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Quasilinear evolution equations of the third order *

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ABSTRACT: The present paper is a survey concerned with certain aspects of solvability and well-posedness of initial and initial-boundary value problems for various quasilinear evolution equations of the third order. This class includes, for example, Korteweg – de Vries (KdV) and Zakharov – Kuznetsov (ZK) equations.

Key Words: quasilinear dispersive equations, Korteweg de Vries equation, boundary value problems, well-posedness.

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1. Korteweg – de Vries equation. Initial value problem

The KdV equation

$$u_t + u_{xxx} + uu_x = 0 \tag{1}$$

is a model equation describing one-dimensional nonlinear wave propagation in dispersive media, [60]. In fact, it became famous in the 60-es of the XX-th century due to discovery of the so-called method of inverse scattering transform just for this equation, [41].

The most typical physical situation, where KdV is used, is wave propagation on a surface of shallow water in a narrow long channel. When this channel can be considered as "infinite", an initial value problem with initial data

$$u(0, x) = u_0(x) \tag{2}$$

for $x \in \mathbb{R}$ appears in a natural way.

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The inverse scattering transform method, in particular, can be used to establish existence results for the problem (1), (2) and it was realized, for example, in [69], [71], [64], [59], [16], [15], [66], [46], [17], [47]. However, this method is beyond the scope of the present paper.

In a parallel way this problem was investigated by methods more traditional for the theory of partial differential equations. First results on global solvability and well-posedness, that is in the strip $\Pi_T = (0,T)\times\mathbb{R}$ for an arbitrary $T>0$, were based on conservation laws for (1)

$$\int_{\mathbb{R}} u^2\,dx = \text{const}, \quad \int_{\mathbb{R}} (u_x^2 - \frac{1}{3}u^3)\,dx = \text{const}, \quad \int_{\mathbb{R}} (u_{xx}^2 + \frac{5}{6}u^2 u_{xx} + \frac{5}{36}u^4)\,dx = \text{const} \tag{3}$$

(in fact, the amount of conservation laws is infinite), [45], [73], [70], [75], [78], [1], [67], [2], [48] etc. In particular, for $u_0 \in H^s(\mathbb{R})$, $s=1$ or $s\geq 2$, existence of global solutions in $L_\infty(0,T;H^s(\mathbb{R}))$ was proved. Uniqueness was obtained via the following simple scheme. If $u(t,x)$ and $\widetilde{u}(t,x)$ are two solutions of the problem (1), (2) with the same initial function u_0, then their difference $w\equiv u-\widetilde{u}$ satisfies the equation

$$w_t + w_{xxx} + w\widetilde{u}_x + uw_x = 0$$

and after multiplying this equality by $2w(t,x)$ and integrating with respect to x it is very easy to derive the inequality

$$\frac{d}{dt}\int_{\mathbb{R}} w^2\,dx \leq \sup_x |2\widetilde{u}_x - u_x| \int_{\mathbb{R}} w^2\,dx, \tag{4}$$

whence u and $\widetilde{u}$ coincide provided $u_x, \widetilde{u}_x \in L_1(0,T;L_\infty(\mathbb{R}))$. Thus the space $L_\infty(0,T;H^s(\mathbb{R}))$ for $s\geq 2$ was proved to be the class of global well-posedness.

For $u_0\in L_2(\mathbb{R})$ such an approach is not effective, because the only first conservation law (3) can be used here, and it is not sufficient to construct a desired solution via certain passing to a limit, because only weak compactness in L_2 is provided.

The further progress in the study of the problem (1), (2) was obtained due to discovery of an effect of local smoothing of solutions, [49], [61]. Multiplying (1) by $2u(t,x)\rho(x)$ for certain smooth, nonnegative and nondecreasing function ρ one can easily derive after integration, that

$$\frac{d}{dt}\int_{\mathbb{R}} u^2\rho\,dx + 3\int_{\mathbb{R}} u_x^2\rho'\,dx - \int_{\mathbb{R}} u^2\rho'''\,dx - \frac{2}{3}\int_{\mathbb{R}} u^3\rho'\,dx = 0, \tag{5}$$

and by the appropriate choice of ρ establish the estimate

$$\lambda(u;T) = \sup_{m\in\mathbb{R}} \int_0^T\int_m^{m+1} u_x^2\,dxdt \leq c(T,\|u_0\|_{L_2(\mathbb{R})}). \tag{6}$$

This estimate made it possible in [49], [61] to prove global existence for $u_0\in L_2(\mathbb{R})$ in the class of functions

$$\{u\in L_\infty(0,T;L_2(\mathbb{R})), \quad \lambda(u;T)<\infty\}.$$

In these papers results on global well-posedness (based on (5)) were also established in weighted L_2 spaces with weight functions increasing as $x \to +\infty$, namely, as $\exp x$ in [49] and x^β for positive β in [61]. Moreover, it was shown, that in the case of sufficiently rapid growth of these weight functions solutions became unique in these classes. In particular, it was proved in [61], that the class of functions

$$\{u \in L_\infty(0,T;L_2(\mathbb{R})), \quad x^\beta u \in L_\infty(0,T;L_2(\mathbb{R}_+)), \beta \geq 3/4\}$$

was a class of global well-posedness for the problem (1), (2). The approach to establish uniqueness in such a class differed essentially from (4) and was based on invertion of the linear part of the equation (1), that is a solution of (1), (2) was considered as a solution of the linear initial value problem for the equation

$$u_t + u_{xxx} = f(t,x) \tag{7}$$

for $f \equiv -uu_x$ with the same initial data (2). The results of [49], [61] were improved and generalized later in [25], [42], [43], [76].

