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Stabilization of Navier-Stokes Equations

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ABSTRACT: We survey here a few recent stabilization results for Navier-Stokes equations.

Key Words: Navier-Stokes equations, Hamilton-Jacobi equation, Brownian motion.

Contents

1. Introduction

We are concerned here with the stabilization of steady state solutions to Navier-Stokes equation

$$\begin{array}{ll} y_t - \nu\Delta y + (y\cdot\nabla)y = \nabla p + mu + f_e & \text{in } D\times R^+ \\ \nabla\cdot y = 0 & \text{in } D\times R^+ \\ y=0 & \text{on } \partial D\times R^+ \\ y(0)=y_0 & \text{in } D \end{array} \tag{1}$$

where D is a bounded and smooth domain of R^d, $d=2,3$ and $f_e\in (L^2(D))^d$. Here $m\in C_0^1(\mathcal{O})$ and $m>1$ on $\mathcal{O}_1\subset\subset\mathcal{O}$ where $\mathcal{O}_1,\mathcal{O}$ are open subdomains of D.

There is a large number of recent works devoted to feedback stabilization of Navier-Stokes equations of the form (1) with internal and boundary controllers.

Let y_e be a solution to stationary equation

$$\begin{array}{ll} -\nu\Delta y_e + (y_e\cdot\nabla)y_e = \nabla p_e + f_e & \text{in } D \\ \nabla\cdot y_e = 0 & \text{in } D \\ y_e=0 & \text{on } \partial D. \end{array} \tag{2}$$

The main result established in [4] (see also [1], [5], [6]) is that *there is a stabilizing feedback controller $u=u(t,x)$ of the form*

$$u(t,x) = -\sum_{i=1}^{M}\psi_i(x)\int_D R(y-y_e)\psi_i m\,dx \tag{3}$$

2000 *Mathematics Subject Classification:* 16N60, 16U80, 16W25 where R is a self-adjoint operator to be precised below and $\{\psi_i\}_{i=1}^M$ is a given system of functions.

We set $y - y_e \Longrightarrow y$, $H = \{y \in (L^2(D))^d;\ \nabla \cdot y = 0,\ y \cdot n = 0 \text{ on } \partial D\}$, $P : (L^2(D))^d \to H$ is the Leray projector and

$$Ay = P\Delta y, \quad By = P(y \cdot \nabla)y,$$

with $D(A) = \{y \in H \cap (H_0^1(D))^d;\ \Delta y \in (L^2(D))^d\}$. Then we may write equation for $y \Longrightarrow y - y_e$ as

$$\begin{aligned} &\frac{dy}{dt}(t) + \nu Ay(t) + A_0 y(t) + By(t) = Fu \\ &y(0) = y_0 - y_e = x \end{aligned} \tag{4}$$

where $A_0 y = P((y_e \cdot \nabla)y + (y \cdot \nabla)y_e)$ and $Fu = \sum_{i=1}^{M} P(m\psi_i)u_i$, $\{\psi_i\}_{i=1}^M \subset D(A)$. We shall denote by $|\cdot|$ the norm of H, $(\cdot,\cdot)$ the scalar product and by A^α, $0 < \alpha < 1$, the fractional power of order α of operator A^α. We set also $|u|_\alpha = |A^{\frac{\alpha}{2}} u|$ for all $\alpha \in (0, 1)$.

The operator R arising in (3) is the symmetric solution to Riccati equation

$$((\nu A + A_0)y, Ry) + \frac{1}{2}|F^* Ry|^2 = \frac{1}{2}\,|A^{\frac{3}{4}} y|^2,\ \forall y \in D(A^{\frac{3}{4}}) \tag{5}$$

and has the following properties (see [4])

$$R \in L(D(A^{\frac{1}{4}}), (D(A^{\frac{1}{4}}))') \cap L(D(A^{\frac{1}{2}}), H) \tag{6}$$

$$(Ry, y) \geq \delta |A^{\frac{1}{4}} y|^2,\ \forall y \in D(A^{\frac{1}{4}}). \tag{7}$$

