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# Mathematical problems of electromagnetoelastic interactions

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ABSTRACT: There are studied nonlinear mathematical problems of the interaction of electromagnetic fields with deformable media. The models are based on combination of the Lam and Maxwell systems coupled through so-called seismomagnetic effect. Several direct and associated with them inverse problems are studied. Then speaking about the inverse problems, electromagnetic and elastic characteristics of a medium are the subject of reconstruction. The values of physical fields are connected through electromagnetoelastic interactions. We consider the processes which are observed when elastic waves propagate in an elastic electroconducting medium. Variations of the seismic and electromagnetic fields in this case are called electromagnetoelastic waves. There are described different statements of mathematical model of the electromagnetoelastic interactions. Then the theoretical results of the analytical solution are discussed for various nonlinear direct and inverse problems for the equations of electromagnetoelasticity.

Key Words: Nonlinear boundary value problems; odd-order dispersive differential equations; existence and uniqueness.

## Contents



## Introduction

The interaction of electromagnetic fields with deformable media is a subject of many theoretical and experimental investigations in the field of continuum mechanics and geophysics in the recent decades. For description of simple enough

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interactions, the theories of magnetohydrodynamics, [1], electroelasticity, [2], [3], and magnetoelasticity, [4], [5], [6], were developed. These theories are, basically, a combination (without introducing new conceptions) of objects and phenomena considered in continuum mechanics and electrodynamics.

Investigation of more complex electromagnetoelastic interactions in a continuous medium requires to consider complex models. For a more profound acquaintance with the modern state of the theory of electromagnetoelastic interactions the reader is referred to, e.g., [7], [8].

The aim of this paper is to study some nonlinear direct and inverse problems connected with electromagnetoelastic interactions. The model considered here is based on a simple variant of combination of the Lamé and Maxwell equations.

Let us give a brief characteristic of basic types of electromagnetoelastic interactions. It is well known that when an electrical-conducting elastic body oscillates in an electromagnetic field, variations of the electrical and magnetic fields are observed as a result of this motion. Similar processes are also observed when seismic waves propagate in the Earth's crust. Variations of seismic and electromagnetic fields arising in this case are called *electromagnetoelastic waves.* Such waves contain a certain information about electromagnetic and elastic parameters of the medium. In this case, as a rule, the following types of electromagnetoelastic interactions are distinguished:

a) Interaction based on the *electrokinetic effect*. It is supposed that generation of electrical signals with elastic waves propagation is connected precisely with manifestation of electrokinetic properties of a medium.
b) Interaction based on the *piezoelectric effect*. This interaction is connected with propagation of elastic waves in crystal rocks when the elastic deformation of a lattice substance produces displacement of electrons and, as consequence, there arises an electrical field induced by such deformations.
c) Interaction based on *slow movement* of a body in an external electromagnetic field. Whereas, for example, the electrokinetic effect is connected with local interactions of elastic waves with a flow in the pore liquid, this effect is based on slow moving of particles in an external electromagnetic field.

In seismics and seismology the third type of interaction leads to so-called *seismomagnetic effect* describing interaction of seismic waves with the Earth's magnetic field. This interaction results in induced electromagnetic waves propagating with speeds commensurable with the speeds of seismic waves.

## 1. Mathematical model of electromagnetoelastic interactions

The interaction of electromagnetic fields with deformable media is considered with point of view of linear elasticity connected with electrodynamic of elastic moving media by means of motion of particles in the electromagnetic field. We do not consider any effects of interactions, which could arise as a result of some kind of relations in constitutive equations besides velocity. We, basically, follow Dunkin and Eringen, [4], when defining a mathematical model for electromagnetoelastic effect.

1.1. ELECTROMAGNETIC THEORY. Let $\mathbb{R}^3$ be a three-dimensional Euclidean space of points $x = (x_1, x_2, x_3)$. The process of propagation of electromagnetic waves in $\mathbb{R}^3$ will be described by the following Maxwell system:

$$\begin{aligned} \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} &= \operatorname{rot} \boldsymbol{H}, \\ \frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{rot} \boldsymbol{E} &= 0, \end{aligned} \tag{1.1.1}$$

$$\operatorname{div} \boldsymbol{D} = \rho_e, \qquad \operatorname{div} \boldsymbol{B} = 0. \tag{1.1.2}$$

