Bol. Soc. Paran. Mat.
(3s.) **v. 24** 1-2 (2006): 41–68.
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Gain in regularity for a coupled nonlinear Schrödinger system

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ABSTRACT: We study the gain of regularity for the initial value problem for a coupled nonlinear Schrödinger system that describes some physical phenomena such as the propagation in birefringent optical fibers, Kerr-like photo refractive media in optics and Bose-Einstein condensates. This study is motivated by the results obtained by N. Hayashi *et al.*

Key Words: Schrödinger equations, gain of regularity.

Contents

1. Introduction

In this paper we consider the initial value problem for the coupled nonlinear Schrödinger system

$$i\,u_t + u_{xx} + |u|^2\,u + \beta\,|v|^2\,u = 0 \tag{1.1}$$
$$i\,v_t + v_{xx} + |v|^2\,v + \beta\,|u|^2\,v = 0 \tag{1.2}$$
$$u(x,\,0) = u_0(x) \tag{1.3}$$
$$v(x,\,0) = v_0(x) \tag{1.4}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$. $u = u(x,\,t)$ is a complex unknown function, $v = v(x,\,t)$ is a complex unknown function and β is a real positive constant which depends on the anisotropy of the fiber. The study of the propagation of pulses in nonlinear optical fibers has been of great interest in the last years. In 1981, I. P. Kaminow [6] showed that single-mode optical fibers are not really "single-mode" but actually bimodal due to the presence of birefringence which can deeply influence the way in which an optical evolves during the propagation along the fiber. Indeed, it can occur that the linear birefringence makes a pulse split in two, while nonlinear

2000 *Mathematics Subject Classification:* Primary 35Q53; Secondary 47J353
Date submission 02-Sept-2006.

birefringence traps them together against splitting. C. R. Menyuk [10,11] showed that the evolution of two orthogonal pulse envelopes in birefringent optical fiber is governed by the coupled nonlinear Schrödinger system (1.1)-(1.4). If $\beta = 0$ the equations in (1.1)-(1.4) are two copies of a single nonlinear Schrödinger equation which is integrable; when $\beta = 1$, (1.1)-(1.4) is known as a Manakov system [9]. In all the other cases the situation is much more complicated from different points of view. The Cauchy problem for the system (1.1)-(1.4) was firstly studied by E. S. P. Siqueira [13,14] for initial data $u_0 \in H^1(\mathbb{R})$ and $v_0 \in H^1(\mathbb{R})$, then the solution $u \in C(\mathbb{R} : H^1(\mathbb{R})) \cap C^1(\mathbb{R} : H^{-1}(\mathbb{R}))$ and $v \in C(\mathbb{R} : H^1(\mathbb{R})) \cap C^1(\mathbb{R} : H^{-1}(\mathbb{R}))$, using the techniques developed in [1,2]. This Schrödinger system has been extensively studied for many authors [6,8,9,10,11] and references therein.

An evolution equation enjoys a gain of regularity if their solutions are smoother for $t > 0$ than its initial data. An equation need not to be hypo-elliptic for this to happen provided that the initial data vanishes at spatial infinity. For instance, for the Schrödinger equation in $\mathbb{R}^n$, it is clear from the explicit formula that a solution is C^∞ as the initial data decay faster than any polynomial. In 1979, T. Kato [7] established a remarkable result for the regularizing property of solutions to the initial value problem for the KdV equation

$$(KdV)\left\{\begin{array}{ll} u_t + u_{xxx} + u\, u_x = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,\, 0) = u_0(x) & \end{array}\right.$$

He proved that if the initial function $u_0(x) \in L^2_b(\mathbb{R}) \equiv H^2(\mathbb{R}) \cap L^2(e^{b\,x}\, dx)$ $(b > 0)$, then the solution $u(t)$ is $C^\infty(\mathbb{R})$ for $t > 0$. A main ingredient in the proof was the fact that formally the semigroup $e^{-\,t\,\partial_x^3}$ in L^2_b is equivalent to $U_b = e^{-\,t\,(\partial_x - b)^3}$ in L^2 when $t > 0$. In 1986, N. Hayashi *et al.* [4,5] showed that if the initial data $u_0(x)$ decreases rapidly enough, then the solution of the Schrödinger equation

$$(S)\left\{\begin{array}{ll} i\, u_t + u_{xx} = \lambda\, |u|^{p-1}\, u, & x \in \mathbb{R},\, t \in \mathbb{R} \\ u(x,\, 0) = u_0(x) & \end{array}\right.$$

with $\lambda \in \mathbb{R}$, $p > 1$ and $u = u(x,\, t)$ is a complex unknown function that becomes smooth for $t \neq 0$, provided that the initial functions in $H^1(\mathbb{R})$ decay rapidly enough as $|x| \to \infty$. On the other hand, the gain of regularity for a higher order Schrödinger equation was been proved in [12]. Thus, it is natural to ask whether the equation (1.1)-(1.4) has a gain in regularity. It might be expected that the Schrödinger systems have an analogous regularizing effect as that of the (S) equation. This is our motivation for the study of gain in regularity. Our aim in this paper is to show that the Schrödinger systems have a regularizing effect. Indeed, that all solutions of finite energy to (1.1)-(1.4) are smooth for $t \neq 0$ provided that the initial functions in $H^1(\mathbb{R})$ decay rapidly enough as $|x| \to \infty$. This paper is organized as follows. Before describing the main results, in section 2 we briefly outline the notation and terminology to be used later on and we present some Lemmas. In section 3 we find estimates of finite energy. In section 4 we obtain a priori estimates and in section 5 we prove the main theorem. Our main result is

Main Theorem. *Let* $(u_0,\, v_0) \in H^1(\mathbb{R})\times H^1(\mathbb{R})$ *and* $(x^n u_0,\, x^n v_0) \in L^2(\mathbb{R})\times L^2(\mathbb{R})$ *for some* $n \in \mathbb{N}$. *Then, there exists a unique solution* $(u(x,\, t),\, v(x,\, t))$ *of* (1.1)-(1.4) *satisfying*

$$(u,\, v) \in C_b(\mathbb{R} :\ H^1(\mathbb{R})) \times C_b(\mathbb{R} :\ H^1(\mathbb{R})) \tag{1.5}$$

$$(J^m u,\, J^m v) \in C(\mathbb{R} :\ L^2(\mathbb{R})) \times C(\mathbb{R} :\ L^2(\mathbb{R})), \quad m = 1,\, 2,\, \dots\, n. \tag{1.6}$$

Moreover $(u,\, v)$ *satisfies the integral identities:*
Densities Conservation

$$||u||_{L^2(\mathbb{R})} = ||u_0||_{L^2(\mathbb{R})} \quad \text{and} \quad ||v||_{L^2(\mathbb{R})} = ||v_0||_{L^2(\mathbb{R})} \tag{1.7}$$

Energy Conservation

$$\begin{aligned} &||u_x||^2_{L^2(\mathbb{R})} + ||v_x||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,||v||^4_{L^4(\mathbb{R})} - \beta \int_{\mathbb{R}} |u|^2\, |v|^2\, dx \\ = \quad &||u_{0\,x}||^2_{L^2(\mathbb{R})} + ||v_{0\,x}||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u_0||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,||v_0||^4_{L^4(\mathbb{R})} - \beta \int_{\mathbb{R}} |u_0|^2\, |v_0|^2\, dx \end{aligned} \tag{1.8}$$

The pseudo conformal identity

$$\begin{aligned} &||\, Ju\, ||^2_{L^2(\mathbb{R})} + ||\, Jv\, ||^2_{L^2(\mathbb{R})} - 2\,t^2\, ||u||^4_{L^4(\mathbb{R})} - 2\,t^2\, ||v||^4_{L^4(\mathbb{R})} - 4\,\beta\, t^2 \int_{\mathbb{R}} |u|^2\, |v|^2\, dx \\ = \quad &||x\, u_0||^2_{L^2(\mathbb{R})} + ||\, v_0||^2_{L^2(\mathbb{R})} - 2\,t^2\, ||u_0||^4_{L^4(\mathbb{R})} - 2\,t^2\, ||v_0||^4_{L^4(\mathbb{R})} - 4\,\beta\, t^2 \int_{\mathbb{R}} |u_0|^2\, |v_0|^2\, dx \\ &+ \int_0^T \left[-\, 2\,t\, ||u||^4_{L^4(\mathbb{R})} - 2\,t\, ||v||^4_{L^4(\mathbb{R})} - 4\,\beta\, t \int_{\mathbb{R}} |u|^2\, |v|^2\, dx \right] dt \end{aligned} \tag{1.9}$$

and the inequality

$$||J^m u||_{L^2(\mathbb{R})} \le c_m\, e^t \qquad \text{and} \qquad ||J^m v||_{L^2(\mathbb{R})} \le c_m\, e^t, \qquad m = 1,\, 2,\, \dots,\, n. \tag{1.10}$$

where c_m *is a positive constant depending only on* m, $||u_0||_{H^1(\mathbb{R})}$, $||v_0||_{H^1(\mathbb{R})}$, $||x^m\, u_0||_{L^2(\mathbb{R})}$ *and* $||x^m\, v_0||_{L^2(\mathbb{R})}$.
The pseudo conformal identity (1.9) was firstly observed by Ginibre and Velo [3] and is useful to obtain a priori estimates of Ju.

