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Weighted Sobolev Spaces and Degenerate Elliptic Equations

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ABSTRACT: In this paper, we survey a number of recent results obtained in the study of weighted Sobolev spaces (with power-type weights, A_p-weights, p-admissible weights, regular weights and the conjecture of De Giorgi) and the existence of entropy solutions for degenerate quasilinear elliptic equations.

Key Words: Degenerate elliptic equations, weighted Sobolev spaces.

Contents

1. Introduction

Let ω be a weight on $\mathbb{R}^n$, i.e., a locally integrable function on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be open, $1 \leq p < \infty$, and k a nonnegative integer. The weighted Sobolev spaces $W^{k,p}(\Omega,\omega)$ consists of all functions u with weak derivatives $D^\alpha u$, $|\alpha| \leq k$ satisfying

$$\| u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \omega \, dx \right)^{1/p} < \infty.$$

In the case $\omega = 1$, this space is denoted $W^{k,p}(\Omega)$. Sobolev spaces without weights occur as spaces of solutions for elliptic and parabolic partial differential equations.

In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is "disturbed" in the sense that some degeneration or singularity appears. This "bad" behaviour can be caused by the coefficients of the corresponding differential operator. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces.

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Example 1.1 Let us start with some well-known facts. We will consider a nonlinear differential operator of order $2k$ in the divergence form

$$(Au)(x) = \sum_{|\alpha| \le k} (-1)^{|\alpha|} D^\alpha a_\alpha(x, u, \nabla u, ..., \nabla^k u) \tag{1.1}$$

for $x \in \Omega \subset \mathbb{R}^n$, where the coefficients $a_\alpha = a_\alpha(x, \xi)$ are defined on $\Omega \times \mathbb{R}^m$, and where $\nabla^j u = \{D^\gamma u : |\gamma| = j\}$ (for $j = 0, 1, ..., k$) is the gradient of the j-th order. Here we suppose that

(i) $a_\alpha(x, \xi)$ satisfy the Carathéodory conditions, i.e., $a_\alpha(., \xi)$ is measurable in Ω for every $\xi \in \mathbb{R}^m$ and $a_\alpha(x, .)$ is continuous in $\mathbb{R}^m$ for a.e. $x \in \Omega$;

(ii) $a_\alpha(x, \xi)$ satisfy the *growth condition*

$$|\, a_\alpha(x, \xi)| \le C_\alpha \left(g_\alpha(x) + \sum_{|\beta| \le k} |\xi_\beta|^{p-1} \right)$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^m$, $C_\alpha > 0$ and $g_\alpha \in L^{p'}(\Omega)$, $(1/p + 1/p' = 1)$;

(iii) $a_\alpha(x, \xi)$ satisfy the *monotonicity condition*

$$\sum_{|\alpha| \le k} (a_\alpha(x, \xi) - a_\alpha(x, \eta))(\xi_\alpha - \eta_\alpha) > 0 \quad \text{for every } \xi, \eta \in \mathbb{R}^m,\ \xi \ne \eta;$$

(iv) $a_\alpha(x, \xi)$ satisfy the *ellipticity condition*

$$\sum_{|\alpha| \le k} a_\alpha(x, \xi)\xi_\alpha \ge C \sum_{|\alpha| \le k} |\xi_\alpha|^p.$$

for every $\xi \in \mathbb{R}^n$ with constant $C > 0$ independent of ξ.

A typical example of a differential operator A satisfying all the foregoing conditions is the so-called p-Laplacian Δ_p defined for $p > 1$ by

$$\begin{aligned} \Delta_p u &= \operatorname{div}(|\nabla u|^{p-2} \nabla u) \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left[\left(\frac{\partial u}{\partial x_1} \right)^2 + ... + \left(\frac{\partial u}{\partial x_n} \right)^2 \right]^{(p-2)/2} \frac{\partial u}{\partial x_i} \right), \end{aligned}$$

or its modification, the operator $\tilde{\Delta}_p$ defined by

$$\tilde{\Delta}_p u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right),$$

or the little more complicated operator

$$Au = -\tilde{\Delta}_p u + |\, u|^{p-2} u.$$

Let us slightly change the foregoing operator into

$$(Au)(x) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) + a_0(x)|u|^{p-2}u \tag{1.2}$$

with given functions (coefficients) $a_i(x)$ $(i = 1, ...n)$ satisfying

$$a_i \in L^\infty(\Omega) \quad (i = 0, 1, ..., n) \tag{1.3}$$

$$\text{and} \qquad a_i(x) \geq C_1 > 0 \quad (i = 0, 1, ..., n), \quad \text{for a.e. } x \in \Omega. \tag{1.4}$$

Then all conditions (ii), (iii), (iv) remain satisfied and we look for a weak solution in the Sobolev space $W^{1,p}(\Omega)$.