Here $\{\psi_i\}_{i=1}^M$ is a system of functions which belongs to space $\operatorname{lin}\{\varphi_j\}_{j=1}^N$ of eigenfunctions far $\mathcal{A} = \nu A + A_0$ (N is the number of unstable eigenvalues) and the dimension M of the system is given by spectral properties of $\mathcal{A}$ ($M = 2$ if all unstable eigenvalues are simple and is maximum $2N$ in general case). As shown in [4], the feedback operator (3) exponentially stabilizes zero solution of (4) in a neighborhood

$$\mathcal{U}_\rho = \{x \in D(A^{\frac{1}{4}});\ |x|_{\frac{1}{2}} < \rho\}. \tag{8}$$

The optimal radius ρ of stability domain $\mathcal{U}_\rho$ is determined by formula

$$\max_{|y|_{\frac{1}{2}} \leq \rho} \frac{2|b(y, y, Ry)|}{|y|_{\frac{3}{2}}^2} < 1 \tag{9}$$

where $b(y, z, w) = \int_D (y \cdot \nabla) zw\, dx$, $\forall y, z, w \in D(A^{\frac{1}{2}})$. It follows that for $d = 3$, ρ might be taken as

$$0 < \rho < \frac{1}{2}\, \|R\|_{L(D(A^{\frac{1}{2}}), H)}. \tag{10}$$

This result is of course not optimal because it uses a linear stabilizing feedback for the linearized equation in nonlinear equation (4) and so our aim here is to find the maximal domain of stability via a nonlinear feedback control.

One might speculate that an optimal control feedback for equation (4) with cost functional

$$J(y,u) = \frac{1}{2}\int_0^\infty (|y(t)|^2_{\frac{3}{2}} + |u(t)|^2_M)dt \tag{11}$$

has a wider domain of stability. The construction of such a feedback law is our aim. Here $|\cdot|_M$ is the norm in R^M.

We shall denote by $\mathcal{A} : D(\mathcal{A}) = D(\mathcal{A}) \to H$ the operator

$$\mathcal{A}y = \nu Ay + A_0 y,\ \ D(\mathcal{A}) = D(A) \tag{12}$$

and by $B : D(B) \subset H \to H$ the operator defined by

$$(By, w) = b(y, y, w),\ \forall w \in D(A^{\frac{1}{2}}). \tag{13}$$

We shall denote by W the space $D(A^{\frac{1}{4}})$ with the norm denoted $|\cdot|_{\frac{1}{2}}$.

In Section 2 below we shall develop this approach and refer to [2] for complete proofs. Section 3 is devoted to a different technique, stabilization by noise developed in forth-coming paper [3].

2. The nonlinear optimal control problem

Consider the control system (4), i.e.,

$$\begin{aligned} &\frac{dy}{dt} + \mathcal{A}y + By = Fu, \quad \forall t \geq 0, \\ &y(0) = x, \end{aligned} \tag{14}$$

where $u(t) = \{u_i(t)\}_{i=1}^M$ and $Fu = \sum_{i=1}^M u_i P(m\psi_i)(x)$. By solution to (14) on $[0,T]$ we mean a function $y \in C([0,T]; H) \cap L^2(0,T; D(A^{\frac{1}{2}}))$ with $\frac{dy}{dt} \in L^2(0,T;(D(A^{\frac{1}{2}}))')$ which satisfies a.e. the equation. For $d = 2$ and $x \in H$ there is a unique such a solution while for $d = 3$ it exists only locally or globally (in time) for x in a suitable chosen neighborhood of origin. We shall denote by $\mathcal{D}$ the stabilizability domain of (14) with respect to cost functional (11), i.e.,

$$\mathcal{D} = \{x \in W;\ \exists (y,u) \in L^2(0,\infty; D(A^{\frac{3}{4}})) \times L^2(0,\infty; R^M) \text{ satisfying } (14)\}. \tag{15}$$

As mentioned earlier $\mathcal{D} \neq \emptyset$ and it contains $\mathcal{U}_\rho$ given by (8).