Here $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}$ and $\boldsymbol{B}$ are the electromagnetic vectors, $\boldsymbol{J}$ is the current density, $\rho_e$ is the charge density, and all quantities are expressed in the MKS units. When a medium is at rest, the electromagnetic constitutive equations of an isotropic medium are

$$\boldsymbol{D}^0 = \epsilon \boldsymbol{E}^0, \qquad \boldsymbol{B}^0 = \mu \boldsymbol{H}^0, \qquad \boldsymbol{J}^0 = \sigma \boldsymbol{E}^0, \tag{1.1.3}$$

where $\epsilon, \mu$ are called the electric and magnetic permeabilities and $\sigma$ is the electrical conductivity. The same equations are assumed to be valid at each point in the reference frame moving with the velocity of a material point, i.e. the *proper* frame, but they are expressed in terms of the field measured in the *laboratory* frame in which motion is observed. For small velocities the proper quantities are related to the laboratory ones by the equations, see [6]

$$\begin{aligned} \boldsymbol{E}^0 &= \boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}, \qquad \boldsymbol{D}^0 = \boldsymbol{D} + c^{-2} \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}, \\ \boldsymbol{H}^0 &= \boldsymbol{H} - \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{D}, \qquad \boldsymbol{B}^0 = \boldsymbol{B} - c^{-2} \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{E}, \\ \boldsymbol{J}^0 &= \boldsymbol{J} - \rho_e \frac{\partial \boldsymbol{u}}{\partial t}, \qquad \rho_e^0 = \rho_e, \qquad c \equiv (\epsilon_0 \mu_0)^{-1/2}, \end{aligned}$$

where $\epsilon_0, \mu_0$ are the dielectric and magnetic permeabilities of the vacuum and $\boldsymbol{u}$ is the displacement vector. Let us substitute these relations into constitutive equations (1.1.3). If the terms of order $|\frac{\partial \boldsymbol{u}}{\partial t}|^2/c^2$ and higher are dropped, the results are as follows, see [6]

$$\begin{aligned} \boldsymbol{D} &= \epsilon \boldsymbol{E} + \alpha \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}, \qquad \alpha \equiv \epsilon\mu - \epsilon_0 \mu_0, \\ \boldsymbol{B} &= \mu \boldsymbol{H} - \alpha \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{E}, \end{aligned} \tag{1.1.4}$$

$$\boldsymbol{J} = \rho_e \frac{\partial \boldsymbol{u}}{\partial t} + \sigma(\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}). \tag{1.1.5}$$

For more details of electromagnetic theory, the reader is referred to many textbooks that treat this research field, e.g., [7], [9], [10].

Thus, we have obtained a complete system for freely moving media. They are Maxwell's equations (1.1.1)–(1.1.2) and the constitutive relations (1.1.4)–(1.1.5). Equation (1.1.5) is a modification of Ohm's law, where appears a term reflecting the influence of particles moving in the magnetic field with a current density.

The electromagnetic matching conditions are obtained in the following manner. First rewrite equations (1.1.1)–(1.1.2) in the equivalent form

$$\begin{aligned}\operatorname{rot}(\boldsymbol{E}+\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{B}) &= -\frac{\partial \boldsymbol{B}}{\partial t}-\frac{\partial \boldsymbol{u}}{\partial t}\operatorname{div}\boldsymbol{B}+\operatorname{rot}(\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{B}),\\ \operatorname{rot}(\boldsymbol{H}-\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{D}) &= \frac{\partial \boldsymbol{D}}{\partial t}+\frac{\partial \boldsymbol{u}}{\partial t}\operatorname{div}\boldsymbol{D}-\operatorname{rot}(\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{D})+\boldsymbol{J}-\rho_e\frac{\partial \boldsymbol{u}}{\partial t},\end{aligned} \tag{1.1.6}$$

$$\operatorname{div}\boldsymbol{B}=0,\qquad \operatorname{div}\boldsymbol{D}=\rho_e. \tag{1.1.7}$$