However, the situation changes dramatically if some of the coefficients $a_i(x)$ violate condition (1.3) and/or condition (1.4) (i.e., with coefficients which are singular ($a_i(x)$ are unbounded) and/or degenerated ($a_i(x)$ are only positive a.e)). The situation can be saved using the weighted Sobolev space $W^{1,p}(\Omega, \omega)$ instead of the classical Sobolev space $W^{1,p}(\Omega)$.

A typical example is the degenerate p-Laplacian $-\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)$, $p \neq 2$.

Example 1.2 In the linear case, we consider the second order, linear, elliptic equations with divergence structure

$$\mathrm{div}(A(x)\nabla u(x)) = 0 \tag{1.5}$$

where $A(x) = [a_{ij}(x)]_{i,j=1,...,n}$ is a symmetric matrix with measurable coefficients, defined in a domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$. We assume the following ellipticity condition

$$0 < \omega(x)|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq |\xi|^2 v(x) \tag{1.6}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$, where ω and v are measurable functions, finite and positive a.e. $x \in \Omega$.

The equation (1.5) is *degenerate* if ω^{-1} is unbounded, and the equation (1.5) is *singular* if v is unbounded.

2. Weighted Sobolev spaces

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ a.e.. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will also denoted by ω. Thus $\omega(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^n$.

Definition 2.1 Let ω be a weight and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions u on Ω such that

$$\|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\,dx\right)^{1/p} < \infty.$$

Definition 2.2 Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. Let ω be a given family of weight functions ω_α, $|\alpha| \leq k$, $\omega = \{\omega_\alpha = \omega_\alpha(x),\, x \in \Omega,\, |\alpha| \leq k\}$. We denote by $W^{k,p}(\Omega, \omega)$

the set of all functions $u \in L^p(\Omega, \omega_0)$ for which the weak derivatives $D^\alpha u$, with $|\alpha| \le k$, belong to $L^p(\Omega, \omega_\alpha)$. The weighted Sobolev space $W^{k,p}(\Omega, \omega)$ is a normed linear space if equipped with the norm

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha u|^p \omega_\alpha \, dx \right)^{1/p}.$$

If $1 < p < \infty$ and $\omega_\alpha^{-1/(p-1)} \in L^1_{\text{loc}}$ ($|\alpha| \le k$) then $W^{k,p}(\Omega, \omega)$ is a uniformly convex Banach space. If we additionally suppose that also $\omega_\alpha \in L^1_{\text{loc}}(\Omega)$ then $C_0^\infty(\Omega)$ is a subset of $W^{k,p}(\Omega, \omega)$, and we can introduce the space $W_0^{k,p}(\Omega, \omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{W^{k,p}(\Omega,\omega)}$.

For more informations see [18].

2.1. EXAMPLES OF WEIGHTS.

2.1.1 Power-type weights

(i) Let Ω be a bounded domain in $\mathbb{R}^n$. Let M be a nonempty subset of $\overline{\Omega} = \Omega \cup \partial\Omega$, and denote $d_M(x) = \text{dist}(x, M)$ for $x \in \Omega$ (the distance of the point x from the set M). Let $\varepsilon \in \mathbb{R}$ and let us denote

$$\omega(x) = [d_M(x)]^\varepsilon \quad (\text{power} - \text{type weight}).$$

So the singularity ($\varepsilon < 0$) or degenerations ($\varepsilon > 0$) can appear on the boundary $\partial\Omega$ of Ω as well as in the interior of the domain. The set M is very often a closed part of the boundary $\partial\Omega$ (i.e., $M \subset \partial\Omega$).

(ii) Let $s = s(t)$ be a continuous positive function defined for $t > 0$ and such that

$$\text{either} \quad \lim_{t \to 0} s(t) = 0 \quad \text{or} \quad \lim_{t \to 0} s(t) = +\infty.$$

Let us denote $\omega(x) = s\,(d_M(x))$. We have that:

(a) if $\lim_{t\to 0} s(t) = 0$ then $W^{k,p}(\Omega) \hookrightarrow W^{k,p}(\Omega, s(d_M))$;

(b) if $\lim_{t\to 0} s(t) = \infty$ then $W^{k,p}(\Omega, s(d_M)) \hookrightarrow W^{k,p}(\Omega)$.

For more informations on properties of spaces with theses weights see [17].

2.1.2 A_p weight

The class of A_p weight was introduced by B. Muckenhoupt (see [20]), where he showed that the A_p weights are precisely those weights ω for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^n, \omega)$ to $L^p(\mathbb{R}^n, \omega)$ ($1 < p < \infty$), that is

$$M: \; L^p(\mathbb{R}^n, \omega) \to L^p(\mathbb{R}^n, \omega)$$
$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy.$$

is bounded if and only if $\omega \in A_p$ ($1 < p < \infty$), i.e. , there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B \omega\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}\,dx\right)^{p-1} \leq C.$$

for every ball $B\subset\mathbb{R}^n$.

