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On the asymptotic behavior for a nonlocal diffusion equation with an absorption term on a lattice

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ABSTRACT: In this paper, we consider the following initial value problem

$$\begin{cases} U_i'(t) = \sum_{j\in B} J_{i-j}(U_j(t) - U_i(t)) - U_i^p(t), \quad t \geq 0, \quad i \in B, \\ U_i(0) = \varphi_i > 0, \quad i \in B, \end{cases}$$

where B is a bounded subset of $\mathbb{Z}^d$, $p > 1$, $J_h = (J_i)_{i\in B}$ is a kernel which is nonnegative, symmetric, bounded and $\sum_{j\in\mathbb{Z}^d} J_j = 1$. We describe the asymptotic behavior of the solution of the above problem.

Key Words: Nonlocal diffusion, asymptotic behavior.

Contents

1. Introduction

Consider the following initial value problem

$$U_i'(t) = \sum_{j\in B} J_{i-j}(U_j(t) - U_i(t)) - U_i^p(t), \quad t \geq 0, \quad i \in B, \tag{1}$$

$$U_i(0) = \varphi_i > 0, \quad i \in B, \tag{2}$$

where B is a bounded subset of $\mathbb{Z}^d$, $p > 1$, $J_h = (J_i)_{i\in B}$ is a kernel which is nonnegative, symmetric, bounded and $\sum_{j\in\mathbb{Z}^d} J_j = 1$. Recently, nonlocal diffusion problems have been the subject of investigations of many authors (see [1], [2], [4]–[7], [13]–[18], [21], [22], [24], [31], [32] and the references cited therein). Nonlocal evolution equations of the form

$$u_t = \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy$$

and variations of it have been used by many authors to model diffusion processes (see [4]–[6], [13], [21]) and neuronal activity (see [19], [23], [28], [29]). Let us notice that certain nonlocal problems are described by discrete equations (see [5], [13])

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and it is also important to have informations about the asymptotic behavior of continuous solutions if we have a continuous nonlocal problems or discrete solutions if the problems are represented by discrete equations. The solution $u(x,t)$ can be interpreted as the density of a single population at the point x at time t and $J(x-y)$ as the probability distribution of jumping from location y to location x. Then the convolution $(J*u)(x,t)=\int_{\mathbb{R}^N} J(y-x)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t)=-\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites (see [21]). For the discrete case

$$U_i^{'}(t)=(J*U)_i(t)-U_i(t),\quad t\geq 0,\quad i\in\mathbb{Z}^d,$$

where $(J*U)_i=\sum_{j\in\mathbb{Z}^d} J_{i-j}U_j$, the component of the solution $U_h(t)=(U_j(t))_{j\in B}$ in i, namely $U_i(t)$ can be interpreted as the density of a single population at the point i at time t and J_{i-j} can be interpreted as the probability distribution of jumping from location i to location j. Then the convolution $(J*U)_i(t)$ is the rate at which individuals are arriving to position i from all other places and $-U_i(t)$ is the rate at which they are leaving location i to travel to all other sites (see [21]). In this paper, we are interested in the asymptotic behavior of the solution of (1)–(2). For local diffusion problems, the asymptotic behavior of solutions has been the subject of investigations of several authors (see [3], [9]–[12], [26], [27] and the references cited therein). For nonlocal problems, in the continuous case, the authors in [14] and [31] have studied the asymptotic behavior of solutions. For our problem, in the case where there is no absorption term, the authors in [25] have studied the asymptotic behavior of the solution when $B=\mathbb{Z}^d$. Our paper is written in the following manner. In the next section, we prove the local existence and the uniqueness of the solution. Finally in the last section, we show that the solution U_h of (1)–(2) tends to zero as t approaches infinity and describe its asymptotic behavior as $t\to+\infty$.