2. Preliminaries

We will use the following standard notation. For $1 \le p \le \infty$, $L^p(\mathbb{R})$ are all complex valued measurable functions on $\mathbb{R}$ such that $|u|^p$ is integrable for $1 \le p < \infty$ and $\sup\limits_{x\in\mathbb{R}} \text{ess}\, |u(x)|$ is finite for $p = \infty$. The norm will be written as

$$||u||_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |u(x)|^p dx \right)^{1/p} \qquad ; \qquad ||u||_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} \text{ess}\, |u(x)|.$$

For a non-negative integer m and $1 \leq p \leq \infty$, we denote by $H^m(\mathbb{R})$ the Sobolev space of functions in $L^2(\mathbb{R})$ having all derivatives of order $\leq m$ belonging to $L^2(\mathbb{R})$. The norm in $H^m(\mathbb{R})$ is given by

$$||u||_{H^m(\mathbb{R})} = \left(\sum_{|\alpha| \leq m} ||D^\alpha u(x)||^2_{L^2(\mathbb{R})} dx \right)^{1/2}.$$

For any interval I of $\mathbb{R}$ and any Banach space X with the norm $|| \cdot ||_X$, we denote by $C(I : X)$(respectively $C_b(I : X)$ the space of continuous(respectively bounded continuous) functions from I to X. We denote $C^k(I : X)(k \geq 1)$ the space of k-times continuously differentiable functions from I to X. For an interval I, the space $L^p(I : X)$ is the space consisting of all strongly measurable X-valued functions $u(t)$ defined on I such that $||u||_X \in L^p(I)$.

Remark. We only consider the case $t > 0$. The case $t < 0$ can be treated analogously.

The following results are going to be used several times from now on.

Lemma 2.1(The Gagliardo-Nirenberg inequality). *Let q, r be any real numbers satisfying $1 \leq q, r \leq \infty$ and let j and m be non-negative integers such that $j \leq m$. Then*

$$||D^j u||_{L^p(\mathbb{R})} \leq M \, ||D^m u||^a_{L^r(\mathbb{R})} \, ||u||^{1-a}_{L^q(\mathbb{R})} \tag{2.1}$$

where $\frac{1}{p} = j + a \left(\frac{1}{r} - m \right) + \frac{(1-a)}{q}$ for all a in the interval $\frac{j}{m} \leq a \leq 1$, and M is a positive constant depending only on m, j, q, r and a.

Lemma 2.2. *For all $u \in H^1(\mathbb{R})$ we have*

$$||u||^2_{L^\infty(\mathbb{R})} \leq 2 \, ||u||_{L^2(\mathbb{R})} \, ||u_x||_{L^2(\mathbb{R})} \tag{2.2}$$

$$||u||^4_{L^4(\mathbb{R})} \leq 2 \, ||u||^3_{L^2(\mathbb{R})} \, ||u_x||_{L^2(\mathbb{R})} \tag{2.3}$$

Lemma 2.3. *Let u and v be the solutions of (1.1)-(1.4), then we have*

$$\frac{d}{dt} (\, |u|^2 \,) = 2 \, Im \, u \, \overline{u}_{xx} \qquad \text{and} \qquad \frac{d}{dt} (\, |v|^2 \,) = 2 \, Im \, v \, \overline{v}_{xx} \tag{2.4}$$

Proof. Multiplying (1.1) by $\overline{u}$ we have

$$\begin{aligned} & i \, \overline{u} \, u_t + \overline{u} \, u_{xx} + |u|^4 + \beta \, |v|^2 \, |u|^2 = 0 \\ & - i \, u \, \overline{u}_t + u \, \overline{u}_{xx} + |u|^4 + \beta \, |v|^2 \, |u|^2 = 0. \qquad \text{(applying conjugate)} \end{aligned}$$

Subtracting,

$$i \, \frac{d}{dt} (\, |u|^2) = u \, \overline{u}_{xx} - \overline{u \, \overline{u}_{xx}} = 2 \, i \, Im \, u \, \overline{u}_{xx} \quad \text{so that} \quad \frac{d}{dt} (\, |u|^2) = 2 \, Im \, u \, \overline{u}_{xx}.$$

The proof works in a similar way for v. The lemma follows. □

In this paper our main tool is the operator J defined by

$$Ju = e^{i \, x^2/4 \, t} \, (2 \, i \, t) \, \partial_x (e^{- \, i \, x^2/4 \, t} \, u) = (x + 2 \, i \, t \, \partial_x) \, u.$$

This operator has the remarkable property that it commutes with the operator L defined by $L = (i\,\partial_t + \partial_x^2)$, namely, $LJ - JL \equiv [L,\, J] = 0$. In general,

$$J^m u = e^{i\,x^2/4\,t}\,(2\,i\,t)^m\,\partial_x^m(e^{-\,i\,x^2/4\,t}\,u) = (x + 2\,i\,t\,\partial_x)^m\,u, \qquad m \in \mathbb{N}$$

where $J^m u = J(J^{m-1}u)$ $(m \in \mathbb{N})$. Hence, applying J^m to the equations (1.1)-(1.4) we have

$$i\,(J^m u)_t + (J^m u)_{xx} + J^m(|u|^2\,u) + \beta\,J^m(|v|^2\,u) = 0, \tag{2.5}$$
$$i\,(J^m v)_t + (J^m v)_{xx} + J^m(|v|^2\,v) + \beta\,J^m|u|^2\,v) = 0, \tag{2.6}$$
$$J^m u(x,\,0) = x^m\,u_0(x), \tag{2.7}$$
$$J^m v(x,\,0) = x^m\,v_0(x), \tag{2.8}$$

which allows us to get the estimates (1.9)-(1.10) of the Main Theorem.
Throughout this paper c is a generic constant, not necessarily the same at each occasion (it will change from line to line), which depends in an increasing way on the indicated quantities.

3. Finite energy solutions

We begin obtaining estimates for the norm-$H^1(\mathbb{R})$. To estimate in $H^1(\mathbb{R})$ is important because in $\mathbb{R}$ we have the following Sobolev immersion $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

Lemma 3.1 (densities conservation). *Let $(u,\,v)$ be the solution to (1.1)-(1.4). Let $(u_0,\,v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, then*

$$||u||_{L^2(\mathbb{R})} = ||u_0||_{L^2(\mathbb{R})} \quad \text{and} \quad ||v||_{L^2(\mathbb{R})} = ||v_0||_{L^2(\mathbb{R})} \tag{3.1}$$

Proof. Multiplying (1.1) by $\overline{u}$ we have

$$\begin{aligned} i\,\overline{u}\,u_t + \overline{u}\,u_{xx} + |u|^4 + \beta\,|v|^2\,|u|^2 &= 0, \\ -\,i\,u\,\overline{u}_t + u\,\overline{u}_{xx} + |u|^4 + \beta\,|v|^2\,|u|^2 &= 0. \qquad \text{(applying conjugate)} \end{aligned}$$

Subtracting and integrating over $x \in \mathbb{R}$ we obtain

$$\frac{d}{dt}\,||u||_{L^2(\mathbb{R})} = 0,$$

and integrating over $t \in [0,\,T]$ we get the first term. Similarly for v, the lemma follows. □

Lemma 3.2 (energy conservation). *Let $(u,\,v)$ be the solutions to (1.1)-(1.4). Let $(u_0,\,v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, then*

$$||u_x||^2_{L^2(\mathbb{R})} + ||v_x||^2_{L^2(\mathbb{R})} \le c \tag{3.2}$$

Proof. Differentiating (1.1) with respect to the x-variable we have

$$i\,u_{x\,t} + u_{xxx} + (|u|^2)_x\,u + |u|^2\,u_x + \beta\,(|v|^2)_x\,u + \beta\,|v|^2\,u_x = 0 \tag{3.3}$$