The union of all Muckenhoupt classes A_p is denoted by A_∞, $A_\infty = \bigcup_{p>1} A_p$.

Example of A_p weights

(i) If $x\in\mathbb{R}^n$, $\omega(x) = |\,x|^{\,\alpha}$ is in A_p if and only if $-n < \alpha < n\,(p-1)$ (see Corollary 4.4 in [26]).

(ii) $\omega(x) = \mathrm{e}^{\lambda\varphi(x)} \in A_2$, with $\varphi\in W^{1,n}(\Omega)$ and λ is sufficiently small (see Corollary 2.18 in [12]).

If $\omega\in A_p$ then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega)\subset L^1_{\mathrm{loc}}(\Omega)$ for every open set Ω. It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$. The *weighted Sobolev space* $W^{k,p}(\Omega,\omega)$ is the set of functions $u\in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u\in L^p(\Omega,\omega)$, $|\,\alpha|\leq k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is given by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{|\,\alpha|\,\leq\,k}\int_\Omega |\,D^\alpha u|^\alpha \omega\,dx\right).$$

We have that:

(i) if $\omega\in A_p$ then $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega,\omega)$ (see Corollary 2.1.6 in [27]).

(ii) if $\omega\in A_p$ then we have the weighted Poincaré inequality, that is:

Let $1 < p < \infty$ and $\omega\in A_p$. Then there are positive constants C and δ such that for all Lipschitz continuous functions φ defined on $\bar{B}$ ($B = B(x_0,R)$) and for all $1\leq\theta\leq n/(n-1)+\delta$,

$$\left(\frac{1}{\omega(B)}\int_B |\,\varphi-\varphi_B|^{\theta p}\omega\,dx\right)^{1/\theta p} \leq C\,R\left(\frac{1}{\omega(B)}\int_B |\nabla\varphi|^p\omega\,dx\right)^{1/p}$$

where $\varphi_B = \dfrac{1}{\omega(B)}\displaystyle\int_B \varphi\,\omega\,dx$ (see Theorem 1.5 in [11]).

For more informations about A_p weights see [11],[12], [26] and [27].

2.1.3 p - admissible weights

Let ω be a locally integrable, nonnegative function in $\mathbb{R}^n$ and $1 < p < \infty$. We say that ω is *p-admissible* if the following four conditions are satisfied:

(I) $0 < \omega(x) < \infty$ a.e. $x\in\mathbb{R}^n$ and ω is *doubling*, i.e., there is a constant $C_I > 0$ such that $\omega(2B)\leq C_I\,\omega(B)$, whenever B is a ball in $\mathbb{R}^n$.

(II) If Ω is an open set and $\varphi_k\in C^\infty(\Omega)$ is a sequence of functions such that

$$\int_\Omega |\varphi_k|^p\omega\,dx \to 0 \quad\text{and}\quad \int_\Omega |\nabla\varphi_k-\vartheta|^p\omega\,dx\to 0$$

as $k\to\infty$ then $\vartheta = 0$.

(III) There are constants $\theta > 1$ and $C_{III} > 0$ such that

$$\left(\frac{1}{\omega(B)}\int_B |\,\varphi|^{\,\theta p}\omega\,dx\right)^{1/\theta p} \leq C_{III} R\left(\frac{1}{\omega(B)}\int_B |\nabla\varphi|^p\omega\,dx\right)^{1/p}.$$

whenever $B = B(x_0, R)$ is a ball in $\mathbb{R}^n$ and $\varphi \in C_0^\infty(B)$.
(IV) There is a constant $C_{IV} > 0$ such that

$$\int_B |\,\varphi - \varphi_B|^p\omega\,dx \leq C_{IV} R^p \int_B |\,\nabla\varphi|^p\omega\,dx$$

whenever $B = B(x_0, R)$ is a ball in $\mathbb{R}^n$ and $\varphi \in C^\infty(B)$ is bounded and $\varphi_B = \dfrac{1}{\omega(B)}\displaystyle\int_B \varphi\,\omega\,dx$.