Then integral analogues of these equations can be obtained by integrating (1.1.6) over the surface $S'$ composed of material particles and bounded by a curve $C$ and (1.1.7) over a volume $V$ of material particles bounded by the surface $S$, see Fig. 1. Note that $C, S', S$ and $V$ move with the material. After applying Stokes' theorem on the left-hand sides of (1.1.6), we obtain

$$\begin{aligned}\int_C(\boldsymbol{E}+\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{B})\cdot d\boldsymbol{c} &= -\frac{d}{dt}\int_{S'}\boldsymbol{B}\cdot d\boldsymbol{s}',\\ \int_C(\boldsymbol{H}-\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{D})\cdot d\boldsymbol{c} &= \frac{d}{dt}\int_{S'}\boldsymbol{D}\cdot d\boldsymbol{s}'+\int_{S'}(\boldsymbol{J}-\rho_e\frac{\partial \boldsymbol{u}}{\partial t})\cdot d\boldsymbol{s}',\end{aligned} \tag{1.1.8}$$

Applying the Gauss-Ostrogradskii theorem to (1.1.7) yields:

$$\begin{aligned}\int_S \boldsymbol{B}\cdot d\boldsymbol{s} &= 0,\\ \int_S \boldsymbol{D}\cdot d\boldsymbol{s} &= \int_V \rho_e dx,\end{aligned} \tag{1.1.9}$$

where we have also used the well-known relation

$$\frac{d}{dt}\int_{S'}\boldsymbol{F}\cdot d\boldsymbol{s}'=\int_{S'}[\frac{\partial \boldsymbol{F}}{\partial t}+\frac{\partial \boldsymbol{u}}{\partial t}\operatorname{div}\boldsymbol{F}-\operatorname{rot}(\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{F})]d\boldsymbol{s}'.$$

Select $S'$ to be a small rectangular area oriented perpendicular to the discontinuity surface such that one side lies in the part of material with one material properties and other one lies in the part with another material properties. As the dimension of $S'$, perpendicular to the boundary, tends to zero, equations (1.1.8) now look like:

$$[\boldsymbol{E}+\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{B}]_t=0,\qquad [\boldsymbol{H}-\frac{\partial \boldsymbol{u}}{\partial t}\times \boldsymbol{D}]_t=J_m^S-\rho_e^S\frac{\partial u_m}{\partial t}, \tag{1.1.10}$$

where the symbol $[\boldsymbol{F}]_t$ means a jump of the tangential components of the vector $\boldsymbol{F}$ across the surface, where the coefficients of equations have breaks, and $J_m^S, \rho_e^S$ represent the surface current and the charge, respectively. Here and in the sequel the subscripts $t, m, n$ denote the vector components in the directions $\boldsymbol{t}, \boldsymbol{m}, \boldsymbol{n}$ which form the right-hand orthogonal triad Now, let us choose $V$ to be a small cylindrical volume whose axis is perpendicular to the discontinuity surface such that one of the circular ends lies in the part of the material with one material properties and another one lies in the part with another material properties. As the height of the volume tends to zero, equations (1.1.9) take the form:

$$[\boldsymbol{B}]_n = 0, \qquad [\boldsymbol{D}]_n = \rho_e^S, \tag{1.1.11}$$

where $[\boldsymbol{F}]_n$ means a jump in the normal component of $\boldsymbol{F}$.

Equations (1.1.10)–(1.1.11) constitute the complete electromagnetic matching conditions on a discontinuity surface.

1.2. Elastic theory. Consider now the equations of motion of a deformable medium. The mechanical equations will be derived by applying the conservation of momentum to the volume of a material, $V$, with the bounding surface $S$ in the absence of a mechanical force, using an assumption that only a mechanical effect of the electromagnetic fields is the introduction of the Lorentz force

$$\boldsymbol{f}^e = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}. \tag{1.2.1}$$

Thus the equation of global conservation of momentum in the rectangular coordinates is the following

$$\int_S T \cdot \boldsymbol{n} ds + \int_V \boldsymbol{f}^e dx = \frac{d}{dt} \int_V \boldsymbol{g}^m dx, \tag{1.2.2}$$

where $T$ is a stress tensor and $\boldsymbol{g}^m$ is momentum per unit volume. Using the Gauss-Ostrogradskii theorem for the surface integral and differentiation of the volume integral according to [11, Eq. 20.9],

$$\frac{d}{dt} \int_V \boldsymbol{g}^m dv = \int_V (\frac{\partial \boldsymbol{g}^m}{\partial t} + \operatorname{Div}(\boldsymbol{g}^m \otimes \frac{\partial \boldsymbol{u}}{\partial t}))dx \tag{1.2.3}$$

we obtain

$$\int_V (\operatorname{Div} T + \boldsymbol{f}^e - \operatorname{Div}(\boldsymbol{g}^m \otimes \frac{\partial \boldsymbol{u}}{\partial t}) - \frac{\partial \boldsymbol{g}^m}{\partial t})dx = 0,$$

where

$$\operatorname{Div} T = \left( \sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij} \right)_{i=1}^{3}. \tag{1.2.4}$$