In the papers [50], [51], [53] due to more careful study of the linear operator $\partial_t + \partial_x^3$ classes of global well-posedness for the problem (1), (2) were constructed when $u_0 \in H^s(\mathbb{R})$), $s \geq 1$. In particular, a sharp version of the local smoothing effect was obtained. Let $S(t,x;u_0)$ be a solution of the problem (7), (2) for $f \equiv 0$, given by the formula

$$S(t,x;u_0) = \mathcal{F}_x^{-1}\left[e^{it\xi^3}\widehat{u}_0(\xi)\right](x). \tag{8}$$

Obviously, for all $t \in \mathbb{R}$

$$\|S(t,\cdot;u_0)\|_{H^s(\mathbb{R})} = \|u_0\|_{H^s(\mathbb{R})}, \quad s \in \mathbb{R}. \tag{9}$$

Moreover, changing variables $\lambda = \xi^3$ one can derive, that

$$S(t,x;u_0) = \frac{1}{3}\mathcal{F}_t^{-1}\left[e^{i\lambda^{1/3}x}\lambda^{-2/3}\widehat{u}_0(\lambda^{1/3})\right](t), \tag{10}$$

and then with the use of the Parseval equality, that for all $x \in \mathbb{R}$

$$\|S_x(\cdot,x;u_0)\|_{L_2(\mathbb{R})} = c\|\lambda^{-1/3}\widehat{u}_0(\lambda^{1/3})\|_{L_2(\mathbb{R})} = c_1\|u_0\|_{L_2(\mathbb{R})}. \tag{11}$$

On the base of both these properties, Strichartz type estimates (obtained in [42]) and maximal function type estimates for the function S, the following class of global well-posedness for (1), (2) was introduced in [51] (for simplicity we consider the case $s = 1$):

$$\begin{aligned} Z_1(\Pi_T) = \{u \in C([0,T];H^1(\mathbb{R})), &\quad u_{xx} \in C_b(\mathbb{R};L_2(0,T)), \\ u_x \in L_4(0,T;L_\infty(\mathbb{R})), &\quad u \in L_2(\mathbb{R};L_\infty(0,T))\}. \end{aligned}$$

The proof of local existence was based on the corresponding estimate on a solution of the linear problem (7), (2)

$$\|u\|_{Z_1(\Pi_T)} \leq c\left(\|u_0\|_{H^1(\mathbb{R})} + \|f\|_{L_1(0,T;H^1(\mathbb{R}))}\right) \tag{12}$$

and on the estimate of the nonlinear term

$$\|uu_x\|_{L_2(0,T;H^1(\mathbb{R}))} \le c\|u\|^2_{Z_1(\Pi_T)}. \tag{13}$$

Note, that in [53] the contraction principle based on (12), (13) was used to construct a solution of (1), (2) in $Z_1(\Pi_T)$ for small T as a fixed point of the map $u = \Lambda v$, where u is a solution of the linear problem (7), (2) for $f \equiv -vv_x$. Therefore, uniqueness of such a solution was automatic. Global well-posedness succeeded from the first two conservation laws (3) (and the third one was not used even for $s > 1$).

In the paper [9] special functional spaces, associated with the linear operator $\partial_t + \partial_x^3$ were introduced, namely, X_s for $s \ge 0$ was a space of functions such, that

$$(1+|\xi|)^s \left[(1+|\lambda-\xi^3|)^{1/2} + \chi(\xi)(1+|\lambda|)^\alpha\right] \widehat{u}(\lambda,\xi) \in L_2(\mathbb{R}) \tag{14}$$

for some $\alpha > 1/2$, where χ was the characteristic function of the interval $(-1,1)$, Y_s was a space of functions such, that

$$(1+|\xi|)^s \left[(1+|\lambda-\xi^3|)^{-1/2} + \chi(\xi)(1+|\lambda|)^{\alpha-1}\right] \widehat{u}(\lambda,\xi) \in L_2(\mathbb{R}), \tag{15}$$

$$(1+|\xi|)^s (1+|\lambda-\xi^3|)^{-1} \widehat{u}(\lambda,\xi) \in L_2(\mathbb{R}^\xi; L_1(\mathbb{R}^\lambda)). \tag{16}$$

It was proved, that for a solution of the linear problem (7), (2) and for an arbitrary function θ from $C_0^\infty(\mathbb{R})$

$$\|\theta(t)u\|_{X_s} \le c(\theta)\left(\|u_0\|_{H^s(\mathbb{R})} + \|f\|_{Y_s}\right) \tag{17}$$

and for the nonlinear term

$$\|uu_x\|_{Y_s} \le c\|u\|^2_{X_s} \tag{18}$$

and so first local well-posedness was established via the contraction principle and then with the use of the first conservation law (3) – the global one in $X_s(\Pi_T)$ for $u_0 \in H^s(\mathbb{R})$, $s \ge 0$ (here and further $X_s(\Omega)$ for a domain $\Omega \subset \mathbb{R}^2$ is interpreted in the restriction sense).