Multiplying (3.3) by $\overline{u}_x$ we have

$$\begin{aligned} i\,\overline{u}_x\,u_{x\,t} + \overline{u}_x\,u_{xxx} + (|u|^2)_x\,u\,\overline{u}_x + |u|^2\,|u_x|^2 + \beta\,(|v|^2)_x\,u\,\overline{u}_x \ + \beta\,|v|^2\,|u_x|^2 = 0 \\ -\,i\,u_x\,\overline{u}_{x\,t} + u_x\,\overline{u}_{xxx} + (|u|^2)_x\,\overline{u\,\overline{u}_x} + |u|^2\,|u_x|^2 + \beta\,(|v|^2)_x\,\overline{u\,\overline{u}_x} + \beta\,|v|^2\,|u_x|^2 = 0. \\ \text{(applying conjugate)} \end{aligned}$$

Subtracting and integrating over $x \in \mathbb{R}$ we have

$$\frac{d}{dt}\int_{\mathbb{R}} |u_x|^2\,dx - 2\,Im\int_{\mathbb{R}} |u|^2\,u\,\overline{u}_{xx}\,dx - 2\,\beta\,Im\int_{\mathbb{R}} |v|^2\,u\,\overline{u}_{xx}\,dx = 0$$

using (2.4) we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} |u_x|^2\,dx - \int_{\mathbb{R}} |u|^2\,\frac{d}{dt}\,(|u|^2)\,dx - \beta\int_{\mathbb{R}} |v|^2\,\frac{d}{dt}\,(|u|^2)\,dx = 0$$

then

$$\frac{d}{dt}\int_{\mathbb{R}} |u_x|^2\,dx - \frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}} |u|^4\,dx - \beta\int_{\mathbb{R}} |v|^2\,\frac{d}{dt}\,(|u|^2)\,dx = 0. \tag{3.4}$$

Performing similar calculations for (1.2) we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} |v_x|^2\,dx - \frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}} |v|^4\,dx - \beta\int_{\mathbb{R}} |u|^2\,\frac{d}{dt}\,(|v|^2)\,dx = 0. \tag{3.5}$$

Adding (3.4) and (3.5)

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} |u_x|^2\,dx + \frac{d}{dt}\int_{\mathbb{R}} |v_x|^2\,dx - \frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}} |u|^4\,dx - \frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}} |v|^4\,dx \\ &-\beta\int_{\mathbb{R}} |v|^2\,\frac{d}{dt}\,(|u|^2)\,dx - \beta\int_{\mathbb{R}} |u|^2\,\frac{d}{dt}\,(|v|^2)\,dx = 0, \end{aligned}$$

thus we deduce

$$\frac{d}{dt}\left[||u_x||^2_{L^2(\mathbb{R})} + ||v_x||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,||v||^4_{L^4(\mathbb{R})} - \beta\int_{\mathbb{R}} |u|^2\,|v|^2\,dx\right] = 0. \tag{3.6}$$

Integrating (3.6) over $t \in [0,\,T]$ we have

$$\begin{aligned} &||u_x||^2_{L^2(\mathbb{R})} + ||v_x||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,||v||^4_{L^4(\mathbb{R})} - \beta\int_{\mathbb{R}} |u|^2\,|v|^2\,dx \\ &= \ ||u_{0\,x}||^2_{L^2(\mathbb{R})} + ||v_{0\,x}||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u_0||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,||v_0||^4_{L^4(\mathbb{R})} - \beta\int_{\mathbb{R}} |u_0|^2\,|v_0|^2\,dx \\ &\leq \ ||u_{0\,x}||^2_{L^2(\mathbb{R})} + ||v_{0\,x}||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u_0||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,||v_0||^4_{L^4(\mathbb{R})} \\ &\quad + \frac{1}{2}\,\beta\,||u_0||^4_{L^4(\mathbb{R})} + \frac{1}{2}\,\beta\,||v_0||^4_{L^4(\mathbb{R})} \\ &\leq \ ||u_{0\,x}||^2_{L^2(\mathbb{R})} + ||v_{0\,x}||^2_{L^2(\mathbb{R})} + \frac{1}{2}\,(1+\beta)\,||u_0||^4_{L^4(\mathbb{R})} + \frac{1}{2}\,(1+\beta)\,||v_0||^4_{L^4(\mathbb{R})}. \end{aligned}$$

Using (2.3) we have

$$
\begin{aligned}
&||u_x||^2_{L^2(\mathbb{R})} + ||v_x||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,||v||^4_{L^4(\mathbb{R})} - \beta \int_{\mathbb{R}} |u|^2\,|v|^2\,dx \\
&\quad\le\ ||u_{0\,x}||^2_{L^2(\mathbb{R})} + ||v_{0\,x}||^2_{L^2(\mathbb{R})} \\
&\qquad + (1+\beta)\,||u_0||^3_{L^2(\mathbb{R})}\,||u_{0\,x}||_{L^2(\mathbb{R})} + (1+\beta)\,||v_0||^3_{L^2(\mathbb{R})}\,||v_{0\,x}||_{L^2(\mathbb{R})} \\
&\quad\le\ ||u_{0\,x}||^2_{L^2(\mathbb{R})} + ||v_{0\,x}||^2_{L^2(\mathbb{R})} \\
&\qquad + \frac{1}{2}(1+\beta)\left[\,||u_0||^6_{L^2(\mathbb{R})} + ||u_{0\,x}||^2_{L^2(\mathbb{R})}\right] + \frac{1}{2}(1+\beta)\left[\,||v_0||^6_{L^2(\mathbb{R})} + ||v_{0\,x}||^2_{L^2(\mathbb{R})}\right] \\
&\quad\le\ \frac{1}{2}(3+\beta)\,||u_{0\,x}||^2_{L^2(\mathbb{R})} + \frac{1}{2}(3+\beta)\,||v_{0\,x}||^2_{L^2(\mathbb{R})} \\
&\qquad + \frac{1}{2}(1+\beta)\,||u_0||^6_{L^2(\mathbb{R})} + \frac{1}{2}(1+\beta)\,||v_0||^6_{L^2(\mathbb{R})}.
\end{aligned}
$$

Then

$$
||u_x||^2_{L^2(\mathbb{R})} + ||v_x||^2_{L^2(\mathbb{R})} - \frac{1}{2}\,||u||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,||v||^4_{L^4(\mathbb{R})} - \beta \int_{\mathbb{R}} |u|^2\,|v|^2\,dx \le c
$$

with $c = c(||u_{0x}||_{L^2(\mathbb{R})}, ||u_{0x}||_{L^2(\mathbb{R})})$. Using again (2.3) we can estimate

$$
\begin{aligned}
&||u_x||^2_{L^2(\mathbb{R})} + ||v_x||^2_{L^2(\mathbb{R})} \le c + \frac{1}{2}\,||u||^4_{L^4(\mathbb{R})} + \frac{1}{2}\,||v||^4_{L^4(\mathbb{R})} + \beta \int_{\mathbb{R}} |u|^2\,|v|^2\,dx \\
&\quad\le\ c + \frac{1}{2}\,||u||^4_{L^4(\mathbb{R})} + \frac{1}{2}\,||v||^4_{L^4(\mathbb{R})} + \frac{1}{2}\,\beta\,||u||^4_{L^4(\mathbb{R})} + \frac{1}{2}\,\beta\,||v||^4_{L^4(\mathbb{R})} \\
&\quad\le\ c + (1+\beta)\,||u||^3_{L^2(\mathbb{R})}\,||u_x||_{L^2(\mathbb{R})} + (1+\beta)\,||v||^3_{L^2(\mathbb{R})}\,||v_x||_{L^2(\mathbb{R})} \\
&\quad\le\ c + \frac{1}{2}(1+\beta)\,||u||^6_{L^2(\mathbb{R})} + \frac{1}{2}\,||u_x||^2_{L^2(\mathbb{R})} + \frac{1}{2}(1+\beta)\,||v||^6_{L^2(\mathbb{R})} + \frac{1}{2}\,||v_x||^2_{L^2(\mathbb{R})},
\end{aligned}
$$

and then we deduce

$$
||u_x||^2_{L^2(\mathbb{R})} + ||v_x||^2_{L^2(\mathbb{R})} \le c + (1+\beta)\,||u||^6_{L^2(\mathbb{R})} + (1+\beta)\,||v||^6_{L^2(\mathbb{R})}.
$$

Using (3.1) we have

$$
||u_x||^2_{L^2(\mathbb{R})} + ||v_x||^2_{L^2(\mathbb{R})} \le c + (1+\beta)\,||u_0||^6_{L^2(\mathbb{R})} + (1+\beta)\,||v_0||^6_{L^2(\mathbb{R})},
$$

and the result follows. □

4. A priori estimates

In the proof stated below it is shown that $L^\infty(\mathbb{R})$ estimates of solutions lead to obtain a priori estimates of $J^m u$. We estimate a Gronwall's inequality type and we establish decay of perturbed solutions.