2. Local existence and uniqueness

In this section, we shall establish the existence and the uniqueness of the solution $U_h(t)$ of (1)–(2) on $(0,T)$ for small T.
Let $t_0>0$ be fixed and define the function space $Y_{t_0}=\{U_h;U_h\in C([0,t_0],\mathbb{Z}^d)\}$ equipped with the norm defined by $\|U_h\|_{Y_{t_0}}=\max_{0\leq t\leq t_0}\|U_h\|_\infty$ for $U_h\in Y_{t_0}$ where $\|U_h(t)\|_\infty=\sup_{i\in B}|U_i(t)|$. It is easy to see that Y_{t_0} is a Banach space. Introduce the set $X_{t_0}=\{U_h;U_h\in Y_{t_0},\|u\|_{Y_{t_0}}\leq b_0\}$, where $b_0=2\|\varphi_h\|_\infty+1$. We observe that X_{t_0} is a nonempty bounded closed convex subset of Y_{t_0}. Define the map R as follows

$$R:X_{t_0}\longrightarrow X_{t_0},$$

$$R(V_h)_i=\varphi_i+\int_0^t(\sum_{j\in B}J_{i-j}(V_j(s)-V_i(s)))ds-\int_0^t V_i^p(s)ds,\quad i\in B.$$

Theorem 2.1. *Assume that $\varphi_h \in Y_{t_0}$. Then R maps X_{t_0} into X_{t_0} and R is strictly contractive if t_0 is appropriately small relative to $\|\varphi_h\|_\infty$.*

Proof. A straightforward computation reveals that

$$|R(V_h)_i(t) - \varphi_i| \leq 2\|V_h\|_{Y_{t_0}} t + \|V_h\|^p_{Y_{t_0}} t,$$

which implies that $\|R(V_h)\|_{Y_{t_0}} \leq \|\varphi_h\|_\infty + 2t_0 + b_0^p t_0$. If

$$t_0 \leq \frac{b_0 - \|\varphi_h\|_\infty}{2b_0 + b_0^p}, \tag{3}$$

then

$$\|R(V_h)\|_{Y_{t_0}} \leq b_0.$$

Therefore if (3) holds, then R maps X_{t_0} into X_{t_0}.
Now, we are going to prove that the map R is strictly contractive. Let $V_h, Z_h \in X_{t_0}$. Setting $\alpha_h = V_h - Z_h$, we discover that

$$|(R(V_h)_i(t) - R(Z_h)_i(t)| \leq |\int_0^t (\sum_{j\in B} J_{i-j}(\alpha_j(s) - \alpha_i(s)))ds| + |\int_0^t (V_i^p(s) - Z_i^p(s))ds|.$$

Use Taylor's expansion to obtain

$$|(R(V_h)_i(t) - R(Z_h)_i(t)| \leq 2\|\alpha_h\|_{Y_{t_0}} t + t\|V_h - Z_h\|_{Y_{t_0}} p\|\beta_h\|^{p-1}_{Y_{t_0}},$$

where β_i is an intermediate value between V_i and Z_i. We deduce that

$$\|R(V_h) - R(Z_h)\|_{Y_{t_0}} \leq 2\|\alpha_h\|_{Y_{t_0}} t_0 + t_0\|V_h - Z_h\|_{Y_{t_0}} p\|\beta_h\|^{p-1}_{Y_{t_0}},$$

which implies that

$$\|R(V_h) - R(Z_h)\|_{Y_{t_0}} \leq (2t_0 + t_0 p b_0^{p-1})\|V_h - Z_h\|_{Y_{t_0}}.$$

If $t_0 \leq \frac{1}{4+2pb_0^{p-1}}$, then $\|R(V_h) - R(Z_h)\|_{Y_{t_0}} \leq \frac{1}{2}\|V_h - Z_h\|_{Y_{t_0}}$.
Hence, we see that $R(V_h)$ is a strict contraction in Y_{t_0} and the proof is complete. □

It follows from the contraction mapping principle that for appropriately chosen $t_0 \in (0,1)$, R has a unique fixed point $U_h(t) \in Y_{t_0}$ which is a solution of (1)–(2). To extend the solution to $[0,\infty)$, we may take as initial data $U_h(t_0) \in \mathbb{Z}^d$ and obtain a solution in $[0, 2t_0]$. Iterating this procedure, we get a solution defined in $[0,\infty)$.

3. Asymptotic behavior of solutions

In this section, we show that the solution $U_h(t)$ of (1)-(2) tends to zero as t approaches infinity. We also give its asymptotic behavior as $t \to \infty$.
Before starting, let us prove the following lemma which is a version of the maximum principle for discrete nonlocal problems.