It follows immediately from condition (I) that the measure ω and Lebesgue measure are mutually absolutely continuous. Condition (II) guarantees that the gradient of a Sobolev functions is well defined. Condition (III) is the *weighted Sobolev inequality* and condition (IV) is the *weighted Poincaré inequality.*

Examples of p-admissible weights

(1) If $\omega \in A_p$ $(1 < p < \infty)$ then ω is a p-admissible weight.
(2) $\omega(x) = |\,x|^\alpha$, $x \in \mathbb{R}^n$, $\alpha > -n$, is a p-admissible weight for all $p > 1$.
(3) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K - quasiconformal mapping and $J_f(x)$ is the determinant of its jacobian matrix, then $\omega(x) = J_f(x)^{1-p/n}$ is p-admissible for $1 < p < n$.
(4) See [2] for non-A_p examples of p-admissible weights.
For more informations about p-admissible weights see [15].

Remark 2.3 Recently P.Hajłasz and P.Koskela (see [13]) showed that conditions (I) - (IV) can be reduced to only two: ω is a p-admissible weights $(1 < p < \infty)$ if and only if ω is doubling and there are constants $C > 0$ and $\lambda \geq 1$ such that

$$\frac{1}{\omega(B)}\int_B |\,\varphi - \varphi_B|\,\omega\,dx \leq C\,R\left(\frac{1}{\omega(\lambda B)}\int_{\lambda B} |\nabla\varphi|^p\,\omega\,dx\right)^{1/p},$$

(the weak $(1, p)$-Poincaré inequality).

2.1.4 Regular weights

Let $\omega(x) \geq 0$ be a weight, with $\omega \in L^1_{\text{loc}}(\Omega)$. We consider the set of functions

$$X \;=\; \left\{u \in W^{1,1}_{\text{loc}}(\Omega) \;\; \text{such that} \;\; \|u\|^2_\omega = \int_\Omega (u^2 + |\nabla u|^2)\,\omega\,dx \;<\; \infty\right\}. \tag{2.1}$$

A weighted Sobolev space can be defined, in general, in two ways:
$W = W(\Omega, \omega)$ is the completion of the set X with respect to the norm $\|.\|_\omega$;
$H = H(\Omega, \omega)$ is the completion of the $\{u \in C^\infty(\Omega) \;:\; \|u\|_\omega \;<\; \infty\}$ with respect to the norm $\|.\|_\omega$ (the energy norm).

We have that $H \subseteq W$. By definition, functions smooth in the interior of Ω are dense in H, while the space W is known to contain all functions of finite well-defined " energy".

The classical Sobolev space corresponds to the weight $\omega(x) \equiv 1$ and is uniquely defined since the spaces $W(\Omega)$ and $H(\Omega)$ are the same for each domain Ω. Of course, if ω is bounded above and away from zero ($0 < c_1 \le \omega(x) \le c_2 < \infty$), the spaces W and H are also the same. However, condition $\omega \in L^1_{\text{loc}}(\Omega)$ do not in general ensure the equality $H = W$.

Example 2.4 Let $n = 2$, $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ |x| < 1\}$ and let

$$\omega(x) = \begin{cases} \Big(\ln(2/|x|)\Big)^{\alpha}, & \text{for } x_1 x_2 > 0 \\ \Big(\ln(2/|x|)\Big)^{-\alpha}, & \text{for } x_1 x_2 < 0 \end{cases}$$

with $\alpha > 1$. Then $H \neq W$. In fact, in the polar variables $r = |x|$ and $\theta = \arccos(x_1/r)$, we set

$$u(x) = \begin{cases} 1, & \text{for } x_1 > 0 \text{ and } x_2 > 0 \\ \operatorname{sen}\theta, & \text{for } x_1 < 0 \text{ and } x_2 > 0 \\ 0, & \text{for } x_1 < 0 \text{ and } x_2 < 0 \\ \cos\theta, & \text{for } x_1 > 0 \text{ and } x_2 < 0. \end{cases}$$

We have that $u \in W(\Omega, \omega)$ and $u \notin H(\Omega, \omega)$ (see counterexample 5.1 in [28]).

If we have only the condition $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ the H-solutions and W-solutions could be different when $n \ge 2$. See the example by F.S. Cassano (in [6]) where the H-solutions and the W-solutions with the same boundary data are different.

Definition 2.5 If $H = W$, we call ω a regular weight.

Sufficient conditions of the equality H=W

(1) F.S. Cassano has established an interesting fact: if the weighted Poincaré inequality

$$\int_B |u - u_B|^2 \omega\, dx \le C(\operatorname{diam}(B))^2 \int_B |\nabla u|^2 \omega\, dx$$

holds for each $u \in W(\mathbb{R}^n, \omega)$ then ω is a regular weight (see [6]).

Example 2.6 An example of a weight function ω for which the weighted Poincaré inequality does not hold (see [5]).