Lemma 4.1. *Let $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Then, for any $t > 0$ we have*

$$
||u||_{L^\infty(\mathbb{R})} \le c\left(\frac{1+t}{t}\right)^{1/4} \quad \text{and} \quad ||v||_{L^\infty(\mathbb{R})} \le c\left(\frac{1+t}{t}\right)^{1/4}. \tag{4.1}
$$

Remark. Using the Gagliardo-Nirenberg inequality we have

$$||u||_{L^p(\mathbb{R})} \leq c\, ||u||^{2/p}_{L^2(\mathbb{R})} ||u||^{(p-2)/p}_{L^\infty(\mathbb{R})}.$$

Thus, using (3.1) and (4.1) we deduce the following L^p estimate

$$||u||_{L^p(\mathbb{R})} \leq c \left(1+\frac{1}{t}\right)^{(p-2)/4p} \quad \text{and} \quad ||v||_{L^p(\mathbb{R})} \leq \left(1+\frac{1}{t}\right)^{(p-2)/4p}$$

for $2 < p \leq +\infty$.

Proof of lemma 4.1. We rewrite the equation (1.1) as $Lu = -\,|u|^2\,u - \beta\,|v|^2\,u$, where $L = i\,\partial_t + \partial_x^2$. We consider the operator $J = x + 2\,i\,t\,\partial_x$ such that $L\,J = J\,L$, then we have $L\,(Ju) = -\,J(|u|^2\,u) - \beta\,J(|v|^2\,u)$. Hence, $i\,(Ju)_t + (Ju)_{xx} = -\,J(|u|^2\,u) - \beta\,J(|v|^2\,u)$. Multiplying the above equation by $(\overline{Ju})$ we have

$$\begin{aligned} i\,(\overline{Ju})\,(Ju)_t + (\overline{Ju})\,(Ju)_{xx} &= -\,J(|u|^2\,u)\,(\overline{Ju}) - \beta\,J(|v|^2\,u)\,(\overline{Ju}) \\ -\,i\,(Ju)\,\overline{(Ju)}_t + (Ju)\,\overline{(Ju)}_{xx} &= -\,\overline{J(|u|^2\,u)\,(\overline{Ju})} - \beta\,\overline{J(|v|^2\,u)\,(\overline{Ju})} \\ &\qquad \text{(applying conjugate).} \end{aligned}$$

Integrating over $x \in \mathbb{R}$, subtracting and dividing by i, we have

$$\frac{d}{dt}\,||\,Ju\,||^2_{L^2(\mathbb{R})} = -\,2\,Im \int_{\mathbb{R}} J(|u|^2\,u)\,(\overline{Ju})\,dx - 2\,\beta\,Im \int_{\mathbb{R}} J(|v|^2\,u)\,(\overline{Ju})\,dx. \quad (4.2)$$

We estimate the two terms on the right hand side of (4.2)

$$J(|u|^2\,u) = x\,|u|^2\,u + 2\,i\,t\,(|u|^2)_x\,u + 2\,i\,t\,|u|^2\,u_x \qquad \text{and} \qquad \overline{Ju} = x\,\overline{u} - 2\,i\,t\,\overline{u}_x.$$

Hence, using straightforward calculations we have

$$\begin{aligned} J(|u|^2\,u)\,(\overline{Ju}) \;&=\; x^2\,|u|^4 + 4\,t^2\,|u|^2\,|u_x|^2 + 4\,t\,x\,|u|^2\,Im\,u\,\overline{u}_x \\ &\quad + i\,t\,x\,(|u|^4)_x + 4\,t^2\,(|u|^2)_x\,u\,\overline{u}_x. \end{aligned}$$

Taking imaginary part

$$Im\,[\,J(|u|^2\,u)\,(\overline{Ju})\,] \;=\; t\,x\,(|u|^4)_x + 4\,t^2\,Im\,[\,(|u|^2)_x\,u\,\overline{u}_x\,],$$

and integrating, we have

$$\begin{aligned} 2\,Im \int_{\mathbb{R}} J(|u|^2\,u)\,(\overline{Ju})\,dx \;&=\; 2\,t \int_{\mathbb{R}} x\,(|u|^4)_x\,dx + 8\,t^2\,Im \int_{\mathbb{R}} (|u|^2)_x\,u\,\overline{u}_x\,dx \\ &=\; -\,2\,t \int_{\mathbb{R}} |u|^4\,dx - 8\,t^2\,Im \int_{\mathbb{R}} |u|^2\,u\,\overline{u}_{xx}\,dx. \end{aligned}$$

Using (2.4) we obtain

$$\begin{aligned} 2\,Im\int_{\mathbb{R}} J(|u|^2\,u)\,(\overline{Ju})\,dx &= -\,2\,t\int_{\mathbb{R}}|u|^4\,dx - 4\,t^2\int_{\mathbb{R}}|u|^2\,\frac{d}{dt}(|u|^2)\,dx \\ &= -\,2\,t\int_{\mathbb{R}}|u|^4\,dx - 2\,t^2\frac{d}{dt}\int_{\mathbb{R}}|u|^4\,dx. \end{aligned}$$

On the other hand, using the identity

$$-\,2\,t^2\frac{d}{dt}\int_{\mathbb{R}}|u|^4\,dx = 4\,t\int_{\mathbb{R}}|u|^4\,dx - \frac{d}{dt}\left[2\,t^2\int_{\mathbb{R}}|u|^4\,dx\right],$$

we deduce

$$2\,Im\int_{\mathbb{R}} J(|u|^2\,u)\,(\overline{Ju})\,dx = 2\,t\,||u||^4_{L^4(\mathbb{R})} - \frac{d}{dt}\left[2\,t^2\,||u||^4_{L^4(\mathbb{R})}\right]. \tag{4.3}$$

Moreover,

$$J(|v|^2\,u) = x\,|v|^2\,u + 2\,i\,t\,(|v|^2)_x\,u + 2\,i\,t\,|v|^2\,u_x \quad \text{and} \quad \overline{Ju} = x\,\overline{u} - 2\,i\,t\,\overline{u}_x.$$

Thus, using straightforward calculations we have

$$\begin{aligned} J(|v|^2\,u)\,(\overline{Ju}) &= x^2\,|v|^2\,|u|^2 + 4\,t^2\,|v|^2\,|u_x|^2 + 4\,t\,x\,|v|^2\,Im\,u\,\overline{u}_x \\ &\quad +2\,i\,t\,x\,(|v|^2)_x\,|u|^2 + 4\,t^2\,(|v|^2)_x\,u\,\overline{u}_x \end{aligned}$$

then $Im\,[\,J(|v|^2\,u)\,(\overline{Ju})\,] = 2\,t\,x\,(|v|^2)_x\,|u|^2 + 4\,t^2\,Im\,[\,(|v|^2)_x\,u\,\overline{u}_x\,]$. Hence

$$\begin{aligned} 2\,Im\int_{\mathbb{R}} J(|v|^2\,u)\,(\overline{Ju})\,dx &= 4\,t\int_{\mathbb{R}} x\,(|v|^2)_x\,|u|^2\,dx + 8\,t^2\,Im\int_{\mathbb{R}}(|v|^2)_x\,u\,\overline{u}_x\,dx \\ &= 4\,t\int_{\mathbb{R}} x\,(|v|^2)_x\,|u|^2\,dx - 8\,t^2\,Im\int_{\mathbb{R}}|v|^2\,u\,\overline{u}_{xx}\,dx. \end{aligned}$$

Using (2.4), we obtain

$$2\,Im\int_{\mathbb{R}} J(|v|^2\,u)\,(\overline{Ju})\,dx = 4\,t\int_{\mathbb{R}} x\,(|v|^2)_x\,|u|^2\,dx - 4\,t^2\int_{\mathbb{R}}|v|^2\,\frac{d}{dt}\,(|u|^2)\,dx. \tag{4.4}$$