Define the function $\varphi : \mathcal{D} \to R$,

$$\varphi(x) = \inf_{(y,u)} \{J(y,u)\},\ \forall x \in \mathcal{D}. \tag{16}$$

We have

Proposition 1 *For each $x \in \mathcal{D}$ there is at least one pair (y^*, u^*) such that*

$$\varphi(x) = J(y^*, u^*). \tag{17}$$

Moreover, $y^*(t) \in \mathcal{D},\ \forall t \geq 0$.

Proof. Existence is standard and so it will be omitted. ∎

Theorem 2 below is a maximum principle type result for problem (16). For the sake of simplicity we shall assume from now on that $d = 2$. The extension to $d = 3$ is however straightforward.

Theorem 2 *Let (y^*, u^*) be optimal in problem (16). Then*

$$u^*(t) = F^*p(t) = \left\{ \int_D m(x)\psi_i(x)p(t,x)dx \right\}_{i=1}^M,\ \forall t > 0, \tag{18}$$

where $p \in L^2(0,\infty; H) \cap C([0,\infty); H) \cap L^\infty(0,\infty; D(A^{\frac{1}{4}})) \cap L^2(0,\infty; D(A^{\frac{3}{4}}))$ is the solution to equation

$$\frac{dp}{dt} - \mathcal{A}^*p - (B'(y^*))^*p = A^{\frac{3}{2}}y^* \ \text{a.e. } t \geq 0. \tag{19}$$

Here $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$ in H and $(B'(y^*))^*$ is defined by

$$((B'(y^*))^*p, w) = b(y^*, w, p) + b(w, y^*, p),\ \forall w \in D(A^{\frac{1}{2}}). \tag{20}$$

Proof: If (y^*, u^*) is optimal in (16), then it is also optimal for problem

$$\begin{aligned} &\text{Min}\Big\{ \int_0^\infty (|y(t)|^2_{\frac{3}{2}} + |v(t) - F^*Ry(t)|^2)dt; \\ &\frac{dy}{dt} + \mathcal{A} + FF^*Ry + By = Fv,\ y(0) = x,\ v \in L^2(0,\infty; R^M)\Big\}, \end{aligned} \tag{21}$$

too, where $R \in L(D(A^{\frac{1}{4}}), (D(A^{\frac{1}{4}})) \cap L(D(A^{\frac{1}{2}}), H)$ is the solution to the algebraic Riccati equation (5).

Next we consider the operator $\mathcal{L} : L^2(0,\infty; H) \to L^2(0,\infty; H)$ defined by

$$(\mathcal{L}z)(t) = \frac{dz}{dt} + \mathcal{A}z(t) + B'((y^*(t))z(t) + FF^*Rz(t)),\ \forall z \in D(\mathcal{L}) \tag{22}$$

$$\begin{aligned} D(\mathcal{L}) = \Big\{ &z \in L^2(0,\infty; D(A^{\frac{3}{4}})) \cap C([0,\infty); D(A^{\frac{1}{4}})); \\ &\frac{dz}{dt} \in L^2(0,\infty; (D(A^{\frac{3}{4}}))'), \\ &\frac{dz}{dt} + \mathcal{A}z \in L^2(0,\infty; H),\ z(0) = 0 \Big\}. \end{aligned} \tag{23}$$

We have also that, if $z \in D(\mathcal{L})$, then $z \in L^2_{loc}(0,\infty; D(A))$, $\frac{dz}{dt} \in L^2_{loc}(0,\infty;H)$. (By $L^2_{loc}(0,\infty;X)$ we mean the space of measurable functions $u:(0,\infty) \to X$ such that $u \in L^2(\delta, T; X)$ for all $0<\delta<T<\infty$.) We set

$$W^{1,2}(0,\infty;H) = \{z \in L^2_{loc}(0,\infty;H);\ \frac{dz}{dt} \in L^2_{loc}(0,\infty;H)\}.$$