Let $\varphi : [0, \infty) \to [0, \infty)$ be a function defined by

$$\varphi(r) = r^{\beta} + \sum_{h=0}^{\infty} \varphi_h(r)$$

where $\varphi_h : [0, \infty) \to [0, \infty)$ $(h = 0, 1, 2, ...)$ are the functions defined by

$$\varphi_h(r) = \begin{cases} 0, & \text{if } 0 \le r < 62^{-(h+3)} \text{ or } r \ge 2^{-h} \\ 2^{-\alpha h}\left(1 - \left(\dfrac{r - 72^{-(h+3)}}{2^{-(h+3)}}\right)^2\right)^2, & \text{if } |r - 72^{-(h+3)}| < 2^{-(h+3)} \end{cases}$$

with α and β are positive numbers such that $0 \le \alpha < \beta$.

Define now the function $\omega : \mathbb{R}^n \to [0, \infty)$ as

$$\omega(x) = \begin{cases} \dfrac{\varphi(|x|)}{|x|^{n-1}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Let $n \geq 2$, $p > 1$, α and β be positive numbers verifying $\alpha + p < \beta < pn - 1$. Then ω is a weight (with ω and $\omega^{-1/(p-1)} \in L^1_{\rm loc}(\mathbb{R}^n)$) such that the Poincaré inequality does not hold.

(2) V.V. Zhikov (see Theorem 4.1 in [28]) showed that: Let F be a closed subset of Ω and let $F_\varepsilon = \{x \in \Omega \;:\; \text{dist}(x, F) \leq \varepsilon\}$. Assume that the weight ω has degeneracies only on F (that is, $0 < c_1(\varepsilon) \leq \omega(x) \leq c_2(\varepsilon) < \infty$ em $\Omega \setminus F_\varepsilon$). If

$$\text{cap}(F) = 0 \quad \text{and} \quad \omega(x) \leq \frac{C}{\text{cap}(F_\varepsilon)} \quad \text{on} \quad \Omega \setminus F_\varepsilon,$$

where

$$\text{cap}\,(F_\varepsilon) = \inf \left\{ \int_\Omega |\nabla u|^2 dx \;:\; u \in C_0^\infty(\Omega),\; u = 1 \text{ in a neighbourhood of } \; F_\varepsilon \right\},$$

then ω is a regular weight, that is, $H(\Omega, \omega) = W(\Omega, \omega)$ (here cap(A) is the Wiener capacity of the set A).

(3) Conjecture of De Giorgi: If $\exp(t\,\omega)$, $\exp(t\,\omega^{-1}) \in L^1_{\rm loc}$ for each $t > 0$ then ω is regular weight (1995) (This, however, remains unproved).

We consider the second order, linear, elliptic equations with divergence structure

$$\text{div}(A(x)\nabla u(x)) = 0 \tag{2.2}$$

where $A(x) = [a_{ij}(x)]_{i,j=1,\ldots,n}$ is a symmetric matrix with measure coefficients defined in a $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$. We assume the following elliptic conditions

$$\omega(x)|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq |\xi|^2 v(x) \tag{2.3}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$.

In 1995, De Giorgi gave a talk in Lecce, Italy, and discussed the natural question: Are size assumptions on ω^{-1} and v sufficient to guarantee the continuity of weak solutions ? He raised the following conjectures on the continuity of weak solutions of equation (2.2).

CONJECTURE 1 Let $n \geq 3$. Suppose that $A(x)$ satisfies condition (2.3) with $\omega = 1$ and v satisfying

$$\int_\Omega \exp(v(x))\,dx < \infty.$$

Then all weak solutions of equation (2.2) are continuous in Ω.

CONJECTURE 2 Let $n \geq 3$. Suppose that $A(x)$ satisfies condition (2.3) with $v(x) = 1$ and with $\omega(x)$ satisfying

$$\int_\Omega \exp(\omega(x)^{-1})\,dx < \infty.$$

Then all weak solutions of equation (2.2) are continuous in Ω.
CONJECTURE 3 ($n = 2$) Suppose that $A(x)$ satisfies condition (2.3) with $v = 1$ and ω satisfying

$$\int_\Omega \exp(\sqrt{\omega(x)^{-1}})\, dx \,<\, \infty.$$

Then all weak solutions of equation (2.2) are continuous in Ω.
The conjectures 1 and 2 are still open. Concerning conjecture 3, J.Onninen and X. Zhong (see Theorem 1.1 in [21]), proved that all weak solutions of equations (2.2) are continuous under the assumption that

$$\int_\Omega \exp(\alpha\sqrt{\omega(x)^{-1}})\, dx \,<\, \infty$$

for some constant $\alpha > 1$ (See also [29]).

2.2. Weighted Sobolev spaces with variable exponent. Most materials can be modelled with sufficient accuracy using classical Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a fixed constant. For some materials with inhomogeneties, for instance eletrorheological fluids, this is not adequate, but rather the exponent p should be able to vary. This leads us to study of variable exponent Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p = p(x)$ is a real-valued function.