Hence, replacing (4.3) and (4.4) in (4.2), we deduce

$$\begin{aligned} \frac{d}{dt}\,||\,Ju\,||^2_{L^2(\mathbb{R})} &= -\,2\,t\,||u||^4_{L^4(\mathbb{R})} + \frac{d}{dt}\left[2\,t^2\,||u||^4_{L^4(\mathbb{R})}\right] \\ &\quad -\,4\,\beta\,t\int_{\mathbb{R}} x\,(|v|^2)_x\,|u|^2\,dx + 4\,\beta\,t^2\int_{\mathbb{R}}|v|^2\,\frac{d}{dt}\,(|u|^2)\,dx. \end{aligned} \tag{4.5}$$

Performing the same calculations for the equation (1.2), we obtain

$$\begin{aligned} \frac{d}{dt}\,||\,Jv\,||^2_{L^2(\mathbb{R})} &= -\,2\,t\,||v||^4_{L^4(\mathbb{R})} + \frac{d}{dt}\left[2\,t^2\,||v||^4_{L^4(\mathbb{R})}\right] \\ &\quad -\,4\,\beta\,t\int_{\mathbb{R}} x\,(|u|^2)_x\,|v|^2\,dx + 4\,\beta\,t^2\int_{\mathbb{R}}|u|^2\,\frac{d}{dt}\,(|v|^2)\,dx. \end{aligned} \tag{4.6}$$

Adding (4.5) and (4.6) we have

$$\begin{aligned}
&\frac{d}{dt}\Big[\,||\,Ju\,||^2_{L^2(\mathbb{R})} + ||\,Jv\,||^2_{L^2(\mathbb{R})} - 2\,t^2\,||u||^4_{L^4(\mathbb{R})} - 2\,t^2\,||v||^4_{L^4(\mathbb{R})}\Big]\\
&= \; -2\,t\,||u||^4_{L^4(\mathbb{R})} - 2\,t\,||v||^4_{L^4(\mathbb{R})}\\
&\quad + 4\,\beta\,t^2\int_{\mathbb{R}}|v|^2\,\frac{d}{dt}\,(|u|^2)\,dx + 4\,\beta\,t^2\int_{\mathbb{R}}|u|^2\,\frac{d}{dt}\,(|v|^2)\,dx\\
&\quad - 4\,\beta\,t\int_{\mathbb{R}}x\,(|v|^2)_x\,|u|^2\,dx - 4\,\beta\,t\int_{\mathbb{R}}x\,(|u|^2)_x\,|v|^2\,dx.
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{d}{dt}\Big[\,||\,Ju\,||^2_{L^2(\mathbb{R})} + ||\,Jv\,||^2_{L^2(\mathbb{R})} - 2\,t^2\,||u||^4_{L^4(\mathbb{R})} - 2\,t^2\,||v||^4_{L^4(\mathbb{R})}\Big]\\
&= \; -2\,t\,||u||^4_{L^4(\mathbb{R})} - 2\,t\,||v||^4_{L^4(\mathbb{R})} + 4\,\beta\,t^2\,\frac{d}{dt}\int_{\mathbb{R}}|u|^2\,|v|^2\,dx + 4\,\beta\,t\int_{\mathbb{R}}|u|^2\,|v|^2\,dx.
\end{aligned}$$

On the other hand, using the identity

$$4\,t^2\,\frac{d}{dt}\int_{\mathbb{R}}|u|^2\,|v|^2\,dx = \frac{d}{dt}\,\left[4\,t^2\int_{\mathbb{R}}|u|^2\,|v|^2\,dx\right] - 8\,t\int_{\mathbb{R}}|u|^2\,|v|^2\,dx,$$

we deduce

$$\begin{aligned}
&\frac{d}{dt}\,\Big[\,||\,Ju\,||^2_{L^2(\mathbb{R})} + ||\,Jv\,||^2_{L^2(\mathbb{R})} - 2\,t^2\,||u||^4_{L^4(\mathbb{R})} - 2\,t^2\,||v||^4_{L^4(\mathbb{R})}\,\Big]\\
&= \; -2\,t\,||u||^4_{L^4(\mathbb{R})} - 2\,t\,||v||^4_{L^4(\mathbb{R})} - 4\,\beta\,t\int_{\mathbb{R}}|u|^2\,|v|^2\,dx + \frac{d}{dt}\,\left[4\,\beta\,t^2\int_{\mathbb{R}}|u|^2\,|v|^2\,dx\right]
\end{aligned}$$

and then, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\,||\,Ju\,||^2_{L^2(\mathbb{R})} + ||\,Jv\,||^2_{L^2(\mathbb{R})} - 2\,t^2\,||u||^4_{L^4(\mathbb{R})} - 2\,t^2\,||v||^4_{L^4(\mathbb{R})} - 4\,\beta\,t^2\int_{\mathbb{R}}|u|^2\,|v|^2\,dx\right]\\
&= -2\,t\,||u||^4_{L^4(\mathbb{R})} - 2\,t\,||v||^4_{L^4(\mathbb{R})} - 4\,\beta\,t\int_{\mathbb{R}}|u|^2\,|v|^2\,dx. \qquad (4.7)
\end{aligned}$$

Let $f(t) \doteq -2\,t\,||u||^4_{L^4(\mathbb{R})} - 2\,t\,||v||^4_{L^4(\mathbb{R})} - 4\,\beta\,t\int_{\mathbb{R}}|u|^2\,|v|^2\,dx$. Then, we have

$$\frac{d}{dt}\,\Big[\,||\,Ju\,||^2_{L^2(\mathbb{R})} + ||\,Jv\,||^2_{L^2(\mathbb{R})} + t\,f(t)\Big] = f(t).$$

Integrating over $t \in [0,\,T]$ we have

$$||\,Ju\,||^2_{L^2(\mathbb{R})} + ||\,Jv\,||^2_{L^2(\mathbb{R})} + t\,f(t) = ||x\,u_0||^2_{L^2(\mathbb{R})} + ||x\,v_0||^2_{L^2(\mathbb{R})} + \int_0^t f(s)\,ds,$$

which we can rewrite as

$$||\,Ju\,||^2_{L^2(\mathbb{R})} + ||\,Jv\,||^2_{L^2(\mathbb{R})} + t\,f(t) \le \alpha + \int_1^t f(s)\,ds \qquad (4.8)$$

where $\alpha \equiv ||x\,u_0||^2_{L^2(\mathbb{R})} + ||x\,v_0||^2_{L^2(\mathbb{R})} + \int_0^1 f(s)\,ds$. From (3.6) and integrating over $t \in [0,\, T]$ we have

$$\begin{aligned} &4\,||u_x||^2_{L^2(\mathbb{R})} + 4\,||v_x||^2_{L^2(\mathbb{R})} - 2\,||u||^4_{L^4(\mathbb{R})} - 2\,||v||^4_{L^4(\mathbb{R})} - 4\,\beta \int_{\mathbb{R}} |u|^2\,|v|^2\,dx \\ &= 4\,||u_{0\,x}||^2_{L^2(\mathbb{R})} + 4\,||v_{0\,x}||^2_{L^2(\mathbb{R})} - 2\,||u_0||^4_{L^4(\mathbb{R})} - 2\,||v_0||^4_{L^4(\mathbb{R})} - 4\,\beta \int_{\mathbb{R}} |u_0|^2\,|v_0|^2\,dx, \end{aligned}$$

and then, we have the estimate

$$\begin{aligned} &-2\,||u||^4_{L^4(\mathbb{R})} - 2\,||v||^4_{L^4(\mathbb{R})} - 4\,\beta \int_{\mathbb{R}} |u|^2\,|v|^2\,dx \\ &\quad\leq\; 4\,||u_{0\,x}||^2_{L^2(\mathbb{R})} + 4\,||v_{0\,x}||^2_{L^2(\mathbb{R})} - 2\,||u_0||^4_{L^4(\mathbb{R})} - 2\,||v_0||^4_{L^4(\mathbb{R})} - 4\,\beta \int_{\mathbb{R}} |u_0|^2\,|v_0|^2\,dx. \end{aligned}$$