Later this result was generalized in [54], [55], [19] for spaces of negative indices. Finally, in [20] global well–posedness in the Bourgain-type spaces was established for $u_0 \in H^s(\mathbb{R})$, $s > -3/4$, but this theory is beyond the scope of the present survey.

Also beyond the scope of the survey are periodic solutions and KdV-type equations with more general nonlinearity $g(u)u_x$ (see, for example, [75], [67], [1], [49], [25], [43], [53], [20], [10]).

2. Korteweg – de Vries equation. Initial-boundary value problems

When a domain of wave propagation, for example, a narrow channel, is considered as bounded (from only one or both sides) the initial value problem for KdV must be substituted by initial-boundary value ones. The most simple and typical domains for such problems are the right half-strip $\Pi_T^+ = (0,T) \times \mathbb{R}_+$, the left half-strip $\Pi_T^- = (0,T) \times \mathbb{R}_-$ and the bounded rectangle $Q_T = (0,T) \times (0,1)$. Besides

the initial profile (2) boundary conditions must be set and they are different for different domains. We consider here the following boundary data:
1) for the problem in Π_T^+ one condition on the left boundary

$$u(t,0) = u_1(t), \tag{19}$$

2) for the problem in Π_T^- two conditions on the right boundary

$$u(t,0) = u_2(t), \quad u_x(t,0) = u_3(t), \tag{20}$$

3) for the problem in Q_T one condition on the left boundary and two conditions on the right boundary

$$u(t,0) = u_1(t), \quad u(t,1) = u_2(t), \quad u_x(t,1) = u_3(t). \tag{21}$$

Such a difference between conditions on the left and the right boundaries originates from different properties of the linear operator $\partial_t + \partial_x^3$ at $+\infty$ and at $-\infty$.

The study of initial-boundary value problems for the KdV equation was performed in a parallel way (with certain delay) to the study of the initial value problem and used the same ideas. However, the presence of boundaries and boundary conditions produced new items and difficulties.

Consider, for example, an estimate in L_2. Let I be either $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$ or $(0,1)$ and let ∂I denotes the finite part of its boundary. Let $u(t,x)$ be a solution of the equation (1) in $(0,T)\times I$, sufficiently smooth and decaying at infinity. Multiplying (1) by $2u$ and integrating over I one obtains the equality

$$\frac{d}{dt}\int_I u^2\,dx + \bigl(2uu_{xx} - u_x^2 + \frac{2}{3}u^3\bigr)\big|_{\partial I} = 0. \tag{22}$$

For $I=\mathbb{R}$ (22) coincides with the first conservation law (3). For the considered initial-boundary value problems in the case $u|_{\partial I}=0$ the estimate on the solution u in $L_2(I)$ uniform with respect to t succeeds from (22). Just in the case of the homogeneous boundary conditions first global existence results were established in [74], [56], [57] (see also [62]). But in the case of non-homogeneous boundary conditions the presence of the term $uu_{xx}|_{\partial I}$ makes it impossible to derive such an estimate directly from (22). Then it is quite natural to introduce an auxiliary function $\varphi(t,x)$ such, that $\varphi|_{\partial I} = u|_{\partial I}$, and define a new function $U(t,x)\equiv u(t,x)-\varphi(t,x)$. The function U satisfies the equation

$$U_t + U_{xxx} + UU_x + (\varphi U)_x = F \equiv -(\varphi_t + \varphi_{xxx} + \varphi\varphi_x), \tag{23}$$

so multiplying (23) by $2U$ and integrating over I we find, that

$$\frac{d}{dt}\int_I U^2\,dx - U_x^2\big|_{\partial I} + \int_I \varphi_x U^2\,dx = 2\int_I FU\,dx. \tag{24}$$

This approach implies, that the function φ, which is an extension of the values of the solution itself at the boundary into the corresponding domain, can be chosen such, that its properties ensure a possibility of derivation of a relevant estimate on the solution in $L_2(I)$ from (24). In particular, it seems natural, that φ must satisfy the following condition

$$\varphi_x \in L_1(0,T;L_\infty(I)). \tag{25}$$

Thus, two combined questions naturally arise: 1) what properties of the boundary data can provide such properties of the function φ, 2) what is the optimal method to construct such an extension?

In fact, in any theory of well-posedness the problem of optimality of conditions on initial and boundary data is very important. In order to try to answer somehow such a question for the considered problems let us consider the function S, defined by the formula (8). Similarly to (11) it can be easily shown, that for all $x \in \mathbb{R}$ and $s \in \mathbb{R}$

$$\|D_t^{1/3} S(\cdot,x;u_0)\|_{H^{s/3}(\mathbb{R})} = \|S_x(\cdot,x;u_0)\|_{H^{s/3}(\mathbb{R})} \sim \|u_0\|_{H^s(\mathbb{R})}. \tag{26}$$

Therefore, one can assume, that conditions of the type $u_0 \in H^s$, $u_1, u_2 \in H^{(s+1)/3}$, $u_3 \in H^{s/3}$ are natural here. Of course, the problem is to establish existence and well-posedness results under such conditions.