The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications. Materials requering such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery in eletrorheological fluids was due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. Electrorheological fluids have been used in robotics and space technology.

For a general account of the underlying physics consult [14] and [22].
Definition 2.7 Let $\Omega \subset \mathbb{R}^n$ be a domain with nonempty boundary $\partial\Omega$. Denote

$$L^\infty_+(\Omega) = \{p \in L^\infty(\Omega) \ :\ \operatorname{ess\,inf}\, p(x) \,>\, 1\}.$$

Definition 2.8 Let ω be a measurable nonnegative and a.e. finite function defined in $\mathbb{R}^n$. For $p \in L^\infty_+(\Omega)$, define

$$L^{p(x)}(\Omega,\omega) = \{u \ :\ u \text{ is measurable on } \Omega \text{ and } \int_\Omega |u(x)|^{\,p(x)}\omega(x)\, dx \,<\infty\},$$

with the norm

$$\| u\|_{L^{p(x)}(\Omega,\omega)} = \inf\left\{\lambda > 0 \ : \ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \omega(x)\,dx \leq 1\right\}.$$

Definition 2.9 Let ω and v are measurable nonnegative and a.e. finite functions defined in $\mathbb{R}^n$. For $p \in L^\infty_+(\Omega)$, define

$$W^{1,p(x)}(\Omega,\omega,v) = \{u \in L^{p(x)}(\Omega,\omega) \ : \ |\nabla u| \in L^{p(x)}(\Omega,v)\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega,\omega,v)} = \| u\|_{L^{p(x)}(\Omega,\omega)} + \|\nabla u\|_{L^{p(x)}(\Omega,v)}$$

or

$$\|u\|_{W^{1,p(x)}(\Omega,\omega,v)} = \inf\left\{\lambda > 0 \ : \ \int_\Omega \left(\left|\frac{u(x)}{\lambda}\right|^{p(x)} \omega(x) + \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} v(x)\right) dx \leq 1\right\}.$$

For more information properties see [10], [16], [19] and [23].

3. Degenerate quasilinear elliptic equations

3.1. ENTROPY SOLUTIONS. Let Ω be an open bounded set of $\mathbb{R}^n$, $n \geq 2$ and the elliptic Dirichlet problem

$$(P)\begin{cases} -\displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f, \quad \text{on } \Omega, \\ u = 0, \quad \text{on } \partial\Omega \end{cases}$$

with $a_{ij} \in L^\infty(\Omega)$ satisfying the ellipticity condition

$$\sum_{i,j=1}^n a_{i,j}\xi_i\xi_j \geq \alpha \sum_{i,j=1}^n \xi_i\xi_j, \quad \forall\, \xi \in \mathbb{R}^n$$

for $\alpha > 0$ and $f \in \mathrm{M}(\Omega)$ (the space of Radon measure, $\mathrm{M}(\Omega) = (C(\overline{\Omega}))'$).

This problem has, for $f \in H^{-1}(\Omega)$, a unique variational solution, which is in $H^1_0(\Omega)$, and verifies

$$\int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\,dx = (f,\varphi)_{H^{-1},H^1_0}, \quad \forall\, \varphi \in H^1_0(\Omega).$$

For $f \notin H^{-1}(\Omega)$, but $f \in \mathrm{M}(\Omega)$, one does not find solutions in $H^1_0(\Omega)$, but in

$$\bigcap_{q<\frac{N}{N-1}} W^{1,q}_0(\Omega)$$

which leads to a weaker formulation, since we need $\varphi \in \bigcup_{p>N} W_0^{1,p}(\Omega)$ for giving a sense to the first integral

$$\int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\,dx = \int_\Omega \varphi\,df, \quad \forall\,\varphi \in \bigcup_{p>N} W_0^{1,p}(\Omega).$$

We have $W_0^{1,p}(\Omega)\subset C(\bar{\Omega})$ for $p > n$, so the right side makes sense. This formulation is weaker than the variational formulation, it does not ensure the uniqueness (see the counter-example of J. Serrin in [24]).

Existence of solutions verifying this formulation has been obtained by several ways: the solutions obtained by duality and the solutions obtained by approximation. The first one is due to Stampacchia (see [25]), the solutions verify a stronger formulation which ensures the uniqueness but can only be applied to linear problem and the second one is due to Boccardo and Gallouët (see [3]), it can be applied to a non linear problem but does not ensure the uniqueness, so entropy conditions have been introduced to precise the formulation (see [1]).