Lemma 3.1. *Let $b_h \in C^0([0,\infty),\mathbb{Z}^d)$ and let $U_h \in C^1([0,\infty),\mathbb{Z}^d)$ satisfying the following inequalities*

$$U_i^{'}(t) - \sum_{j\in B} J_{i-j}(U_j(t) - U_i(t)) + b_i(t)U_i(t) \geq 0, \quad i \in B, \qquad t > 0,$$

$$U_i(0) \geq 0, \quad i \in B.$$

Then, we have $U_i(t) \geq 0, \quad i \in B,\ t > 0.$

Proof. Let $T_0 < \infty$ and let λ be such that $b_i(t) - \lambda > 0$ for $t \in [0, T_0]$, $i \in B$. Introduce the vector $Z_h(t) = e^{\lambda t}U_h(t)$ and let $m = \min_{t\in[0,T_0]} \|Z_h(t)\|_{\inf}$ where $\|Z_h(t)\|_{\inf} = \min_{0\leq i\leq I} Z_i(t)$. Then, there exists $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$ for a certain $i_0 \in B$. We get $Z_{i_0}(t_0) \leq Z_{i_0}(t)$ for $t \leq t_0$ and $Z_{i_0}(t_0) \leq Z_j(t_0)$ for $j \in B$, which implies that

$$Z_{i_0}^{'}(t_0) = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \tag{4}$$

and

$$\sum_{j\in B} J_{i_0-j}(Z_j(t_0) - Z_{i_0}(t_0)) \geq 0. \tag{5}$$

Using the first inequality of the lemma, it is not hard to see that

$$Z_{i_0}^{'}(t_0) - \sum_{j\in B} J_{i_0-j}(Z_j(t_0) - Z_{i_0}(t_0)) + (b_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \tag{6}$$

It follows from (4)–(6) that $(b_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$ because $b_{i_0}(t_0) - \lambda > 0$. We deduce that $U_h(t) \geq 0$ for $t \in [0, T_0]$, which leads us to the result. □
Another version of the maximum principle for discrete nonlocal problems is the following comparison lemma.

Lemma 3.2. *Let $U_h, V_h \in C^1([0,\infty),\mathbb{Z}^d)$ such that*

$$U_i^{'}(t) - \sum_{j\in B} J_{i-j}(U_j(t) - U_i(t)) + U_i^p(t)$$
$$> V_i^{'}(t) - \sum_{j\in B} J_{i-j}(V_j(t) - V_i(t)) + V_i^p(t), \quad i \in B, \quad t > 0,$$

$$U_i(0) > V_i(0), \quad i \in B.$$

Then, we have $U_i(t) > V_i(t), \quad i \in B,\ t > 0.$

Proof. Let $W_h(t) = U_h(t) - V_h(t)$ and let t_0 be the first $t \in (0,\infty)$ such that $W_h(t) > 0$ for $t \in [0,t_0)$ but $W_{i_0}(t_0) = 0$ for a certain $i_0 \in B$. We have

$$\sum_{j\in B} J_{i_0-j}(W_j(t_0) - W_{i_0}(t_0)) = \sum_{j\in B} J_{i_0-j}W_j(t_0) > 0$$

because $W_j(t_0) \geq 0$ for $j \in B$ and $W_h(t_0) \neq 0$. Obviously

$$W_{i_0}^{'}(t_0) = \lim_{k\to 0} \frac{W_{i_0}(t_0) - W_{i_0}(t_0-k)}{k} \leq 0.$$

It follows that

$$W_{i_0}^{'}(t_0) - \sum_{j\in B} J_{i_0-j}(W_j(t_0) - W_{i_0}(t_0)) + U_{i_0}^p(t_0) - V_{i_0}^p(t_0) \leq 0.$$

But, this contradicts the first strict differential inequality of the lemma and the proof is complete. □

Remark 3.1. *If we modify slightly the proof of Lemma 3.2, it is not hard to see that $U_h(t) > 0$ for $t \geq 0$ where $U_h(t)$ is the solution of (1)–(2).*

Introduce the function

$$\mu(x) = (C_0 + x)^p - \lambda(C_0 + x),$$

where $C_0 = (\frac{1}{p-1})^{\frac{1}{p-1}}$ and $\lambda = \frac{1}{p-1}$, which is crucial for the asymptotic behavior of solutions. We have $\mu(0) = 0$ and $\mu'(0) = 1$. We deduce that $\mu(\varepsilon) > 0$ and $\mu(-\varepsilon) < 0$ for ε small enough.
The lemma below shows that the solution U_h of the problem (1)-(2) tends to zero as t approaches infinity.