Another problem is to obtain a priori estimates on solutions of the considered problems in more smooth spaces, for example, analogous to the last two conservation laws (3). The difficulties on this way can be shown even for the linear equation (7) in the case $f \equiv 0$ and for the zero boundary data (19)–(21). Multiplying (7) by $-2u_{xx}(t,x)$ and integrating over I we derive the equality

$$\frac{d}{dt}\int_I u_x^2\,dx - u_{xx}^2\big|_{\partial I} = 0 \tag{27}$$

and so the desired estimate on u_x in $L_2(I)$ can be obtained only for the problem in Π_T^+. Next, multiplying (7) by $2u_{xxxx}(t,x)$ and integrating over I we derive the equality

$$\frac{d}{dt}\int_I u_{xx}^2\,dx - 2u_{tx}u_{xx}\big|_{\partial I} = 0 \tag{28}$$

and here the estimate on u_{xx} in $L_2(I)$ can be obtained only for the problem in Π_T^-.

The problems of global solvability and well-posedness for the considered initial-boundary value problems under non-homogeneous boundary data were considered in [3], [26], [4], [29], [30], [31], [32], [33], [6], [21], [7], [36], [37], [44], [8], [39]. These papers consequently developed the theory and improved results of the preceding ones.

The Bourgain-type spaces for initial-boundary value problems for KdV were first introduced in [21] in the following way. Instead of (14) the spaces X_s were defined by the property

$$(1+|\lambda|^{1/3}+|\xi|)^s\left[(1+|\lambda-\xi^3|)^b+\chi(\xi)(1+|\lambda|)^\alpha\right]\widehat{u}(\lambda,\xi) \in L_2(\mathbb{R}) \tag{29}$$

for some $b \in (0,1/2)$, $\alpha > 1/2$, while instead of (15), (16) the spaces Y_s were defined by

$$(1+|\lambda|^{1/3}+|\xi|)^s\left[(1+|\lambda-\xi^3|)^{-b}+\chi(\xi)(1+|\lambda|^{\alpha-1}\right]\widehat{u}(\lambda,\xi) \in L_2(\mathbb{R}) \tag{30}$$

(in fact, in [21] the Sobolev spatial weight $(1+|\xi|)^s$ was used as for the initial value problem, but in [37] and [39] it was proposed to use the joint spatial and temporal weight $(1+|\lambda|^{1/3}+|\xi|)^s$ for more convenience). It was shown in [21], that the inequality (18) was valid for such modified spaces if $b \geq 7/16$.

Now we want to describe the modern state of the theory. In [37], [39] the following results were established.

The problem (1), (2), (19) is globally well-posed in $X_s(\Pi_T^+)$ (where $7/16 \leq b < 1/2$) for $s \geq 0$, $s \neq 3m+1/2$, $m = 0, 1, \ldots$, if $u_0 \in H^s(\mathbb{R}_+)$, $u_1 \in H^{(s+1)/3+\varepsilon}(0,T)$, where $\varepsilon > 0$ is arbitrary small for $s = 0$ and $\varepsilon = 0$ for $s > 0$, and certain compatibility conditions are satisfied in the point $(0,0)$.

The problem (1), (2), (20) is globally well-posed in $X_s(\Pi_T^-)$ (where $7/16 \leq b < 1/2$) for $s \geq 0$, $s \neq 3m+1/2$, $s \neq 3m+3/2$, $m = 0, 1, \ldots$, if $u_0 \in H^s(\mathbb{R}_-)$, $u_2 \in H^{(s+1)/3+\varepsilon}(0,T)$, $u_3 \in H^{s/3}(0,T)$, where $\varepsilon > 0$ is arbitrary small for $s = 0$ and $\varepsilon = 0$ for $s > 0$, and certain compatibility conditions are satisfied in the point $(0,0)$.

The same result is valid for the problem in Q_T (1), (2), (21) (where also $u_1 \in H^{(s+1)/3+\varepsilon}(0,T)$).

So global well-posedness is established under natural assumptions on the boundary data for $s > 0$ and ε-close to natural for $s = 0$. Note, that in [44] local well–posedness was proved for all three problems under natural assumptions for $-3/4 < s \leq 0$. Note also, that certain results under these natural assumptions for the first time appeared in [6], [21].

As for the initial value problem the proof of global well-posedness in [37], [39] is based on local well-posedness, which is established via the contraction principle, and global a priory estimates. Such an approach requires, first of all, the study of the corresponding initial-boundary value problems for the linear equation (7). Consider, for example, the problem in the right half-strip Π_T^+. Then a solution is constructed in the form

$$u(t,x) = w(t,x) + J(t,x;u_1 - w|_{x=0}), \tag{31}$$

where w is a solution of the initial value problem (7), (2) and $J(t,x;\mu)$ is a solution of the problem (7), (2), (19) for $f \equiv 0$, $u_0 \equiv 0$, $u_1 \equiv \mu$. Such a function can be referred as a "boundary potential" for the homogeneous equation (7).

