Bol. Soc. Paran. Mat.
(3s.) **v. 26** 1-2 (2008): 133–137.

Well and ill-posed problems for the KdV and Kawahara equations

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ABSTRACT: Well and ill-posedness of initial-boundary value problems for the KdV and Kawahara equations posed on a finite interval are discussed. Non-existence of solutions to ill-posed problem for the KdV equation is proved as well as solvability, uniqueness, exponential decay and asymptotics of regular solutions to the Kawahara equation subject to reasonable boundary conditions.

Key Words: KdV and Kawahara equations, solvability, stability.

Contents

1. Introduction

Current discussion mainly concerns the existence and uniqueness of global-in-time regular solutions to the scaled Kawahara equation

$$u_t + uu_x - u_{xxxxx} + u_{xxx} = 0,$$

posed on a bounded interval. Our study is motivated by physics and numerics: the general nonlinear relation known as the Kawahara equation is a fifth-order dispersive partial differential equation describing one-dimensional propagation of small-amplitude long waves in various problems of fluid dynamics and plasma physics, [5]. The Kawahara equation is also known as the fifth-order KdV or a special version of the Benney-Lin equation, [1]. This model was originally developed for unbounded regions of wave propagation when the coefficient of the third derivative in the generalised KdV equation is close to zero. If, however, one is interested in implementing a numerical scheme to calculate solutions to the KdV and/or Kawahara equations in infinite regions, the issue of cutting off the spatial domain arises. In this situation some boundary conditions are needed to specify the solution, [2]. Therefore, precise mathematical analysis of boundary value problems in bounded domains for the KdV and Kawahara equations is to be welcomed.

2000 *Mathematics Subject Classification:* 35Q35, 35M20, 35Q72, 76N10.

2. Ill-posedness example

Initial-boundary value problems in bounded domains for odd-order dispersive equations are also interesting from the purely mathematical point of view: a type of the boundary conditions, needed to ensure the well-posedness of a problem, depends on a sign of the higher derivative coefficient, [6]. The KdV equation is probably one of the most studied dispersive models in this context. However, the results treating the KdV equation posed on bounded intervals usually deal with "well-chosen" problems when one condition at the left boundary and two conditions at the right-hand endpoint are imposed. This is explained commonly by physical arguments. On the other hand, it is not clear what happens if the boundary data are "wrongly-chosen". Above situation is typical for odd-order differential equations. To clarify these questions, we show below an example which classifies the type of ill-posedness for a "wrongly-posed" problem, and brings a reason for results provided in the next section.

Consider the following IBVP for the nonlinear dimensionless KdV equation:

$$u_t + uu_x + u_{xxx} = 0, \quad x \in (0,1) \subset \mathbb{R}, \quad t > 0;$$

$$u(0,t) = u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0;$$
$$u(x,0) = u_0(x), \quad x \in (0,1).$$

Suppose this problem has solution $u = \overline{u}(x,t)$, sufficiently regular. Then u satisfies the linear equation

$$u_t + u_{xxx} = f(x,t) \equiv -\overline{u}(x,t)\overline{u_x}(x,t)$$

with the same conditions as above. Taking the Laplace transform, say $\mathcal{L}$, in $t \geq 0$ with a real parameter $p > 0$, we get the following BVP:

$$pv + v''' = F(x), \quad x \in (0,1),$$

$$v(0) = v'(0) = 0, \quad v(1) = 0,$$

where $v(x) = \mathcal{L}u$ and F depends on u_0 and $\mathcal{L}f$.

This problem is ill-posed by non-existence of solution. In fact, a general solution for homogeneous equation $pv + v''' = 0$ is

$$v^0(x) = C_1 \mathrm{e}^{-ax} + C_2 \mathrm{e}^{ax/2} \cos bx + C_3 \mathrm{e}^{ax/2} \sin bx$$

with $a = \sqrt[3]{p}$ and $b = a\sqrt{3}/2$. Hence,

$$v(x) = v^0(x) + \frac{1}{W_0} \int_0^x F(s) \sum_{k=1}^{3} W_k(s)\, ds$$

where $W_0 \neq 0$ and $W_k(x)$ are the Wronskian and related matrixes depending on $a > 0$. Boundary conditions and simple computations give

$$A \cdot (C_1, C_2, C_3) = (0, F_2, F_3) \not\equiv (0,0,0)$$

with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -a & a/2 & b \\ \mathrm{e}^{-a} & \mathrm{e}^{a/2}\cos b & \mathrm{e}^{a/2}\sin b \end{bmatrix}.$$

The hypothesis of an existence of $v(x)$ then implies

$$\det A = \frac{a\sqrt{3}}{2}\mathrm{e}^{a/2}\left[2\cos\left(\frac{a\sqrt{3}}{2}+\frac{\pi}{3}\right) - \mathrm{e}^{-3a/2}\right] \neq 0.$$