If the mechanical momentum is locally conserved, then

$$\operatorname{Div} T + \boldsymbol{f}^e = \operatorname{Div}(\boldsymbol{g}^m \otimes \frac{\partial \boldsymbol{u}}{\partial t}) + \frac{\partial \boldsymbol{g}^m}{\partial t}. \tag{1.2.5}$$

In an elastic solid $\boldsymbol{g}^m = \rho \frac{\partial \boldsymbol{u}}{\partial t}$, where $\rho$ is the material density, and the assumption of infinitesimal strains and rotations (1.2.5) reduces to

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + \boldsymbol{f}^e. \tag{1.2.6}$$

The elastic matching conditions on stress are obtained by applying (1.2.6) to appropriate differential elements. Introduce Maxwell's stresses $T^e$ and the electromagnetic momentum $\boldsymbol{g}^e$ according to Minkowski, see [12]

$$\begin{gathered} T^e = \boldsymbol{E} \otimes \boldsymbol{D} + \boldsymbol{H} \otimes \boldsymbol{B} - \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})I, \\ \boldsymbol{g}^e = \boldsymbol{D} \times \boldsymbol{B}, \end{gathered} \tag{1.2.7}$$

where $I$ is the unit matrix of order $3 \times 3$. Let us show that the Lorentz force $\boldsymbol{f}^e$ can be represented in the following form

$$\boldsymbol{f}^e = \operatorname{Div} T^e - \frac{\partial \boldsymbol{g}^e}{\partial t}. \tag{1.2.8}$$

It is easy to check the correctness of formula (1.2.8) taking into account equations (1.1.1)–(1.1.2) and constitutive relations

$$\boldsymbol{D} = \epsilon \boldsymbol{E}, \qquad \boldsymbol{B} = \mu \boldsymbol{H}.$$

After simple transformations we come to the equation

$$\operatorname{Div} T = \rho_e \boldsymbol{E} + \operatorname{rot} \boldsymbol{E} \times \boldsymbol{D} + \operatorname{rot} \boldsymbol{H} \times \boldsymbol{B}.$$

Using Maxwell's equations (1.1.1)–(1.1.2) we obtain

$$\operatorname{Div} T = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} + \frac{\partial \boldsymbol{D}}{\partial t} \times \boldsymbol{B} - \frac{\partial \boldsymbol{B}}{\partial t} \times \boldsymbol{D}$$

from which next formula is followed

$$\operatorname{Div} T = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} + \frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) = \boldsymbol{f}^e + \frac{\partial \boldsymbol{g}^e}{\partial t}.$$

This formula proves the representation (1.2.8).

The volume integral containing $\boldsymbol{f}^e$ can be written down as

$$\int_V \boldsymbol{f}^e dx = \int_S (T + \boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}) \cdot \boldsymbol{n} ds - \int_V (\frac{\partial \boldsymbol{g}^e}{\partial t} + \operatorname{Div}(\boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}))dx,$$

where $\operatorname{Div}(\boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t})$ was added and subtracted, and the Gauss-Ostrogradskii theorem was used to convert the volume integral to the surface one. Using this expression and (1.2.3) in (1.2.2) gives us

$$\int_S (T + T^e + \boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}) \cdot \boldsymbol{n} ds = \frac{d}{dt} \int_V (\boldsymbol{g}^m + \boldsymbol{g}^e) dx, \tag{1.2.9}$$

which is the form appropriated for obtaining the matching conditions on surface fractions.

Now let $S$ and $V$ be the surface and the volume of a small cylindrical element, whose axis is perpendicular to the discontinuity surface such that one end of the cylinder lies in the part of a material with certain material properties and another one lies in the part with other material properties (Fig. 1). Applying (1.2.9) to this cylindrical region and allowing the axial dimension to approach zero, (1.2.9) becomes

$$[T + T^e + \boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}] \cdot \boldsymbol{n} = 0. \tag{1.2.10}$$

In the case of a body surrounded by vacuum $T = 0$ outside the body and (1.2.10) reduces to

$$T \cdot \boldsymbol{n} = -[T^e + \boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}] \cdot \boldsymbol{n}, \quad \text{on} \quad \Omega, \tag{1.2.11}$$

where $\Omega$ is the body surface.