and it turns out that (see [2]) *the operator $\mathcal{L}$ is surjective.* □

Proof of Theorem 2. For each $f \in L^2(0,\infty;H)$, the solution $q \in L^2(0,H;H)$ to equation

$$\frac{dq}{dt} - \mathcal{A}^* q - (B'(y^*))^* q - (FF^*R)^* q = f, \quad t \geq 0 \tag{24}$$

is defined by

$$\langle q, \psi \rangle_{L^2(0,\infty;H)} = -\langle f, \mathcal{L}^{-1}\psi \rangle_{L^2(0,\infty;H)},\ \forall \psi \in L^2(0,\infty;H) \tag{25}$$

and so $\mathcal{L}^*(q) = -f$ where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$.

According to this definition, the solution p to equation (19) is defined by

$$\langle p, \psi \rangle_{L^2(0,\infty;H)} = (A^{\frac{3}{2}} y^* - RFF^* p, \mathcal{L}^{-1}\psi)_{L^2(0,\infty;H)},\ \forall \psi \in L^2(0,\infty;H). \tag{26}$$

Since, as seen earlier, $FF^*Rz = FF^*R\mathcal{L}^{-1}\psi \in L^2(0,\infty;H)$, (26) makes sense.

Now, coming back to problem (21), we see that for $v^*(t) = u^*(t) + F^*Ry^*(t)$ (optimal) we have

$$\int_0^\infty ((y^*(t), z(t))_{\frac{3}{2}} + (v^*(t) - F^*Ry^*(t), v(t) - F^*Rz(t)))dt = 0 \tag{27}$$

for all $v \in L^2(0,\infty;R^M)$, where z is the solution to equation

$$\mathcal{L}(z) = Fv. \tag{28}$$

(Here $(\cdot,\cdot)_{\frac{3}{2}}$ is the scalar product in $D(A^{\frac{3}{4}})$.) Then, if p is the solution to equation

$$\mathcal{L}p^* = -(A^{\frac{3}{2}} y^* - RFu^*), \tag{29}$$

we obtain by (27) that

$$\langle \mathcal{L}^* p, z \rangle_{L^2(0,\infty;H)} - \langle u^*, v \rangle_{L^2(0,\infty;H)} = 0,\ \forall v \in L^2(0,\infty;H)$$

and we get (18) as claimed.

Theorem 3 *For $x \in \mathcal{U}_\rho$ and ρ sufficiently small the solution (y^*, u^*) to problem (16) is unique and $\varphi : W \to R$ is Gâteaux differentiable on $\mathcal{U}_\rho$. Moreover, the semigroup $t \to y^*(t,x)$ leaves invariant the set $\mathcal{U}_\rho$ and*

$$u^*(t) = -F^* \nabla \varphi(y^*(t)), \quad \forall t \geq 0. \tag{30}$$

Proof. The proof of uniqueness for solution (y^*,p) to system (14), (19), (20) for $x\in\mathcal{U}_\rho$ with ρ small enough follows as in [7] by standard estimates of the type used above for the solutions to system (4), (20). Moreover, by (18) we see that for all $h\in\mathcal{U}_\rho$,

$$\lim_{\lambda\downarrow 0}\frac{\varphi(x)+\lambda h)-\varphi(x)}{h}=\int_0^\infty(\langle y^*(t),z(t)\rangle_{\frac{3}{2}}+(u^*(t),v(t))_{R^M})dt$$

where (z,v) is the solution to system

$$\frac{dz}{dt}+\mathcal{A}z+B'(y^*)z=Fv,\quad t\geq 0$$
$$z(0)=h.$$

Then, we obtain that

$$-(p(0),h)=\lim_{h\downarrow 0}\frac{1}{h}(\varphi(x+\lambda h)-\varphi(x)). \tag{31}$$

Hence $-p(0)=\nabla\varphi(x)$, where p is the solution to equation (20). By the dynamic programming principle the latter implies also that (30) holds and the flow $t\to y^*(t,x)$ leaves invariant $\mathcal{U}_\rho$. ∎