For the first time this potential was introduced in [14], where via the Laplace transform it was shown, that for $x > 0$

$$J(t,x;\mu) = \int_0^t \frac{3}{t-\tau} A''\left(\frac{x}{(t-\tau)^{1/3}}\right) \mu(\tau)\, d\tau, \tag{32}$$

where

$$A(\vartheta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(\xi^3+\vartheta\xi)}\, d\xi \equiv \mathcal{F}^{-1}\left[e^{i\xi^3}\right](\vartheta) \tag{33}$$

was the well-known Airy function. The alternative representation for the function J is the following:

$$J(t,x;\mu) = \mathcal{F}_t^{-1}\left[e^{r(\lambda)x}\widehat{\mu}(\lambda)\right](t), \tag{34}$$

where

$$r(\lambda) = -\frac{1}{2}(\sqrt{3}|\lambda|^{1/3} + i\lambda^{1/3})$$

is the unique root of the algebraic equation

$$r^3 + i\lambda = 0, \quad \lambda \in \mathbb{R} \setminus \{0\}, \tag{35}$$

with the negative real part and a function μ is extended by zero for $t < 0$, [36].

It was shown in [37], that for $b < 1/2$ and an arbitrary function θ from $C_0^\infty(\mathbb{R})$

$$\|\theta(t)J(\cdot,\cdot;\mu)\|_{X_s(\mathbb{R}^t\times\mathbb{R}^x_+)} \le c(\theta)\|\mu\|_{H^{(s+1)/3}} \tag{36}$$

and, therefore, similarly to (17) on the base of the representation (31)

$$\|u\|_{X_s(\Pi_T^+)} \le c\left(\|u_0\|_{H^s(\mathbb{R}_+)} + \|f\|_{Y_s(\Pi_T^+)} + \|u_1\|_{H^{(s+1)/3}(0,T)}\right). \tag{37}$$

This estimate combined with (18) provided an opportunity to establish local well-posedness of the problem (1), (2), (19) under natural assumptions on the boundary data by a standard argument.

Note, that the estimate of the (11), (26) type easily follows for the boundary potential J from the formula (34), namely, for $x \ge 0$, $s \ge 0$

$$\|\partial_x^j J(\cdot,x;\mu)\|_{H^s(\mathbb{R})} \le c\|\mu\|_{H^{j/3+s}(\mathbb{R})}. \tag{38}$$

Moreover, the following analogue of (9) for the positive half-line is valid for the function J: for $t \in \mathbb{R}$, $s \ge 0$

$$\|J(t,\cdot;\mu)\|_{H^s(\mathbb{R}_+)} \le c\|\mu\|_{H^{(s+1)/3}(\mathbb{R})}. \tag{39}$$

The estimate (39) easily follows from the following fundamental inequality, established in [6]: if a certain continuous function $\gamma(\xi)$ satisfies an inequality $\Re\gamma(\xi) \le -\varepsilon|\xi|$ for some $\varepsilon > 0$ and all $\xi \in \mathbb{R}$, then

$$\Big\|\int_{\mathbb{R}} e^{\gamma(\xi)x} f(\xi)\,d\xi\Big\|_{L_2(\mathbb{R}^x_+)} \le c(\varepsilon)\|f\|_{L_2(\mathbb{R})}, \tag{40}$$

by the simple change of variables $\lambda = \xi^3$ in (34).

Similar boundary potentials Q and R for the homogeneous linear equation (7) were constructed with the use of the rest roots of the equation (35) in [39] for the problem in Π_T^- ($Q(\cdot,0-0;\mu) = R_x(\cdot,0-0;\mu) = \mu$, $Q_x(\cdot,0-0;\mu) = R(\cdot,0-0;\mu) = 0$), and local well-posedness for the problem (1), (2), (20) was proved under natural assumptions on the boundary data.

Moreover, the potentials J, Q, R were used in [39] to construct a suitable solution of the linear problem in Q_T (7), (2), (21) (the idea of such an approach appeared first in [44]) and similar result on local well-posedness of the problem (1), (2), (21) was established under natural assumptions on the boundary data.

The boundary potential J was also used in [37], [39] to construct the auxiliary function φ and use it in (23), (24) to obtain a global estimate in L_2. More precisely,

$$\varphi(t,x) \equiv \begin{cases} J(t,x;u_1) & \text{for } \Pi_T^+, \\ J(-t,-x;\widetilde{u}_2) & \text{for } \Pi_T^-, \\ J(t,x;u_1)\eta(1-x) + J(-t,1-x;\widetilde{u}_2)\eta(x) & \text{for } Q_T, \end{cases} \tag{41}$$

where $\widetilde{u}_2(t) \equiv u_2(-t)$ and η is a certain "cut-off" function, $\eta(0) = 0$, $\eta(1) = 1$.

The properties (38), (39) for $s = 0$ were used there and besides them one more, which had no analogue for the potential Q, namely,

$$\|J(\cdot,\cdot;\mu)\|_{L_2(0,T;H^s(\mathbb{R}_+))} \le c(T)\|\mu\|_{H^{(2s-1)/6}(\mathbb{R})}, \tag{42}$$

(this estimate can be derived from (34) via the Parseval equality). The well-known embedding theorem yields from (42), that

$$\|J_x(\cdot,\cdot;\mu)\|_{L_2(0,T;L_\infty(\mathbb{R}_+))} \le c(T,\varepsilon)\|\mu\|_{H^{1/3+\varepsilon}(\mathbb{R})} \tag{43}$$

and so the condition (25) is satisfied for $u_1, u_2 \in H^{1/3+\varepsilon}(0,T)$. Just the necessity of realization of (25) caused ε-worsening of the assumptions on the boundary data (in comparison with the natural ones) in the case $s = 0$. The idea to use the property (42) to provide (25) for the problem in Π_T^+ goes back to [8].

