are nonempty semi-open and hence $int(A)$ and $int(B)$ are nonempty and disjoint. Therefore, by using Lemma 2 of [9], we obtain that $int(cl(A))$ and $int(cl(B))$ are nonempty disjoint regular open sets in Y. Since f is almost semi-continuous, $f^{-1}(int(cl(A)))$ and $f^{-1}(int(cl(B)))$ are nonempty disjoint semi-open sets in X. By Lemma 14, (X, τ) is not δ-semiconnected. □

Corollary 3.3A. *([3]) If $f : (X, \tau) \to (Y, \sigma)$ is a δ-semicontinuous surjection and (X, τ) is δ-semiconnected, then (Y, σ) is connected.*

Definition 3.3. *A space X is said to be δ-semi-T_2 [3] if for each pair of distinct points x and y in X, there exist disjoint δ-semiopen sets A and B in X such that $x \in A$ and $y \in B$.*

Theorem 3.4. *If $f : (X, \tau) \to (Y, \sigma)$ is an almost semi-continuous injection and (Y, σ) is T_2, then (X, τ) is δ-semi-T_2.*

Proof. Let x and y be any pair of distinct points of X. There exist disjoint open sets U and V in Y such that $f(x) \in A$ and $f(y) \in B$. The sets $int(cl(A))$ and $int(cl(B))$ are disjoint regular open sets in Y. Since f is almost semi-continuous, $f^{-1}(int(cl(A)))$ and $f^{-1}(int(cl(B)))$ is semi-open in X containing x and y, respectively. Thus, $f^{-1}(int(cl(A))) \cap f^{-1}(int(cl(B))) = \emptyset$ and hence by Theorem 6.5 of [12] (X, τ) is δ-semi-T_2. □

Corollary 3.4B. *([3]) If $f : (X, \tau) \to (Y, \sigma)$ is a δ-semicontinuous injection and (Y, σ) is T_2, then (X, τ) is δ-semi-T_2.*

Definition 3.4. *A space X is said to be r-T_1 [2] if for each pair of distinct points x and y of X, there exist regular open sets A and B containing x and y respectively such that $y \notin A$ and $x \notin B$.*

Definition 3.5. *A space X is said to be δ-semi-T_1 [3] if for each pair of distinct points x and y of X, there exist δ-semiopen sets A and B containing x and y, respectively such that $y \notin A$ and $x \notin B$.*

Theorem 3.5. *If $f : (X, \tau) \to (Y, \sigma)$ is an almost δ-semicontinuous injection and (Y, σ) is r-T_1, then (X, τ) is δ-semi-T_1.*

Proof. Let Y be r-T_1 and x, y be distinct points in X. There exist regular open subsets A, B in Y such that $f(x) \in A$, $f(y) \notin A$, $f(x) \notin B$ and $f(y) \in B$. Since f is almost δ-semicontinuous, $f^{-1}(A)$ and $f^{-1}(B)$ are δ-semiopen subsets of X such that $x \in f^{-1}(A)$, $y \notin f^{-1}(A)$, $x \notin f^{-1}(B)$ and $y \in f^{-1}(B)$. Hence, X is δ-semi-T_1. □

Corollary 3.5C. *([3]) If $f : (X, \tau) \to (Y, \sigma)$ is a δ-semicontinuous injection and (Y, σ) is T_1, then (X, τ) is δ-semi-T_1.*

4. Almost δ-semicontinuous functions

In this section, we point out that if we use Theorem 12 then several properties of almost δ-semicontinuous functions established in [1] follow from the corresponding known properties of semi-continuity. For example, we can obtain the following two theorems by Lemma 24 and Theorems 6 and 5 of [9].

Lemma 4.1. *([5]) Let $\{X_i : i \in I\}$ be any families of spaces. For the product space $\prod_{i \in I} X_i$, $(\prod_{i \in I} X_i)_s = \prod_{i \in I} (X_i)_s$.*

Theorem 4.1. *([1]) If a function $f : X \to \prod Y_i$ is almost δ-semicontinuous, then $p_i \circ f : X \to Y_i$ is almost δ-semicontinuous for each $i \in I$, where p_i is the projection of $\prod Y_i$ onto Y_i.*

Proof. Let $f : X \to \prod Y_i$ be almsot δ-semicontinuous. Then, by Theorem 12, $f : X_s \to (\prod Y_i)_s$ is semi-continuous and hence by Theorem 6 of [9] and Lemma 24 $p_i \circ f : X_s \to (Y_i)_s$ is semi-continuous. By Theorem 12, $p_i \circ f : X \to Y_i$ is almost δ-semicontinuous. □

Theorem 4.2. *([1]) The product function $f : \prod X_i \to \prod Y_i$ is almost δ-semicontinuous if and only if $f_i : X_i \to Y_i$ is almost δ-semicontinuous for each $i \in I$.*

Proof. Let $f : \prod X_i \to \prod Y_i$ be almost δ-semicontinuous. Then, by Theorem 12 $f : (\prod X_i)_s \to (\prod Y_i)_s$ is semi-continuous and by Lemma 24 $f : \prod (X_i)_s \to \prod (Y_i)_s$ is semi-continuous. It follows from Theorem 5 of [9] that $f_i : (X_i)_s \to (Y_i)_s$ is semi-continuous for each $i \in I$. Therefore, $f_i : X_i \to Y_i$ is almost δ-semicontinuous for each $i \in I$. The converse is similarly proven. □

Remark 4.1. *Theorem 30 (resp. Theorem 35, Theoem 37, Theorem 40) in [1] follows from Theorem 2 of [10] (resp. Theorem 3 of [9], Theorem 4 of [9], Theorem 2.6 of [4]).*

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