Using straightforward calculations as in Lemma 3.2, we have

$$-2\,||u||^4_{L^4(\mathbb{R})} - 2\,||v||^4_{L^4(\mathbb{R})} - 4\,\beta \int_{\mathbb{R}} |u|^2\,|v|^2\,dx \leq \widetilde{\alpha},$$

with $\widetilde{\alpha} = \widetilde{\alpha}(\,||u_0||_{L^2(\mathbb{R})},\, ||v_0||_{L^2(\mathbb{R})},\, ||u_{0\,x}||_{L^2(\mathbb{R})},\, ||v_{0\,x}||_{L^2(\mathbb{R})})$. Hence, $\alpha \leq \widetilde{\alpha}$ which is finite by hypothesis. Then (4.8) can be written as

$$F(t) \leq \widetilde{\alpha} + \int_1^t G(s)\,F(s)\,ds,$$

where $F(t) \equiv t\,f(t)$ and $G(t) \equiv 1/t$. Since F and G are continuous on $[1,\, \infty[$, Gronwall's inequality implies that $F(t) \leq \widetilde{\alpha}\,t$, for all $t > 1$. We simplify to get that $f(t) \leq \widetilde{\alpha}$, for all $t > 1$. The hypothesis on $(u_0,\, v_0)$ and integration of (3.6) together imply that $f(t)$ is uniformly bounded for all t, in particular for $0 \leq t \leq 1$. Hence, there exists a constant $c(\,||u_0||_{L^2(\mathbb{R})},\, ||u_0||_{L^2(\mathbb{R})},\, ||u_{0\,x}||_{L^2(\mathbb{R})},\, ||v_{0\,x}||_{L^2(\mathbb{R})})$ such that $f(t) \leq c$, for any $t > 0$. Therefore,

$$||\,Ju\,||^2_{L^2(\mathbb{R})} + ||\,Jv\,||^2_{L^2(\mathbb{R})} + t\,f(t) \leq c\,t, \qquad \text{for any} \quad t > 0$$

which we can written as

$$||\,Ju\,||^2_{L^2(\mathbb{R})} + ||\,Jv\,||^2_{L^2(\mathbb{R})} - 2\,t^2\,||u||^4_{L^4(\mathbb{R})} - 2\,t^2\,||v||^4_{L^4(\mathbb{R})} - 4\,\beta\,t^2 \int_{\mathbb{R}} |u|^2\,|v|^2\,dx \leq c\,t,$$

for any $t > 0$. Hence using that $Ju = e^{i\,x^2/4\,t}\,(2\,i\,t)\,\partial_x(e^{-\,i\,x^2/4\,t}\,u)$ and $Jv = e^{i\,x^2/4\,t}\,(2\,i\,t)\,\partial_x(e^{-\,i\,x^2/4\,t}\,v)$ we obtain

$$\begin{aligned} &4\,t^2\,||\,\partial_x(e^{-\,i\,x^2/4\,t}\,u)\,||^2_{L^2(\mathbb{R})} + 4\,t^2\,||\,\partial_x(e^{-\,i\,x^2/4\,t}\,v)\,||^2_{L^2(\mathbb{R})} \\ &\qquad - 2\,t^2\,||u||^4_{L^4(\mathbb{R})} - 2\,t^2\,||v||^4_{L^4(\mathbb{R})} - 4\,\beta\,t^2 \int_{\mathbb{R}} |u|^2\,|v|^2\,dx \leq c\,t, \end{aligned}$$

for any $t > 0$. Thus, simplifying by $2t^2$, and passing the 3rd, 4th and 5th terms of the left to the right, we obtain the estimate

$$\begin{aligned} & 2\,||\,\partial_x(e^{-\,i\,x^2/4\,t}\,u)\,||^2_{L^2(\mathbb{R})} + 2\,||\,\partial_x(e^{-\,i\,x^2/4\,t}\,v)\,||^2_{L^2(\mathbb{R})} \\ & \quad \le \quad c\,t^{-\,1} + ||u||^4_{L^4(\mathbb{R})} + ||v||^4_{L^4(\mathbb{R})} + 2\,\beta \int_{\mathbb{R}} |u|^2\,|v|^2\,dx \\ & \quad \le \quad c\,t^{-\,1} + ||u||^4_{L^4(\mathbb{R})} + ||v||^4_{L^4(\mathbb{R})} + \beta\,||u||^4_{L^4(\mathbb{R})} + \beta\,||v||^4_{L^4(\mathbb{R})} \\ & \quad \le \quad c\,t^{-\,1} + (1+\beta)\,||u||^4_{L^4(\mathbb{R})} + (1+\beta)\,||v||^4_{L^4(\mathbb{R})}, \end{aligned}$$

for any $t > 0$. Using similar calculations as in Lemma 3.2, and using (3.1) and (3.2) we obtain

$$||\,\partial_x(e^{-\,i\,x^2/4\,t}\,u)\,||^2_{L^2(\mathbb{R})} + ||\,\partial_x(e^{-\,i\,x^2/4\,t}\,v)\,||^2_{L^2(\mathbb{R})} \le c\,t^{-\,1} + c_1 \le c\left(\frac{1+t}{t}\right),$$

for any $t > 0$. On the other hand, using the Gagliardo-Nirenberg inequality, we can estimate

$$\begin{aligned} ||\,e^{-\,i\,x^2/4\,t}\,u\,||_{L^\infty(\mathbb{R})} &\le c\,||\,\partial_x(e^{-\,i\,x^2/4\,t}\,u)\,||^{1/2}_{L^2(\mathbb{R})}\,||u||^{1/2}_{L^2(\mathbb{R})}, \\ ||\,e^{-\,i\,x^2/4\,t}\,v\,||_{L^\infty(\mathbb{R})} &\le c\,||\,\partial_x(e^{-\,i\,x^2/4\,t}\,v)\,||^{1/2}_{L^2(\mathbb{R})}\,||v||^{1/2}_{L^2(\mathbb{R})}, \end{aligned}$$

and then, we deduce

$$||\,e^{-\,i\,x^2/4\,t}\,u\,||^4_{L^\infty(\mathbb{R})} + ||\,e^{-\,i\,x^2/4\,t}\,v\,||^4_{L^\infty(\mathbb{R})} \le c\left(\frac{1+t}{t}\right),$$

for any $t > 0$. We obtain

$$||\,e^{-\,i\,x^2/4\,t}\,u\,||_{L^\infty(\mathbb{R})} \le c\left(\frac{1+t}{t}\right)^{1/4} \quad \text{and} \quad ||\,e^{-\,i\,x^2/4\,t}\,v\,||_{L^\infty(\mathbb{R})} \le c\left(\frac{1+t}{t}\right)^{1/4}.$$

for any $t > 0$. Therefore, we obtain

$$||u||_{L^\infty(\mathbb{R})} \le c\left(\frac{1+t}{t}\right)^{1/4} \qquad \text{and} \qquad ||v||_{L^\infty(\mathbb{R})} \le c\left(\frac{1+t}{t}\right)^{1/4},$$

for any $t > 0$. The lemma follows. □

Lemma 4.2. *Let* $(u_0,\,v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ *and* $(x^n\,u_0,\,x^n\,v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$. *Then, there exists a positive constant* c_m *depending on* $||u_0||_{H^1(\mathbb{R})}$, $||v_0||_{H^1(\mathbb{R})}$ *and* $||x^n\,u_0||_{L^2(\mathbb{R})}$, $||x^n\,v_0||_{L^2(\mathbb{R})}$ *but independent of* t *such that*

$$||\,J^m\,u\,||_{L^2(\mathbb{R})} \le c_m\,e^t \qquad \textit{and} \qquad ||\,J^m\,v\,||_{L^2(\mathbb{R})} \le c_m\,e^t, \tag{4.9}$$

for $m = 1,\,2,\,\ldots\,,\,n$.