The mechanical constitutive equations are taken to be the usual Hook's Law for an isotropic elastic medium, i.e.

$$T = \lambda \operatorname{tr} S \cdot I + 2\varkappa S, \tag{1.2.12}$$

where $S$ is the strain tensor defined by the formula

$$S_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, 2, 3.$$

In the above formulas, $\lambda, \varkappa$ are the Lamé coefficients. When using these relations it is assumed that the stresses and strains for this combined system in proper and laboratory frames are the same. Due to the fact that the system has been split to two parts, the mechanical part and the electromagnetic part, as expressed by the Minkowski energy-momentum tensor, this question needs further consideration. For the present purposes we simply assume that constitutive relations (1.2.12) for purely elastic medium are unaffected by the electromagnetic fields. For very large fields or finite deformations the interaction terms will enter in the constitutive relations thereby coupling together the elastic and electromagnetic constitutive equations, see [7].

1.3. Summary of equations and matching conditions. Here we summarize the basic field equations and matching conditions for an electromagnetoelastic medium.

**Field equations**

$$\begin{aligned} &\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} = \operatorname{rot} \boldsymbol{H}, \qquad \operatorname{div} \boldsymbol{D} = \rho_e, \\ &\frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{rot} \boldsymbol{E} = 0, \qquad \operatorname{div} \boldsymbol{B} = 0, \end{aligned} \tag{1.3.1}$$

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}. \tag{1.3.2}$$

**Constitutive equations**

$$\begin{gathered}\boldsymbol{D} = \epsilon \boldsymbol{E} + \alpha \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}, \qquad \alpha \equiv \epsilon\mu - \epsilon_0\mu_0, \\ \boldsymbol{B} = \mu \boldsymbol{H} - \alpha \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{E}, \qquad \boldsymbol{J} = \rho_e \frac{\partial \boldsymbol{u}}{\partial t} + \sigma(\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}),\end{gathered} \tag{1.3.3}$$

$$\begin{gathered}T = \lambda \operatorname{tr} S \cdot I + 2\varkappa S, \\ S_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = 1,2,3.\end{gathered} \tag{1.3.4}$$

**Matching conditions**

$$\begin{gathered}[\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}]_t = 0, \qquad [\boldsymbol{H} - \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{D}]_t = J_m^S - \rho_e^S \frac{\partial u_m}{\partial t}, \\ [\boldsymbol{B}]_n = 0, \qquad [\boldsymbol{D}]_n = \rho_e^S,\end{gathered} \tag{1.3.5}$$

$$[T + \boldsymbol{E} \otimes \boldsymbol{D} + \boldsymbol{H} \otimes \boldsymbol{B} - \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})I + (\boldsymbol{D} \times \boldsymbol{B}) \otimes \frac{\partial \boldsymbol{u}}{\partial t}] \cdot \boldsymbol{n} = 0. \tag{1.3.6}$$

## 2. Direct problems

2.1. BASIC EQUATIONS. Consider the case of diffusion approximation of Maxwell's system. This means that in field equations (1.3.1)–(1.3.2) we neglect by displacement current $\frac{\partial \boldsymbol{D}}{\partial t}$ formally assuming $\epsilon = 0$, and set $\rho_e = 0$. Simultaneously we put in constitutive equations (1.3.3)–(1.3.4) $\alpha = 0$ and $\rho_e = 0$, too. It is easy to show that in this case in the presence of external electromagnetic $\boldsymbol{j}$ and elastic $\boldsymbol{f}$ sources of oscillations we can form the following electromagnetoelasticity system

$$\begin{gathered}\sigma \boldsymbol{E} + \sigma\mu \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H} + \boldsymbol{j} = \operatorname{rot} \boldsymbol{H}, \\ \mu \frac{\partial \boldsymbol{H}}{\partial t} + \operatorname{rot} \boldsymbol{E} = 0, \qquad \operatorname{div} \mu \boldsymbol{H} = 0,\end{gathered} \tag{2.1.1}$$

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + \mu \operatorname{rot} \boldsymbol{H} \times \boldsymbol{H} + \boldsymbol{f}. \tag{2.1.2}$$