Lemma 3.3. *Let $U_h(t)$ be the solution of (1)–(2). Then, we have*

$$\lim_{t\to\infty} U_i(t) = 0, \quad i \in B.$$

Proof. Since $\varphi_i \geq 0$, $i \in B$, Lemma 3.1 implies that $U_i(t) \geq 0$, $i \in B$, $t > 0$. Introduce the vector $Z_h(t)$ defined as follows $Z_i(t) = C_0 t^{-\lambda}$, $i \in B$, $t \geq 1$, where $\lambda = \frac{1}{p-1}$ and $C_0 = (\frac{1}{p-1})^{\frac{1}{p-1}}$. A straightforward computation reveals that

$$Z_i^{'}(t) - \sum_{j\in B} J_{i-j}(Z_j(t) - Z_i(t)) + Z_i^p(t) = 0, \quad t \geq 1,$$

$$Z_i(1) = C_0, \quad i \in B.$$

Let $k > 1$ be so large that

$$kZ_i(1) = kC_0 > U_i(1), \quad i \in B.$$

Obviously $kZ_i^p < (kZ_i)^p$ which implies that

$$kZ_i^{'} - \sum_{j\in B} J_{i-j}(kZ_j(t) - kZ_i(t)) + (kZ_i)^p > 0, \quad i \in B, \quad t \geq 1,$$

$$kZ_i(1) = kC_0 > U_i(1), \quad i \in B.$$

Lemma 3.2 implies that

$$0 \leq U_i(t) < kZ_i(t), \quad i \in B, \quad t \geq 1.$$

Hence, we have

$$0 \leq U_i(t) \leq kC_0 t^{-\lambda}, \quad i \in B, \quad t \geq 1.$$

We deduce that

$$\lim_{t\to+\infty} U_i(t) = 0, \quad i \in B,$$

and the proof is complete. □
Now, let us give the asymptotic behavior of the solution U_h. We have the following result.

Theorem 3.1. *Let $U_h(t)$ be the solution of (1)-(2). Then, we have*

$$U_i(t) = C_0 t^{-\lambda}(1 + o(1)), \quad i \in B \quad as \quad t \to \infty,$$

where $C_0 = (\frac{1}{p-1})^{\frac{1}{p-1}}$ *and* $\lambda = \frac{1}{p-1}$.

The proof of the above theorem is based on the following lemmas.

Lemma 3.4. *Let $U_h(t)$ be the solution of (1)-(2). Then, for any $\varepsilon > 0$ small enough, there exist two times $\tau \geq T \geq 1$ such that*

$$U_i(t+\tau) \leq (C_0+\varepsilon)(t+T)^{-\lambda} + (t+T)^{-\lambda-1}, \quad i \in B, \quad t \geq 0.$$

Proof. Introduce the vector W_h defined as follows

$$W_i(t) = (C_0+\varepsilon)t^{-\lambda} + t^{-\lambda-1}, \quad i \in B, \quad t \geq 1.$$

A direct calculation yields

$$\begin{aligned} W_i^{'}(t) \quad &- \quad \sum_{j\in B} J_{i-j}(W_j(t) - W_i(t)) + W_i^p(t) \\ &= \quad t^{-\lambda-1}\left(-\lambda(C_0+\varepsilon) - (\lambda+1)t^{-1}\right) + t^{-\lambda p}(C_0+\varepsilon+t^{-1})^p. \end{aligned}$$

Due to the fact that $p\lambda = \lambda + 1$, we arrive at

$$\begin{aligned} W_i^{'}(t) \quad &- \quad \sum_{j\in B} J_{i-j}(W_j(t) - W_i(t)) + W_i^p(t) \\ &= \quad t^{-\lambda-1}\left(-\lambda(C_0+\varepsilon) - (\lambda+1)t^{-1} + (C_0+\varepsilon+t^{-1})^p\right). \end{aligned}$$

Applying Taylor's expansion, we get $(C_0+\varepsilon+t^{-1})^p=(C_0+\varepsilon)^p+M(t)t^{-1}$, where $M(t)$ is a bounded function for $t\geq 1$. Hence, for $t\geq 1$, we find that

$$\begin{aligned} W_i^{'}(t) &- \sum_{j\in B} J_{i-j}(W_j(t)-W_i(t))+W_i^p(t) \\ &= t^{-\lambda-1}(\mu(\varepsilon)-(\lambda+1)t^{-1}+M(t)t^{-1}). \end{aligned}$$