Proof. From equation (1.1) we have $Lu = -\,|u|^2\,u - \beta\,|v|^2\,u$ where $L = i\,\partial_t + \partial_x^2$. We consider the operator $J^m u = (x + 2\,i\,t\,\partial_x)^m u = e^{i\,x^2/4\,t}\,(2\,i\,t)^m\,\partial_x^m (e^{-\,i\,x^2/4\,t}\,u)$ with $J^m u = J(J^{m-1}u)$ $(m \in \mathbb{N})$ such that $L\,J = J\,L$. Hence $L\,(J^m u) = -\,J^m(|u|^2\,u) - \beta\,J^m(|v|^2\,u)$. Thus

$$i\,(J^m u)_t + (J^m u)_{xx} = -\,J^m(|u|^2\,u) - \beta\,J^m(|v|^2\,u).$$

Multiplying the above equation by $(\overline{J^m u})$ we have

$$\begin{aligned} i\,(\overline{J^m u})\,(J^m u)_t + (\overline{J^m u})\,(J^m u)_{xx} &= -\,J^m(|u|^2\,u)\,(\overline{J^m u}) - \beta\,J^m(|v|^2\,u)\,(\overline{J^m u}) \\ -\,i\,(J^m u)\,\overline{(J^m u)}_t + (J^m u)\,\overline{(J^m u)}_{xx} &= -\,\overline{J^m(|u|^2\,u)\,(\overline{J^m u})} - \beta\,\overline{J^m(|v|^2\,u)\,(\overline{J^m u})}. \end{aligned}$$

(applying conjugate)

Integrating over $x \in \mathbb{R}$, subtracting and dividing by i, we have

$$\begin{aligned} \frac{d}{dt}\,||\,J^m u\,||^2_{L^2(\mathbb{R})} &= -\,2\,Im \int_{\mathbb{R}} J^m(|u|^2\,u)\,(\overline{J^m u})\,dx \\ &\quad -2\,\beta\,Im \int_{\mathbb{R}} J^m(|v|^2\,u)\,(\overline{J^m u})\,dx. \end{aligned} \tag{4.10}$$

In a similar way we estimate the same but using (1.2)

$$\begin{aligned} \frac{d}{dt}\,||\,J^m v\,||^2_{L^2(\mathbb{R})} &= -\,2\,Im \int_{\mathbb{R}} J^m(|v|^2\,v)\,(\overline{J^m v})\,dx \\ &\quad -2\,\beta\,Im \int_{\mathbb{R}} J^m(|u|^2\,v)\,(\overline{J^m v})\,dx. \end{aligned} \tag{4.11}$$

Adding (4.10) and (4.11) we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\,||\,J^m u\,||^2_{L^2(\mathbb{R})} + ||\,J^m v\,||^2_{L^2(\mathbb{R})}\right] \\ &= -\,2\,Im \int_{\mathbb{R}} J^m(|u|^2\,u)\,(\overline{J^m u})\,dx - 2\,Im \int_{\mathbb{R}} J^m(|v|^2\,v)\,(\overline{J^m v})\,dx \\ &\quad -\,2\,\beta\,Im \int_{\mathbb{R}} J^m(|v|^2\,u)\,(\overline{J^m u})\,dx - 2\,\beta\,Im \int_{\mathbb{R}} J^m(|u|^2\,v)\,(\overline{J^m v})\,dx. \end{aligned} \tag{4.12}$$

We estimate the first two terms in the right hand side. We have that

$$\begin{aligned} J^m(\,|u|^2\,u\,) &= e^{i\,x^2/4\,t}\,(2\,i\,t)^m\,\partial_x^m\left(e^{-\,i\,x^2/4\,t}\,|u|^2\,u\right) \\ &= e^{i\,x^2/4\,t}\,(\,2\,i\,t\,)^m\,\partial_x^m\left(\,|e^{-\,i\,x^2/4\,t}\,u|^2\,e^{-\,i\,x^2/4\,t}u\,\right). \end{aligned}$$

Let $w = e^{-\,i\,x^2/4\,t}u$, . Then $\partial_x^m\left(\,|w|^2\,w\,\right) = \partial_x^m\left(\,w\,\overline{w}\,w\,\right) = \sum_{i+j+k=m} \partial_x^i w\;\partial_x^j \overline{w}\;\partial_x^k w,$

and hence

$$\begin{aligned}
&\|\, J^m(\,|u|^2\, u\,)\,\|_{L^2(\mathbb{R})}\\
&= (2\,t)^m \left\| \sum_{i+j+k=m} \partial_x^i w\; \partial_x^j \overline{w}\; \partial_x^k w \right\|_{L^2(\mathbb{R})}\\
&\leq (2\,t)^m \sum_{i+j+k=m} \left\|\, \partial_x^i w\; \partial_x^j \overline{w}\; \partial_x^k w \,\right\|_{L^2(\mathbb{R})}\\
&\leq (2\,t)^m \sum_{i+j+k=m} \|\partial_x^i w\|_{L^{2m/i}(\mathbb{R})}\, \|\partial_x^j w\|_{L^{2m/j}(\mathbb{R})}\, \|\partial_x^k w\|_{L^{2m/k}(\mathbb{R})} \qquad (4.13)
\end{aligned}$$

using the Gagliardo-Nirenberg inequality we have

$$\begin{aligned}
\|\partial_x^i w\|_{L^{2m/i}(\mathbb{R})} &\leq c\,\|\partial_x^m w\|_{L^2(\mathbb{R})}^{i/m}\, \|w\|_{L^\infty(\mathbb{R})}^{(m-i)/m},\\
\|\partial_x^j w\|_{L^{2m/j}(\mathbb{R})} &\leq c\,\|\partial_x^m w\|_{L^2(\mathbb{R})}^{j/m}\, \|w\|_{L^\infty(\mathbb{R})}^{(m-j)/m},\\
\|\partial_x^k w\|_{L^{2m/k}(\mathbb{R})} &\leq c\,\|\partial_x^m w\|_{L^2(\mathbb{R})}^{k/m}\, \|w\|_{L^\infty(\mathbb{R})}^{(m-k)/m},
\end{aligned}$$

for every $i \in \{0, \ldots, m\}$. Then

$$\sum_{i+j+k=m} \|\partial_x^i w\|_{L^{2m/i}(\mathbb{R})}\, \|\partial_x^j w\|_{L^{2m/j}(\mathbb{R})}\, \|\partial_x^k w\|_{L^{2m/k}(\mathbb{R})} \leq c\,\|\partial_x^m w\|_{L^2(\mathbb{R})}\, \|w\|_{L^\infty(\mathbb{R})}^2.$$

Thus, in (4.13) we obtain

$$\begin{aligned}
\|J^m(|u|^2\, u)\|_{L^2(\mathbb{R})} &\leq c\,(2\,t)^m\, \|\partial_x^m w\|_{L^2(\mathbb{R})}\, \|w\|_{L^\infty(\mathbb{R})}^2\\
&\leq c\,\|J^m u\|_{L^2(\mathbb{R})}\, \|u\|_{L^\infty(\mathbb{R})}^2. \qquad (4.14)
\end{aligned}$$

In a similar way

$$\|J^m(|v|^2 v)\|_{L^2(\mathbb{R})} \leq c\,\|J^m v\|_{L^2(\mathbb{R})}\, \|v\|_{L^\infty(\mathbb{R})}^2. \qquad (4.15)$$

Using Hölder's inequality in (4.12) and replacing (4.14) and (4.15) in (4.12) we have

$$\begin{aligned}
&\frac{d}{dt}\left[\|J^m u\|_{L^2(\mathbb{R})}^2 + \|J^m v\|_{L^2(\mathbb{R})}^2 \right]\\
&\leq 2\,\|J^m(|u|^2\, u)\|_{L^2(\mathbb{R})}\, \|J^m u\|_{L^2(\mathbb{R})} + 2\,\|J^m(|v|^2\, v)\|_{L^2(\mathbb{R})}\, \|J^m v\|_{L^2(\mathbb{R})}\\
&\quad - 2\,\beta\, Im \int_{\mathbb{R}} J^m(|v|^2\, u)\, (\overline{J^m u})\, dx - 2\,\beta\, Im \int_{\mathbb{R}} J^m(|u|^2\, v)\, (\overline{J^m v})\, dx\\
&\leq c\,\|J^m u\|_{L^2(\mathbb{R})}\, \|u\|_{L^\infty(\mathbb{R})}^2\, \|J^m u\|_{L^2(\mathbb{R})} + c\,\|J^m v\|_{L^2(\mathbb{R})}\, \|v\|_{L^\infty(\mathbb{R})}^2\, \|J^m v\|_{L^2(\mathbb{R})}\\
&\quad - 2\,\beta\, Im \int_{\mathbb{R}} J^m(|v|^2\, u)\, (\overline{J^m u})\, dx - 2\,\beta\, Im \int_{\mathbb{R}} J^m(|u|^2\, v)\, (\overline{J^m v})\, dx\\
&\leq c\,\|J^m u\|_{L^2(\mathbb{R})}^2\, \|u\|_{L^\infty(\mathbb{R})}^2 + c\,\|J^m v\|_{L^2(\mathbb{R})}^2\, \|v\|_{L^\infty(\mathbb{R})}^2\\
&\quad - 2\,\beta\, Im \int_{\mathbb{R}} J^m(|v|^2\, u)\, (\overline{J^m u})\, dx - 2\,\beta\, Im \int_{\mathbb{R}} J^m(|u|^2\, v)\, (\overline{J^m v})\, dx. \qquad (4.16)
\end{aligned}$$