We make the following assumptions about functions $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{u}, \boldsymbol{j}, \boldsymbol{f}$:

$$\begin{gathered}\boldsymbol{E} = (0,1,0)E(z,t), \quad \boldsymbol{H} = (1,0,0)H(z,t), \quad \boldsymbol{u} = (0,0,1)u(z,t) \\ \boldsymbol{j} = (0,1,0)j(z,t), \qquad \boldsymbol{f} = (0,0,1)f(z,t),\end{gathered} \tag{2.1.3}$$

where the variable $z$ stands to the variable $x_3$. Under such assumptions for the case $\rho = const, \mu = const$ we can form the following non-dimensional model system

$$h_t = (rh_z)_z - (hu_t)_z - (rj)_z, \tag{2.1.4}$$

$$u_{tt} = (\nu^2 u_z)_z - phh_z + f, \tag{2.1.5}$$

where $h, u, j, f$ are dimensionless analogues of the functions introduced by formulas (2.1.3), $r^{-1} = \mu L V_0 \sigma$ is the magnetic Reynolds number, $p = \mu H_0^2 \rho^{-1} V_0^{-2}$, $\nu = \sqrt{(\lambda + 2\varkappa)/\rho V_0^2}$ is dimensionless velocity of the elastic waves propagation; and $L, V_0, H_0$ are characteristic values of length, seismic velocity and magnetic field, respectively.

2.2. THE PROBLEM STATEMENT. WEAK SOLUTIONS. Now we can formulate the direct problem. Let be $Q_T = (t,z) : t \in (0,T) \times \Omega$, where $\Omega = (-l,l)$.

**Direct Problem 2.2.1** *Determine a set of the functions*

$$h : \overline{Q}_T \to \mathcal{R}, \quad u : \overline{Q}_T \to \mathcal{R}$$

*such that*

$$h_t = (rh_z)_z - (hu_t)_z - (rj)_z, \quad (z,t) \in Q_T, \tag{2.2.1}$$
$$u_{tt} = (\nu^2 u_z)_z - phh_z + f, \quad (z,t) \in Q_T, \tag{2.2.2}$$
$$h(z,0) = h_0(z), \quad z \in \Omega,, \tag{2.2.3}$$
$$u(z,0) = u_0(z), \quad u_t(z,0) = u_1(z), \quad z \in \Omega, \tag{2.2.4}$$
$$h(\pm l,t) = 0, \quad t \in (0,T), \tag{2.2.5}$$
$$u(\pm l,t) = 0, \quad t \in (0,T). \tag{2.2.6}$$

Here $r(z), \nu(z)$ are positive piecewise smooth functions and $j(z,t), f(z,t)$ are piecewise smooth functions, discontinuous at the points $z = z_k, k = 1,2,\ldots,m$, $-l < z_1 < z_2 < \cdots < z_m < l$; $p$ is a positive number.

Direct Problem (2.2.1) can be considered as a diffraction problem, i.e., as the problem with $Q_T$ partitioned into several domains $Q_T^{(k)}, Q_T^{(k)} = \Omega^{(k)} \times (0,T), \Omega^{(k)} = (z_k, z_{k+1}), k = 0,1,\ldots,m, z_0 = -l, z_{m+1} = l$, in each of which there is given parabolic-hyperbolic system (2.2.1)–(2.2.2) with smooth coefficients and free terms. We wish to find in $\overline{Q}_T$ a solution of this system satisfying:

- in $Q_T^{(k)}, k = 1,2,\ldots,m$, the corresponding equations (2.2.1)–(2.2.2);
- on the lower base of $Q_T$ the initial conditions (2.2.3)–(2.2.4);
- on the lateral surface of $Q_T$ the boundary conditions (2.2.5)–(2.2.6);
- at the jump points $z_k, k = 1,2,\ldots,m$, the following compatibility conditions

$$[h] = [u] = 0, \tag{2.2.7}$$
$$[r(h_z - j)] = \left[\nu^2 u_z\right] = 0\,. \tag{2.2.8}$$

The symbol $[v]$ denotes the jump of the function $v$ as it passes through $z_k$.

Problems of this type can be reduced by means of a simple technique to problems for the determination of weak (generalized) solutions of ordinary initial boundary-value problems with discontinuous coefficients, see [21, pp. 224-232]. This fact will be used for the analysis of Direct Problem 2.3.1.