Corollary 1 *The function φ is the unique solution on $\mathcal{U}_\rho$ to operatorial (Hamilton–Jacobi) equation*

$$(\mathcal{A}x+Bx,\nabla\varphi(x))+\frac{1}{2}|F^*\nabla\varphi(x)|^2=\frac{1}{2}|x|^2_{\frac{3}{2}},\ \forall x\in\mathcal{U}_\rho\cap D(A). \tag{32}$$

Moreover, φ is convex for a sufficiently small ρ, $\varphi(x)\geq\gamma|x|^2_{\frac{1}{2}}$, $\forall x\in\mathcal{U}_\rho$ and $(D^2\varphi(0)h,h)=(Rh,h)\geq\gamma|h|^2_{\frac{1}{2}}$, $\forall h\in W$, where γ is a positive constant.

Proof. Equation (32) follows by (30) and the obvious relation

$$\varphi(y^*(t))=\frac{1}{2}\int_t^\infty(|y^*(s)|^2_{\frac{3}{2}}+|u^*(s)|^2_M)ds,\ \forall t\geq 0. \tag{33}$$

Conversely, if φ is a solution to (32), then (33) holds and this proves uniqueness of solution φ to (32).

We note also that $D^2\varphi(0)=R\in L(W,W')\cap L(D(A^{\frac{1}{2}}),H)$ is the solution to the algebraic Riccati equation (5) and $D^2\varphi\in C_b(\mathcal{U}_\rho;L(D(A^{\frac{1}{2}}),H)$. In particular this implies that φ is convex in the neighborhood $\mathcal{U}_\rho$ of the origin for ρ sufficiently small. ∎

3. Stabilization by noise

Throughout in the following β_i, $i=1,\dots$, are independent Brownian motions in a probability space $\{\Omega,\mathbb{P},\mathcal{F},\mathcal{F}_t\}_{t>0}$.

Consider the Navier-Stokes equation (1), i.e.,

$$\begin{aligned} & X_t - \nu\Delta X + (X\cdot\nabla)X = f_e + \nabla p \quad \text{in } D\times(0,\infty) \\ & \nabla\cdot X = 0, \quad X\big|_{\partial D} = 0 \\ & X(0) = x, \qquad\qquad\qquad D\subset R^d,\ d\ge 2. \end{aligned} \tag{34}$$

Let X_e be a steady-state to (34), i.e. (see (2)),

$$\begin{aligned} & -\nu\Delta X_e + (X_e\cdot\nabla)X_e = f_e + \nabla p_e \quad \text{in } D \\ & \nabla\cdot X_e = 0, \quad X_e\big|_{\partial D} = 0. \end{aligned} \tag{35}$$

If $X \Longrightarrow X - X_e$, equation (34) reduces to

$$\begin{aligned} & X_t - \nu\Delta X + (X\cdot\nabla)X_e + (X_e\cdot\nabla)X + (X\cdot\nabla)X = \nabla p \\ & \nabla\cdot X = 0, \quad X\big|_{\partial D} = 0 \\ & X(0) = x \end{aligned} \tag{36}$$

where X is $x - X_e$.

Or, in the space H,

$$\begin{aligned} & \dot{X}(t) + \mathcal{A}X(t) + BX(t) = 0,\ t\ge 0, \\ & X(0) = x. \end{aligned} \tag{37}$$

To stabilize the linearized part of (37), we associate the control stochastic problem

$$\begin{aligned} & dX + \mathcal{A}X\,dt = v\,dW_t \\ & X(0) = x \end{aligned} \tag{38}$$

where W_t is a Wiener process in a probability space $\{\Omega,\mathbb{P},\mathcal{F},\mathcal{F}_t\}_{t>0}$ and v a control input to be precised below.

We recall a few properties of the Stokes–Oseen operator $\mathcal{A}$ defined above.