By the other hand, $\det A$ vanishes at a countable set of values of $a > 0$. Contradiction. ■

3. Main results

For real $T > 0$ denote $Q_T = \{(x,t) \in \mathbb{R}^2 : \ x \in (0,1) \subset \mathbb{R}, \ t \in (0,T)\}$. In Q_T there considered is a nonlinear equation

$$u_t + uDu - D^5u + D^3u = 0 \tag{1}$$

subject to initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in (0,1); \tag{2}$$

$$D^iu(0,t) = D^iu(1,t) = D^2u(1,t) = 0, \quad i = 0,1; \quad t \in (0,T). \tag{3}$$

Here and henceforth $u : (0,1)\times(0,T) \to \mathbb{R}$ is the unknown function, u_t denotes its partial derivative with respect to $t > 0$, $D^j = \partial^j/\partial x^j$ are the derivatives with respect to x of order $j \in \mathbb{N}$, $D^0u := u$ and $u_0(x) \in H^5(0,1)$ is the given function satisfying

$$D^iu_0(0) = D^iu_0(1) = D^2u_0(1) = 0, \quad i = 0,1. \tag{4}$$

We adopt the usual notations $\|\cdot\|$ and $(\cdot,\cdot)$ for the norm and inner product in $L^2(0,1)$.

The main results are the following theorems.

Theorem 1 *Let $u_0 \in H^5(0,1)$ satisfy (4). Then for all finite $T > 0$ problem (1)-(3) has a unique regular solution $u(x,t)$:*

$$u \in L^\infty(0,T;H^5(0,1)) \cap L^2(0,T;H^7(0,1)),$$
$$u_t \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1)).$$

Theorem 2 *Let*

$$11 - \frac{2}{3}\|u_0\| = \kappa > 0.$$

Then for all $t > 0$ the regular solution given by Theorem 1 satisfies the following inequality

$$\|u\|^2(t) \leq 4\|u_0\|^2 e^{-\kappa t}. \tag{5}$$

To formulate the next theorem, for real $\mu > 0$ and for a fixed $m \in \mathbb{N}$ we consider in Q_T the following problems:

$$u_t^\mu + u^\mu D u^\mu + D^3 u^\mu - \mu D^5 u^\mu = 0, \quad (x,t) \in Q_T; \tag{6}$$
$$D^i u^\mu(0,t) = D^i u^\mu(1,t) = D^2 u^\mu(1,t) = 0, \quad i = 0,1; \tag{7}$$
$$u^\mu(x,0) = u_0^m(x), \quad x \in (0,1) \tag{8}$$

and

$$u_t + uDu + D^3 u = 0, \quad (x,t) \in Q_T; \tag{9}$$
$$u(0,t) = u(1,t) = Du(1,t) = 0, \quad t \in (0,T); \tag{10}$$
$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{11}$$

Theorem 3 *Let $u_0^m \in H^5(0,1)$ and $u_0 \in H^3(0,1)$ satisfy the consistency conditions related to (7) and (10) correspondingly. Suppose*

$$\|u_0^m - u_0\|_{H^3(0,1)} \to 0 \quad as \quad m \to \infty.$$

Then for all finite $T > 0$ there exists a unique solution $u(x,t)$ to (9)-(11) such that

$$u \in L^\infty(0,T; H^3(0,1)) \cap L^2(0,T; H^4(0,1)),$$
$$u_t \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1)).$$

Moreover, if $\mu \to 0$, and $m \to \infty$, then solutions $u^{\mu,m}(x,t)$ of (6)-(8) given by Theorem 1 satisfy

$$u^{\mu,m} \to u \ \text{*weak in } L^\infty(0,T; L^2(0,1)) \text{ and weakly in } L^2(0,T; H^1(0,1)),$$
$$u_t^{\mu,m} \to u_t \ \text{*weak in } L^\infty(0,T; L^2(0,1)) \text{ and weakly in } L^2(0,T; H^1(0,1)).$$

Proof. First, we treat the stationary case: an explicit representation for a solution to a linear stationary problem is used in order to solve a nonlinear stationary equation by the method of continuation with respect to a parameter. Then, we construct regular solutions to a linear evolution problem exploiting the method of semi-discretization with respect to t. In the sequel, the existence and uniqueness of a local regular solution to the nonlinear evolution problem are proved by using the contraction mapping arguments. Necessary a priori estimates are obtained in the following to extend the local solution to the whole time interval $(0,T)$ with arbitrary $T > 0$. Finally, we prove Theorems 2 and 3 dealing with L^2-stability as $t \to \infty$ and asymptotics of obtained solutions while the coefficient of the higher derivative approaches zero. For the detailed proof, see [3] and [4]. ■

Acknowledgment

The first author was partially supported by a grant from Fundação Araucária.

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