2.3. MAIN RESULTS. Suppose that the functions $r, \nu, j, f$, the constant $p$ and the initial data $h_0, u_0, u_1$ in Direct Problem **??** enjoy the properties

(a) $r, \nu, j, f$ are supposed to be piecewise smooth functions with jumps at the points $z_m : -l < z_1 < z_2 < \cdots < z_m < l$; $0 < r_0 \le r(z) \le r_1 < \infty$, $0 < \nu_0 \le \nu(z) \le \nu_1 < \infty$ and $p$ is a positive number;

(b) $h_0 \in C^{\alpha}(\overline{\Omega})$, $\alpha \in (0,1), h_0(\pm l) = 0$, and $u_0 \in \overset{o}{W}{}_2^1(\Omega)$, $u_1 \in L_2(\Omega)$.

Direct Problem 2.3.1 is solvable.

**Theorem 2.3.1** *If conditions $(a) - (b)$ are fulfilled, then Direct Problem 2.3.1 has a weak solution*

$$h(z,t) \in \overset{o}{V}_2(Q_T), \qquad u(z,t) \in \overset{o}{W}{}_2^{1,1}(Q_T).$$

## 3. An inverse problem for electromagnetoelasticity equations with partially nonlinear interaction

In this section, following the original work [31], we present the results of the solution to an inverse problem for a electromagnetoelasticity system in the case of complete nonlinear interaction of electromagnetic and elastic waves.

3.1. FORMULATION OF AN INVERSE PROBLEM. Let us consider one of possible formulations of inverse problems for the direct problem earlier studied in section 2. In this section we assume that the free member of equation (2.2.2) has representation $f(z,t) = \phi(t)g(z,t)$, where the function $\phi$ is *unknown*. Let us now formulate inverse problem which will be studied now.

**Inverse Problem 3.1.1** *Determine a set of the functions*

$$h : Q_T \to \mathbb{R}, \quad u : Q_T \to \mathbb{R}, \quad \phi : [0,T] \to \mathbb{R}$$

*from equations (2.2.1)–(2.1.6) and*

$$\int_\Omega \rho(z) h h_z dz = -\frac{1}{2}\int_\Omega \rho_z h^2 dz = \psi(t), \quad t \in [0,T], \tag{3.1.1}$$

where $\rho \in \overset{o}{W}{}_2^1(\Omega)$ and $\psi : [0,T] \to \mathbb{R}$ are giving functions having sufficient smoothness.

The functions $r, \nu, g, j$ are supposed to be smooth functions with possible jumps in points $z_m : -l < z_1 < z_2 < \cdots < z_m < l$, $0 < r_0 \le r(z) \le r_1 < \infty$, $0 < \nu_0 \le \nu(z) \le \nu_0 < \infty$; $p$ is a positive number, and

$$\int_\Omega \rho(z) g(z,t) dz \ge \rho_0 > 0, \quad t \in [0,T]. \tag{3.1.2}$$

At the points of discontinuity we assume the fulfilment of the transmission conditions (2.2.7)–(2.2.8).

From (3.1.1) we obtain

$$\psi(t) = W(\phi), \tag{3.1.3}$$

where $W : L_2(0,T) \mapsto L_2(0,T)$ is operator defined by formula

$$W(\phi) = \frac{\langle u_{tt}, \rho\rangle + \int_\Omega \nu^2 u_z \rho' \, dz + p\psi}{\int_\Omega \rho g \, dz}. \tag{3.1.4}$$

Here $u = u(z,t;\phi), h = h(z,t;\phi), \phi = \phi(t)$ are the solution of the inverse problem and $\psi$ is the additional information (3.1.1).

3.2. MAIN RESULTS. Now we are ready to formulate and prove our main results.

**Theorem 3.2.1** *Let $\phi$ be a fixed point of the operator $W(\phi)$, i. e., $\phi = W(\phi)$. Then $u, h, \phi$ are a solution of Inverse Problem 3.1. The reciprocal statement is valid: let $u, h, \phi$ be a solution of Inverse Problem 3.1.1, then $\phi = W(\phi)$.*

There is valid the following existence and uniqueness theorem.

**Theorem 3.2.2** *For sufficiently small values $T > 0$ Inverse Problem 3.1.1 has a unique solution, which can be obtained by the method of successive approximations.*

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