We shall denote by $\widetilde{H}$ the complexified space $H+iH$ with scalar product denoted $\langle\cdot,\cdot\rangle$ and norm $|\cdot|_{\widetilde{H}}$. Denote again $\mathcal{A}$ the extension of $\mathcal{A}$ to this space. The operator $\mathcal{A}$ has a compact resolvent $(\lambda I+\mathcal{A})^{-1}$ and $-\mathcal{A}$ generates a C_0-analytic semigroup $e^{-\mathcal{A}t}$ in $\widetilde{H}$. Consequently, it has a countable number of eigenvalues $\{\lambda_j\}_{j=1}$ with corresponding eigenfunctions φ_j each with finite algebraic multiplicity m_j.

We shall denote by N the number of eigenvalues λ_j with $\operatorname{Re}\lambda_j \le -\gamma$, $j = 1,...,N$, where γ is a fixed positive number.

Denote by $\widetilde{P}_N$ the projector on the finite dimensional space

$$\widetilde{\mathcal{X}}_u = \operatorname{lin\,span}\{\varphi_j\}_{j=1}^N. \tag{39}$$

We have $\widetilde{\mathcal{X}}_u = \widetilde{P}_N\widetilde{H}$ and

$$\widetilde{P}_N = \frac{1}{2\pi i}\int_\Gamma (\lambda I+\mathcal{A})^{-1}d\lambda,$$

where Γ is a closed smooth curve in $\mathbb{C}$ which is the boundary of a domain containing in interior the eigenvalues $\{\lambda_j\}_{j=1}^N$.

Let $\mathcal{A}_u = P_N\mathcal{A},\ \mathcal{A}_s = (I-\widetilde{P}_N)\mathcal{A}$. Then $\mathcal{A}_u, \mathcal{A}_s$ leave invariant spaces $\widetilde{\mathcal{X}}_u, \widetilde{\mathcal{X}}_s = (I-\widetilde{P}_N)\widetilde{H}$ and the spectra $\sigma(\mathcal{A}_u),\ \sigma(\mathcal{A}_s)$ are given by (see [9])

$$\sigma(\mathcal{A}_u) = \{\lambda_j\}_{j=1}^N, \quad \sigma(\mathcal{A}_s) = \{\lambda_j\}_{j=N+1}^\infty.$$

Since $\sigma(\mathcal{A}_s) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda > -\gamma\}$ and $\mathcal{A}_s$ generates an analytic C_0-semigroup on $\widetilde{H}$, we have

$$|e^{-\mathcal{A}_s t}x|_{\widetilde{H}} \le Ce^{-\gamma t}|x|_{\widetilde{H}}, \quad \forall x \in \widetilde{H},\ t \ge 0. \tag{40}$$

We set

$$\psi_j^1 = \mathrm{Re}\,\varphi_j,\ \psi_j^2 = \mathrm{Im}\,\varphi_j,\ j = 1, ..., N, \tag{41}$$

and

$$\mathcal{X}_u = \mathrm{lin\ span}\{\{\psi_j^1\} \cup \{\psi_j^2\}\}_{j=1}^N$$
$$\mathcal{X}_s = \mathrm{lin\ span}\{\{\psi_j^1\} \cup \{\psi_j^2\}\}_{j=N+1}^\infty.$$

Clearly, $\mathcal{A}$ leaves invariant the real spaces $\mathcal{X}_u$ and $\mathcal{X}_s$. More precisely, we have

$$\widetilde{\mathcal{X}}_u = \mathcal{X}_u + i\mathcal{X}_u,\ \widetilde{\mathcal{X}}_s = \mathcal{X}_s + i\mathcal{X}_s$$

and therefore, $H = \mathcal{X}_u \oplus \mathcal{X}_s$, the direct sum of $\mathcal{X}_u$ and $\mathcal{X}_s$ (see, e.g., [9]). Since the system $\{\lambda_j\}_{j=1}^N$ is of the form

$$\{\xi_j \pm i\eta_j\}_{j=1}^{2M},\ \{\delta_j\}_{j=1}^{M_0},\ 2M + M_0 = N,\ \xi_j, \eta_j, \delta_j \in R$$

it follows that

$$\mathcal{X}_u = \mathrm{lin\ span}\{\psi_j\}_{j=1}^N, \tag{42}$$

where

$$\begin{aligned} &\psi_j = \psi_j^1,\ 1 \le j \le M,\ \psi_j = \psi_j^2, \quad M < j \le 2M, \\ &\psi_j = \varphi_j \text{ (real eigenfunctions)}, \quad 2M < j \le N. \end{aligned} \tag{43}$$