3.2. Existence of entropy solutions: degenerate quasilinear elliptic equations. The main purpose is to establish the existence of entropy solutions for the Dirichlet problem

$$(P)\begin{cases} -\mathrm{div}[\omega(x)\mathcal{A}(x,u,\nabla u)] = f(x), & \text{in } \Omega \\ u(x) = 0, & \text{in } \partial\Omega \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded open set, $f\in L^1(\Omega)$, ω is a weight function (i.e., a locally integrable function on $\mathbb{R}^N$ such that $0<\omega(x)<\infty$ a.e. $x\in\mathbb{R}^N$) and the function $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ satisfies the following conditions

(H1) $x\mapsto\mathcal{A}(x,s,\xi)$ is measurable on Ω for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and $(s,\xi)\mapsto\mathcal{A}(x,s,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^N$ for almost all $x\in\Omega$.

(H2) $\langle\mathcal{A}(x,s,\xi_1)-\mathcal{A}(x,s,\xi_2),\xi_1-\xi_2\rangle > 0$, whenever $\xi_1,\xi_2\in\mathbb{R}^N$, $\xi_1\neq\xi_2$ (where $\langle .,.\rangle$ denotes the usual inner product in $\mathbb{R}^N$)

(H3) $\langle\mathcal{A}(x,s,\xi),\xi\rangle\geq\lambda\,|\xi|^p$, with $1<p<\infty$.

(H4) $|\mathcal{A}(x,s,\xi)|\leq K(x)+h_1(x)\,|s|^{p/p'}+h_2(x)\,|\xi|^{p/p'}$, where K, h_1 and h_2 are positive functions, with $h_1\in L^\infty(\Omega)$, $h_2\in L^\infty(\Omega)$ and $K\in L^{p'}(\Omega,\omega)$ (where $1/p+1/p'=1$).

We propose to solve the problem (P) by approximation with variational solutions: we take $f_n\in C_0^\infty(\Omega)$ such that $f_n\to f$ in $L^1(\Omega)$, we find a solution $u_n\in W_0^{1,p}(\Omega,\omega)$ for the problem with right-hand side f_n and we will try to pass to the limit as $n\to\infty$.

Definition 3.1 Let $\Omega \subset \mathbb{R}^N$ a bounded open set, $1 < p < \infty$, k a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)\, dx + \sum_{1\leq|\alpha|\leq k} \int_\Omega |D^\alpha u(x)|^p \omega(x)\, dx \right)^{1/p}. \quad (3.1)$$

We also define the space $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega,\omega)} = \left(\sum_{1\leq|\alpha|\leq k} \int_\Omega |D^\alpha u(x)|^p \omega(x)\, dx \right)^{1/p}.$$

We need the following basic result.

Theorem 3.2 (The weighted Sobolev inequality) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p-weight, $1 < p < \infty$. Then there exist positive constants C_Ω and δ such that for all $f \in C_0^\infty(\Omega)$ and $1 \leq \eta \leq N/(N-1) + \delta$*

$$\|f\|_{L^{\eta p}(\Omega,\omega)} \leq C_\Omega \||\nabla f|\|_{L^p(\Omega,\omega)}. \quad (3.2)$$

Proof. See Theorem 1.3 in [11]. □

Definition 3.3 We say that $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$ if $T_k(u) \in W_0^{1,p}(\Omega,\omega)$, for all $k > 0$, where the function $T_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k \\ k\ \text{sign}(s), & \text{if } |s| > k. \end{cases}$$

Remark 3.4 If $u \in W_{\text{loc}}^{1,1}(\Omega,\omega)$ then we have

$$\nabla T_k(u) = \chi_{\{|u|<k\}} \nabla u$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^N$. □

Definition 3.5 Let $f \in L^1(\Omega)$ and $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$. We say that u is an entropy solution to problem (P) if for all $k > 0$ and all $\varphi \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \langle \mathcal{A}(x,u,\nabla u), \nabla T_k(u-\varphi) \rangle\, \omega\, dx = \int_\Omega f\, T_k(u-\varphi)\, dx. \quad (3.3)$$

We recall that the gradient of u which appears in (3.3) is defined as in Remark 2.8 in [7], that is to say that $\nabla u = \nabla T_k(u)$ on the set where $|u| < k$.

Definition 3.6 Let $0 < p < \infty$ and let ω be a weight function. We define the weighted Marcinkiewicz space $\mathcal{M}^p(\Omega,\omega)$ as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that the function

$$\Gamma_f(k) = \omega(\{x \in \Omega \;:\; |f(x)| > k\}),\ k > 0,$$

satisfies an estimate of the form $\Gamma_f(k) \leq Ck^{-p}$, $0 < C < \infty$.
Remark 3.7 If $1 \leq q < p$ and $\Omega \subset \mathbb{R}^N$ is a bounded set, we have that

$$L^p(\Omega,\omega) \subset \mathcal{M}^p(\Omega,\omega), \text{ and } \mathcal{M}^p(\Omega,\omega) \subset L^q(\Omega,\omega).$$