Since $\varepsilon>0$, we discover that $\mu(\varepsilon)>0$. Therefore, there exists a positive time $T\geq 1$ such that

$$W_i^{'}(t)-\sum_{j\in B} J_{i-j}(W_j(t)-W_i(t))+W_i^p(t)>0,\quad i\in B,\quad t\geq T.$$

Since $U_i(t)$ goes to zero as t approaches infinity for $i\in B$, owing to Lemma 3.3, there exists $\tau\geq T$ such that

$$U_i(\tau)<W_i(T),\quad i\in B.$$

Setting $Z_i(t)=U_i(t+\tau-T)$, we easily see that

$$Z_i^{'}(t)-\sum_{j\in B} J_{i-j}(Z_j(t)-Z_i(t))+Z_i^p(t)=0,\quad i\in B,\quad t\geq T,$$

$$Z_i(T)=U_i(\tau)<W_i(T),\quad i\in B.$$

Comparison Lemma 3.2 implies that

$$Z_i(t)\leq W_i(t),\quad i\in B,\quad t\geq T.$$

Hence

$$U_i(t+\tau-T)\leq (C_0+\varepsilon)t^{-\lambda}+t^{-\lambda-1},\quad i\in B,\quad t\geq T.$$

We deduce that

$$U_i(t+\tau)\leq (C_0+\varepsilon)(t+T)^{-\lambda}+(t+T)^{-\lambda-1},\quad i\in B,\quad t\geq 0,$$

and the proof is complete. □

Lemma 3.5. *Let $U_h(t)$ be the solution of (1)-(2). Then, for any $\varepsilon>0$ small enough, there exists a time $\tau\geq 1$ such that*

$$U_i(t+1)\geq (C_0+\varepsilon)(t+\tau)^{-\lambda}+(t+\tau)^{-\lambda-1},\quad i\in B,\quad t\geq 0.$$

Proof. From Remark 3.1, we know that $U_h(t)>0$ for $t\geq 0$. Introduce the vector $W_h(t)$ such that

$$W_i(t+1)=(C_0-\varepsilon)t^{-\lambda}+t^{-\lambda-1},\quad i\in B,\quad t\geq 1.$$

As in the proof of Lemma 3.4, we find that

$$W_i'(t) - \sum_{j\in B} J_{i-j}(W_j(t) - W_i(t)) + W_i^p(t)$$
$$= t^{-\lambda-1}(\mu(-\varepsilon) - (\lambda+1)t^{-1} + M(t)t^{-1})$$

where $M(t)$ is a bounded function for $t \geq 1$. Since $-\varepsilon < 0$, we discover that $\mu(-\varepsilon) < 0$. Consequently, there exists a positive time $T \geq 1$ such that

$$W_i'(t) - \sum_{j\in B} J_{i-j}(W_j(t) - W_i(t)) + W_i^p(t) < 0, \quad i \in B, \quad t \geq T.$$

Since $\lim_{t\to\infty} W_i(t) = 0$, $i \in B$, there exists a time $\tau \geq T$ such that

$$W_i(\tau) < U_i(1), \quad i \in B.$$

Setting $Z_i(t) = W_i(t+\tau-1)$, $i \in B$, it is not difficult to see that

$$Z_i'(t) - \sum_{j\in B} J_{i-j}(Z_j(t) - Z_i(t)) + Z_i^p(t) < 0, \quad i \in B, \quad t \geq 1,$$

$$Z_i(1) = W_i(\tau) < U_i(1), \quad i \in B.$$

It follows from Comparison Lemma 3.2 that

$$Z_i(t) < U_i(t), \quad i \in B, \quad t \geq 1,$$

which implies that

$$U_i(t) \geq (C_0 - \varepsilon)(t+\tau-1)^{-\lambda} + (t+\tau-1)^{-\lambda-1}, \quad i \in B, \quad t \geq 1.$$

We deduce that

$$U_i(t+1) \geq (C_0 - \varepsilon)(t+\tau)^{-\lambda} + (t+\tau)^{-\lambda-1}, \quad i \in B, \quad t \geq 0$$

and the proof is complete. □
Now, we are in a position to prove the main result of this section.
Proof of Theorem 3.1. It follows from Lemmas 3.4 and 3.5 that

$$C_0 - \varepsilon \leq \lim_{t\to\infty} \inf(\frac{U_i(t)}{t^{-\lambda}}) \leq \lim_{t\to\infty} \sup(\frac{U_i(t)}{t^{-\lambda}}) \leq C_0 + \varepsilon,$$

which gives the desired result. □

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