The infinite dimensional space $\mathcal{X}_s$ is similarly generated and estimate (40) remains valid in the $|\cdot|_N$-norm for $\mathcal{A}_s$ defined on $\mathcal{X}_s \subset H$. We shall denote by P_N the projector corresponding to the decomposition $H = \mathcal{X}_u \oplus \mathcal{X}_s$, i.e.,

$$\mathcal{X}_u = P_N H,\ \mathcal{X}_s = (I - P_N)H.$$

Consider the orthonormal system $\{\phi_j\}_{j=1}^N$ obtained from $\{\psi_j\}_{j=1}^N$ by the Schmidt orthogonalization procedure.

Consider the following stochastic perturbation of the linearized system (5) (see (6))

$$\begin{aligned} &dX + \mathcal{A}X\,dt = \eta\sum_{i=1}^N (X(t), \phi_i)\, P(m\phi_i)d\beta_i(t) \\ &X(0) = x, \end{aligned} \tag{44}$$

where $\eta \in R$ and $m = \chi_{\mathcal{O}_0}$ is the characteristic of the open subset $\mathcal{O}_0 \subset D$.

Equation (44) can be seen as a closed loop system associated to the controlled equation

$$\begin{aligned} &dX + \mathcal{A}X\,dt = \eta \sum_{i=1}^{N} v_i(t) P(m\phi_i) d\beta_i(t),\ t \geq 0, \\ &X(0) = x, \end{aligned} \tag{45}$$

with the feedback controller $v_i = (X(t), \phi_i)$, $i = 1, ..., N$.

Theorem 4 *Let $X = X(t, x)$ be the solution to (45). Then, for $|\eta|$ sufficiently large, we have*

$$\mathbb{P}\left\{\lim_{t\to\infty} e^{\gamma t} X(t,x) = 0\right\} = 1,\ \forall x \in H, \tag{46}$$

The closed loop system (44) can be equivalently written as

$$\begin{aligned} &dX(t) - \nu\Delta X(t)dt + (X(t)\cdot\nabla)X_e dt + (X_e \cdot \nabla)X(t)dt \\ &\qquad = \eta m \sum_{i=1}^{N} (X(t), \phi_i)\phi_i d\beta_i(t) + \nabla p(t)dt \text{ in } (0,\infty)\times D \\ &\nabla \cdot X(t) = 0,\ X(t)\big|_{\partial D} = 0 \\ &X(0) = x \text{ in } D. \end{aligned} \tag{47}$$

In particular, it follows by Theorem 4 that the feedback controller $u_i = \eta m(X - X_e, \phi_i)\phi_i$, $i = 1, ..., N$, stabilizes exponentially the stationary solution X_e, i.e., we have

Corollary 2.1. *The solution X to closed loop system*

$$\begin{aligned} &dX(t) - \nu\Delta X(t)dt = \eta m \sum_{i=1}^{N} (X(t) - X_e, \phi_i)\phi_i d\beta_i(t) + \nabla p dt,\ t \geq 0 \\ &\nabla \cdot X = 0,\ X\big|_{\partial D} = 0 \\ &X(0) = x \end{aligned} \tag{48}$$

satisfies

$$\mathbb{P}\left[\lim_{t\to\infty} (X(t) - X_e)e^{\gamma t} = 0\right] = 1,\ \forall x \in H. \tag{49}$$

The proof of Theorem 4 follows from same sharp arguments involving the martingale theory and is given in [3].

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