Global a prori estimates (and consequently, global well-posedness) in more smooth spaces were obtained in [37], [39] with regard to (27), (28), that is first in H^1 for the problem in Π_T^+, in H^2 for the problem in Π_T^- (these estimates were analogues to the last two conservation laws (3)) and then via differentiation with respect to t in H^{3k}, H^{3k+1} for the first problem, in H^{3k}, H^{3k+2} for the second problem and in H^{3k} for the problem in the bounded rectangle, k – natural. Global well-posedness for intermediate values of s was established via nonlinear interpolation theory from [72] (for the first time for KdV-like problems this ides was used in [2]).

Note, that the rate of growth of the nonlinearity in KdV, which does not exceed quadratic, is essential for a global estimate in L_2 in the case of non-homogeneous boundary data (see (24)). Certain global results for the considered problems in the case of the greater rate of growth can be found in [58], [5], [34], [21], [11], but they are beyond the scope of this survey.

Intial-boundary value problems for the KdV equation with different from (19)–(21) boundary conditions or in other domains (for example, with moving boundaries) are considered in [12], [13], [18], [80], [24].

3. Zakharov – Kuznetsov equation

The ZK equation

$$u_t + u_{xxx} + u_{xyy} + uu_x = 0 \tag{44}$$

is one on the variants of a multidimensional generalization of the KdV equation. It describes nonlinear wave processes in dispersive media, when waves propagate in the x-direction and can be deformated in the transverse y-direction. In particular, it is a model equation for ion-acoustic waves in magnetized plasma, [79].

The study of initial and initial-boundary value problems for ZK in comparison with KdV, besides traditional difficulties originating from the transfer from the line to the plane, has some additional obstacles. First of all, in contrast to (3) only two conservation laws are known for (44):

$$\iint_{\mathbb{R}^2} u^2\,dxdy = \text{const}, \quad \iint_{\mathbb{R}^2} (u_x^2 + u_y^2 - \frac{1}{3}u^3)\,dxdy = \text{const}. \tag{45}$$

Next, Bourgain-type spaces for the ZK equation, for which certain analogues of (17) and (18) are valid, are not found yet.

On the other hand, the local smoothing effect is also valid for this equation as for KdV. In particular, similarly to (6) it can be shown, that

$$\lambda_2(u;T) = \sup_{m\in\mathbb{R}} \int_0^T \int_m^{m+1} \int_{\mathbb{R}} (u_x^2 + u_y^2)\,dydxdt \le c(T, \|u_0\|_{L_2(\mathbb{R}^2)}), \tag{46}$$

if a smooth and decaying at infinity solution $u(t,x,y)$ of the initial value problem in $\Pi_T^2 = (0,T)\times\mathbb{R}^2$ with the initial profile

$$u(0,x,y) = u_0(x,y) \tag{47}$$

is considered.

By virtue of the first conservation law (45) and the estimate (46) global existence (without uniqueness) of weak solutions of the problem (44), (47) for $u_0 \in L_2(\mathbb{R}^2)$ can be established by the methods similar to [49], [61], [25] for KdV. Such results were proved in [27] for more general multidimentional generalizations of the KdV equation and in more details are described in the next section of this survey.

Global well-posedness of the problem (44), (47) was established in [28] in more smooth classes by the method similar to the one from [51], [53]. First the linear equation

$$u_t + u_{xxx} + u_{xyy} = f(t,x,y) \tag{48}$$

was considered in [28] and estimates similar to (9), (11) and other ones from [51], [53] were obtained. In particular, for a solution of the problem (48), (47) when $f \equiv 0$

$$S_2(t,x,y;u_0) = \mathcal{F}_{x,y}^{-1}\left[e^{it(\xi^3+\xi\eta^2)}\widehat{u}_0(\xi,\eta)\right](x,y) \tag{49}$$

for all $t\in\mathbb{R}$

$$\|S_2(t,\cdot,\cdot;u_0)\|_{H^s(\mathbb{R}^2)} = \|u_0\|_{H^s(\mathbb{R}^2)} \tag{50}$$

and for all $x\in\mathbb{R}$

$$\|S_{2x}(\cdot,x,\cdot;u_0)\|_{L_2(\mathbb{R}^2)}^2 + \|S_{2y}(\cdot,x,\cdot;u_0)\|_{L_2(\mathbb{R}^2)}^2 \sim \|u_0\|_{L_2(\mathbb{R}^2)}^2. \tag{51}$$

A functional space (for simplicity we consider here only the case $s=1$)

$$\begin{aligned} Z_1(\Pi_T^2) = \{&u\in C([0,T];H^1(\mathbb{R}^2)), \quad u_{xx}, u_{xy}, u_{yy} \in C_b(\mathbb{R}^x; L_2((0,T)\times\mathbb{R})),\\ &u_x, u_y \in L_2(0,T;L_\infty(\mathbb{R}^2)), \quad u\in L_2(\mathbb{R}^x; L_\infty((0,T)\times\mathbb{R}))\}\end{aligned}$$

was introduced and estimates similar to (12), (13) were established. On the base of these estimates first local well-posedness and then with the help of the conservation laws (45) global well-posedness of the problem (44), (47) in $Z_1(\Pi^2_T)$ for $u_0 \in H^1(\mathbb{R}^2)$ was proved . Similar results were obtained in [28] for $u_0 \in H^k(\mathbb{R}^2)$, $k \geq 2$ – natural.