Lemma 3.8 *Let $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$ and $\omega \in A_p$, $1 < p < \infty$, be such that*

$$\frac{1}{k}\int_{\{|u|<k\}} |\nabla u|^p \omega \, dx \leq M, \tag{3.4}$$

for every $k > 0$. Then

(i) $u \in \mathcal{M}^{p_1}(\Omega,\omega)$, where $p_1 = \eta\,(p-1)$ (where η is the constant in the weighted Sobolev inequality). More precisely, there exists $C > 0$ such that

$$\Gamma_u(k) \leq CM^{\eta} k^{-p_1}.$$

(ii) $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega)$, where $p_2 = p\,p_1/(p_1+1)$. More precisely, there exists $C > 0$ such that

$$\Gamma_k(|\nabla u|) \leq CM^{(p_1+\eta)/(p_1+1)} k^{-p_2}.$$

Proof. See Lemma 2.13 and Lemma 2.14 in [9]. □

We need the following results.
Lemma 3.9 *Let $\omega \in A_p$, $1 < p < \infty$ and a sequence $\{u_n\}$, $u_n \in W_0^{1,p}(\Omega,\omega)$ satisfies*
(i) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega)$ and ω-a.e. in Ω.
(ii) $\displaystyle\int_\Omega \langle \mathcal{A}(x,u_n,\nabla u_n) - \mathcal{A}(x,u_n,\nabla u), \nabla(u_n-u)\rangle\, \omega\, dx \to 0$ with $n \to \infty$.
Then $u_n \to u$ in $W_0^{1,p}(\Omega,\omega)$.
Proof. The proof of this lemma follows the lines of Lemma 5 in [4]. □
Theorem 3.10 *Let ω and v be weights, and assume*
(I) *$x \mapsto \mathcal{A}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$*
$(\eta,\xi) \mapsto \mathcal{A}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x \in \Omega$.
(II) *$[\mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\eta',\xi')].(\xi-\xi') \geq 0$, whenever $\xi,\xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, where*

$$\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi),...,\mathcal{A}_n(x,\eta,\xi)).$$

(III) *$\mathcal{A}(x,\eta,\xi).\xi \geq \lambda|\xi|^p + \Lambda|\eta|^p - g_1(x)|\eta| - g_2(x)$,*
with $1 < p < \infty$, $g_1 \in L^{p'}(\Omega,v)$ and $g_2 \in L^1(\Omega,v)$, where λ and Λ are positive constants.
(IV) *$|\mathcal{A}(x,\eta,\xi)| \leq K(x) + h_1(x)|\eta|^{1/p'} + h_2(x)|\xi|^{p/p'}$, where K, h_1 and h_2 are positive functions, with h_1 and $h_2 \in L^\infty(\Omega)$, and $K \in L^{p'}(\Omega,v)$, and v is a weight function.*
If $\omega \leq v$, $\omega \in A_p$, $v \in A_p$ with $1 < p < \infty$, $v/\omega \in L^{p'}(\Omega,\omega)$, and $f_0/\omega \in L^{p'}(\Omega,\omega)$, $f_j/v \in L^{p'}(\Omega,v)$ $(j = 1,...,n)$ then problem

$$(P1)\begin{cases} -\sum_{j=1}^{n} D_j\big[v(x)\mathcal{A}_j(x,u(x),\nabla u(x))\big] = f_0(x) - \sum_{j=1}^{n} D_j f_j(x), \quad \text{on } \Omega \\ u(x) = 0, \quad \text{on } \partial\Omega \end{cases}$$

has a solution $u \in W_0^{1,p}(\Omega,\omega,v)$*, where* $W_0^{1,p}(\Omega,\omega,v)$ *as the closure of* $C_0^\infty(\Omega)$ *with respect to the norm*

$$\|u\|_{W_0^{1,p}(\Omega,\omega,v)} = \left(\int_\Omega |u(x)|^p\omega(x)dx + \sum_{j=1}^{n}\int_\Omega |D_j u(x)|^p v(x)dx\right)^{1/p}.$$

Proof. See Theorem 1.1 in [8]. □

Our main result is the following theorem:

Theorem 3.11 *Let* $\omega \in A_p$*,* $1 < p < \infty$*, and* $\mathcal{A}(x,s,\xi)$ *satisfies the conditions (H1),(H2), (H3) and (H4). Then, there exists an entropy solutions* u *of problem (P).*

Moreover, $u \in \mathcal{M}^{p_1}(\Omega,\omega)$ *and* $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega)$*, with* $p_1 = \eta\,(p-1)$ *and* $p_2 = p_1\,p/(p_1+1)$ *(where* η *is the constant in the weighted Sobolev inequality).*

Proof. See Theorem 3.2 in [9]. □

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