Initial-boundary value problems for the ZK equation up to today have been considered only in domains, which were generalizations of the domains Π^+_T, Π^-_T, Q_T for the case $y \in \mathbb{R}$, namely, $\Pi^{2+}_T = \Pi^+_T \times \mathbb{R}^y$, $\Pi^{2-}_T = \Pi^-_T \times \mathbb{R}^y$, $Q^2_T = Q_T \times \mathbb{R}^y$ (with the only exception in [65], which is described in the next section). Boundary conditions were similar to (19)–(21):
for the problem in Π^{2+}_T

$$u(t,0,y) = u_1(t,y), \tag{52}$$

for the problem in Π^{2-}_T

$$u(t,0,y) = u_2(t,y), \quad u_x(t,0,y) = u_3(t,y), \tag{53}$$

for the problem in Q^2_T

$$u(t,0,y) = u_1(t,y), \quad u(t,1,y) = u_2(t,y), \quad u_x(t,1,y) = u_3(t,y). \tag{54}$$

Of course, difficulties related to nonzero boundary data for L_2 estimates similar to (22)–(25) also appear for these problems. Moreover, similarly to (26) it can be shown, that conditions of the type $u_0 \in H^s$, $u_1, u_2 \in H^{(s+1)/3,s+1}_{t,y}$, $u_3 \in H^{s/3,s}_{t,y}$ are natural here (see also (51)).

As for KdV boundary potentials for the homogeneous linearized ZK equation turned out to be very important for the theory of initial-boundary value problems.

In [38] such a boundary potential for the problem in Π^{2+}_T was constructed in a form

$$J_2(t,x,y;\mu) \equiv \mathcal{F}^{-1}_{t,y}\Big[e^{r(\lambda,\eta)x}\widehat{\mu}(\lambda,\eta)\Big](t,y), \quad x>0, \tag{55}$$

where $r(\lambda,\eta)$ is the unique root of the algebraic equation

$$r^3 - r\eta^2 + i\lambda = 0, \quad (\lambda,\eta) \in \mathbb{R}^2 \setminus \{(0,0)\}, \tag{56}$$

with the negative real part. The estimates, similar to (38), (39) were established for the function J_2 and, moreover, it was proved, that

$$\|J_{2x}(\cdot,\cdot,\cdot;\mu)\|_{L_2(0,T;L_\infty(\mathbb{R}^2_+))} \leq c(T)\|\mu\|_{H^{2/3,2}_{t,y}(\mathbb{R}^2)}. \tag{57}$$

With the use of the boundary potential J_2 a solution of the linear problem (48), (47), (52) in a form similar to (31) was constructed and the following estimate was proved:

$$\|u\|_{Z_1(\Pi^{2+}_T)} \leq c\big(\|u_0\|_{H^1(\mathbb{R}^2_+)} + \|u_1\|_{H^{2/3,2}_{t,y}((0,T)\times\mathbb{R})} + T^{1/6}\|f\|_{L_2(0,T;H^1(\mathbb{R}^2_+))}\big), \tag{58}$$

where $Z_1(\Pi^{2+}_T)$ was a space analogous to $Z_1(\Pi^2_T)$ with the addition

$$u \in C_b(\overline{\mathbb{R}}^x_+; H^{2/3,2}_{t,y}((0,T)\times\mathbb{R})), \quad u_x \in C_b(\overline{\mathbb{R}}^x_+; H^{1/3,1}_{t,y}((0,T)\times\mathbb{R})).$$

Local well-posedness for the problem (44), (47), (52) was proved via the contraction principle on the basis of the estimate (58) and then also global well-posedness in $Z_1(\Pi_T^{2+})$ was established under natural assumptions on initial and boundary data for $s = 1$ with the use the boundary potential J_2 as an auxiliary function to make boundary data zero (the estimate (57) was essential here). This result can be generalized for all natural $s \geq 2$.

For other domains Π_T^{2-} and Q_T^2 local well-posedness under natural assumptions can be obtained by similar arguments with the help of corresponding boundary potentials. But the absence of an analogue for ZK of the third conservation law (3) has not allowed yet to establish global well-posedness for the problem (44), (47), (53). The only known global result here is existence of a weak solution in the class

$$\{u \in L_\infty(0,T;L_2(\mathbb{R}^2_-)), \quad \lambda_2^-(u;T) = \sup_{m\geq 0}\int_0^T\int_{-m-1}^{-m}\int_{\mathbb{R}}(u_x^2+u_y^2)\,dydxdt < \infty\}$$

for $u_0 \in L_2(\mathbb{R}^2_-)$, $u_2 \in H^{s/3,s}_{t,y}((0,T)\times\mathbb{R})$, $s > 3/2$, $u_3 \in L_2((0,T)\times\mathbb{R})$, proved in [40]. This result is based on the first conservation law (45), the local smoothing effect of the (46) type and certain properties of the boundary potential J_2. For the problem (44), (47), (54) global well-posedness can be established for natural $s \geq 3$.

4. General equations

We consider here equations of such a form

$$u_t - Pu + \operatorname{div}_x g(u) = 0, \tag{59}$$

where in this section $x = (x_1, \dots, x_n)$, $g = (g_1, \dots, g_n)$ for $n \geq 2$,

$$P = \sum_{|\alpha|=3} a_\alpha \partial_x^\alpha,$$

$\alpha = (\alpha_1, \dots, \alpha_n)$ – multiindex, $\alpha_j \geq 0$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}$, a_α – real constants.

First consider the initial value problem in the domain $\Pi_T^n = (0,T)\times\mathbb{R}^n$ with the initial condition (2) for $x \in \mathbb{R}^n$. The first conservation law for this problem is obvious:

$$\int_{\mathbb{R}^n} u^2\,dx = \text{const}. \tag{60}$$

The problem is to generate additional assumptions on the equation, which can provide other global estimates on solutions. In [68] it was assumed, that

$$P = \sum_{j=1}^n \frac{\partial}{\partial x_j}L, \quad g_1 = g_2 = \dots = g_n,$$

where L was an elliptic operator. Then multiplying (59) by $2(Lu - g(u))$ and integrating over $\mathbb{R}^n$ one can derive the following analogue of the second conservation law (3)

$$\int_{\mathbb{R}^n}(Lu\cdot u - 2g^*(u))\,dx = \text{const}, \tag{61}$$

where g^* is the primitive to g such, that $g^*(0) = 0$. So if g satisfies the following restriction on its growth:

$$|g'(u)| \le c(|u|^b + 1), \quad 0 \le b < 4/n, \tag{62}$$

then an estimate in $H^1(\mathbb{R}^n)$ for a solution is provided and as a result existence of a global solution in $L_\infty(0,T;H^1(\mathbb{R}^n))$ for $u_0 \in H^1(\mathbb{R}^n)$ can be proved, [68].

In [27] another condition on the operator P was proposed: there existed a vector $e = (e_1, \dots, e_n)$ such, that

$$Q_e(\xi) = (\operatorname{grad}_\xi P(\xi), e) < 0 \quad \forall \xi \ne 0, \tag{63}$$

where

$$P(\xi) = \sum_{|\alpha|=3} a_\alpha \xi^\alpha, \quad \xi = (\xi_1, \dots, \xi_n), \quad \xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n},$$

was the symbol of the operator P. Of course, the operator Q_e with the symbol $Q_e(\xi)$ is elliptic. If $\rho(\vartheta)$ is a smooth, positive and increasing function such, that $0 < \rho'(\vartheta) \le c\rho(\vartheta)$, $|\rho^{(j)}(\vartheta)| \le c(j)\rho'(\vartheta)$, $j \ge 2$, for all $\vartheta \in \mathbb{R}$, $\rho_e(x) \equiv \rho(x_1 e_1 + \dots + x_n e_n)$, then

$$-\int_{\mathbb{R}^n} Pu \cdot u\rho_e \, dx \ge c_1 \int_{\mathbb{R}^n} |\operatorname{grad}_x u|^2 \rho_e' \, dx - c_2 \int_{\mathbb{R}^n} u^2 \rho_e \, dx \tag{64}$$

for certain positive constants c_1 and c_2. The inequality (64) provides a local smoothing effect of the (6) type and thus in [27] a result on existence of a global solution in

$$L_\infty(0,T;L_2(\mathbb{R}^n)) \cap L_2(0,T;H^{1,loc}(\mathbb{R}^n))$$

of the problem (59), (2) was established if $u_0 \in L_2(\mathbb{R}^n)$ and the functions g_j satisfied the inequality (62). In particular, this result is valid for the ZK equation. Some results on uniqueness of the constructed solutions in weighted spaces under more restrictive assumptions on the functions g_j were also obtained in [27], but they excluded ZK. Other conditions, that provide local smoothing effect for dispersive equations can be found in [22], [52], [63].

Note, that in the case $n = 2$ the condition (63) is necessary and sufficient to reduce the operator P to the form $-\partial_{x_1}^3 - \partial_{x_1}\partial_{x_2}^2$ by certain linear change of variables, [23].

Note also, that in [68] and [27] more general, than (59) equations were considered (in particular, of an arbitrary high odd order).

In [35] one initial-boundary value problem for the equation (59) was considered in the domain $\Pi_T^{n+} = (0,T) \times \mathbb{R}_+^n$, where $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_1 > 0\}$, with a boundary condition

$$u(t,0,x') = u_1(t,x'), \quad x' = (x_2, \dots, x_n). \tag{65}$$

The operator P was assumed to satisfy the condition (63) for $e = (1,0,\dots,0)$. For $u_0 \in L_2(\mathbb{R}_+^n)$, restrictions on the functions g_j of the (62) type, where, in addition, $b \le 1$, and certain assumptions on the function u_1 global existence of weak solutions of the problem (59), (2), (65) was proved similarly to the results from [27]. Uniqueness of the constructed solutions in certain weighted spaces was also established under more restrictive assumptions on nonlinearity, but unlike the initial value problem this result included the ZK equation. A boundary potential for the corresponding linear equation in a form similar to (32) was constructed, studied and used in [38].

Certain initial-boundary value problem in a bounded domain for an equation of the (59) type in the case of two spatial variables was studied in [65]. By some linear change of variables this equation can be reduced to the ZK equation.

Well-posedness of various initial-boundary value problems for linear evolution equations of an arbitrary high odd order was studied in [77]. Assumptions on the considered equations there were of the (63) type.

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