Now we estimate the third term in (4.16). Using the Hölder inequality we have

$$-\ 2\,\beta\, Im \int_{\mathbb{R}} J^m(|v|^2\,u)\,(\overline{J^m u})\,dx \le 2\,\beta\,||J^m(|v|^2\,u)||_{L^2(\mathbb{R})}\,||J^m u||_{L^2(\mathbb{R})}. \quad (4.17)$$

We estimate the $||J^m(|v|^2\,u)||_{L^2(\mathbb{R})}$ term. From the definition of J^m we have

$$\begin{aligned} J^m(|v|^2\,u) &= e^{i\,x^2/4\,t}\,(2\,i\,t)^m\,\partial_x^m(|v|^2\,e^{-\,i\,x^2/4\,t}\,u) \\ &= e^{i\,x^2/4\,t}\,(2\,i\,t)^m\,\partial_x^m(e^{-\,i\,x^2/4\,t}\,v \quad \overline{e^{-\,i\,x^2/4\,t}\,v} \quad e^{-\,i\,x^2/4\,t}\,u) \\ &= e^{i\,x^2/4\,t}\,(2\,i\,t)^m\,\partial_x^m(w\,\overline{w}\,z) \qquad (4.18) \end{aligned}$$

where $w = e^{-\,i\,x^2/4\,t}\,v$ and $z = e^{-\,i\,x^2/4\,t}\,u$. On the other hand, using $\partial_x^m(w\,\overline{w}\,z) = \sum_{i+j+k=m} \partial_x^i w\;\partial_x^j\overline{w}\;\partial_x^k z$. and the Hölder's inequality, we have

$$\begin{aligned} &||\partial_x^m(w\,\overline{w}\,z)||_{L^2(\mathbb{R})} \\ &\le \sum_{i+j+k=m} ||\partial_x^i w\;\partial_x^j\overline{w}\;\partial_x^k z\,||_{L^2(\mathbb{R})} \\ &\le \sum_{i+j+k=m} ||\partial_x^i w||_{L^{2m/i}(\mathbb{R})}\,||\partial_x^j w||_{L^{2m/j}(\mathbb{R})}\,||\partial_x^k z||_{L^{2m/k}(\mathbb{R})}. \qquad (4.19) \end{aligned}$$

Using the Gagliardo-Nirenberg inequality we have

$$\begin{aligned} ||\partial_x^i w||_{L^{2\,m/i}(\mathbb{R})} &\le c\,||\partial_x^m w||_{L^2(\mathbb{R})}^{i/m}\,||w||_{L^\infty(\mathbb{R})}^{(m-i)/m}, \\ ||\partial_x^j w||_{L^{2\,m/j}(\mathbb{R})} &\le c\,||\partial_x^m w||_{L^2(\mathbb{R})}^{j/m}\,||w||_{L^\infty(\mathbb{R})}^{(m-j)/m}, \\ ||\partial_x^k z||_{L^{2\,m/k}(\mathbb{R})} &\le c\,||\partial_x^m z||_{L^2(\mathbb{R})}^{k/m}\,||z||_{L^\infty(\mathbb{R})}^{(m-k)/m} \end{aligned}$$

for every $i \in \{0,\,\ldots,\,m\}$. Hence, in (4.19), we obtain

$$\begin{aligned} &||\partial_x^m(w\,\overline{w}\,z)||_{L^2(\mathbb{R})} \\ &\le \sum_{i+j+k=m} ||\partial_x^i w||_{L^{2m/i}(\mathbb{R})}\,||\partial_x^j w||_{L^{2m/j}(\mathbb{R})}\,||\partial_x^k z||_{L^{2m/k}(\mathbb{R})} \\ &\le c \sum_{i+j+k=m} ||\partial_x^m w||_{L^2(\mathbb{R})}^{(i+j)/m}\,||\partial_x^m z||_{L^2(\mathbb{R})}^{k/m}\,||w||_{L^\infty(\mathbb{R})}^{[2m-(i+j)]/m}\,||z||_{L^\infty(\mathbb{R})}^{(m-k)/m}. \end{aligned} \qquad (4.20)$$

Replacing in (4.18) and using Young's inequality, we have

$$\begin{aligned} &||J^m(|v|^2\,u)||_{L^2(\mathbb{R})} \\ &\le c \sum_{i+j+k=m} ||J^m v||_{L^2(\mathbb{R})}^{(i+j)/m}\,||J^m u||_{L^2(\mathbb{R})}^{k/m}\,||v||_{L^\infty(\mathbb{R})}^{[2m-(i+j)]/m}\,||u||_{L^\infty(\mathbb{R})}^{(m-k)/m} \\ &\le c \sum_{i+j+k=m} \left[\frac{(i+j)}{m}\,||J^m v||_{L^2(\mathbb{R})} + \frac{k}{m}\,||J^m u||_{L^2(\mathbb{R})}\right]||v||_{L^\infty(\mathbb{R})}^{[2m-(i+j)]/m}\,||u||_{L^\infty(\mathbb{R})}^{(m-k)/m}. \end{aligned} \qquad (4.21)$$

This way we have

$$
\begin{aligned}
&||J^m(|v|^2\,u)||_{L^2(\mathbb{R})}\;||J^m u||_{L^2(\mathbb{R})}\\
&\quad\le\; c\sum_{i+j+k=m}\frac{(i+j)}{m}\;||J^m v||_{L^2(\mathbb{R})}\;||J^m u||_{L^2(\mathbb{R})}\;||v||_{L^\infty(\mathbb{R})}^{[2m-(i+j)]/m}\;||u||_{L^\infty(\mathbb{R})}^{(m-k)/m}\\
&\qquad+c\sum_{i+j+k=m}\frac{k}{m}\;||J^m u||_{L^2(\mathbb{R})}^2\;||v||_{L^\infty(\mathbb{R})}^{[2m-(i+j)]/m}\;||u||_{L^\infty(\mathbb{R})}^{(m-k)/m}\\
&\quad\le\; c\sum_{i+j+k=m}\frac{(i+j)}{2m}\;\Big[||J^m v||_{L^2(\mathbb{R})}^2+||J^m u||_{L^2(\mathbb{R})}^2\Big]\;||v||_{L^\infty(\mathbb{R})}^{[2m-(i+j)]/m}\;||u||_{L^\infty(\mathbb{R})}^{(m-k)/m}\\
&\qquad+c\sum_{i+j+k=m}\frac{k}{m}\;||J^m u||_{L^2(\mathbb{R})}^2\;||v||_{L^\infty(\mathbb{R})}^{[2m-(i+j)]/m}\;||u||_{L^\infty(\mathbb{R})}^{(m-k)/m},
\end{aligned}
$$

Rearranging the terms of this last estimate, we have

$$
\begin{aligned}
&||J^m(|v|^2\,u)||_{L^2(\mathbb{R})}\;||J^m u||_{L^2(\mathbb{R})}\\
&\quad\le\; c\,||J^m v||_{L^2(\mathbb{R})}^2\sum_{i+j+k=m}\frac{(i+j)}{2m}\;||v||_{L^\infty(\mathbb{R})}^{[2m-(i+j)\,]/m}\;||u||_{L^\infty(\mathbb{R})}^{(m-k)/m}\\
&\qquad+c\,||J^m u||_{L^2(\mathbb{R})}^2\sum_{i+j+k=m}\frac{(i+j)}{2m}\;||v||_{L^\infty(\mathbb{R})}^{[2m-(i+j)\,]/m}\;||u||_{L^\infty(\mathbb{R})}^{(m-k)/m}\\
&\qquad+c\,||J^m u||_{L^2(\mathbb{R})}^2\sum_{i+j+k=m}\frac{k}{m}\;||v||_{L^\infty(\mathbb{R})}^{[2m-(i+j)\,]/m}\;||u||_{L^\infty(\mathbb{R})}^{(m-k)/m}.
\end{aligned}
$$

Using Lemma 2.2, Lemma 3.1, and Lemma 3.2 we obtain

$$||J^m(|v|^2\,u)||_{L^2(\mathbb{R})}\;||J^m u||_{L^2(\mathbb{R})}\le c_m\;||J^m u||_{L^2(\mathbb{R})}^2+c_m\;||J^m v||_{L^2(\mathbb{R})}^2. \tag{4.22}$$

In a similar way we have

$$||J^m(|u|^2\,v)||_{L^2(\mathbb{R})}\;||J^m v||_{L^2(\mathbb{R})}\le c_m\;||J^m u||_{L^2(\mathbb{R})}^2+c_m\;||J^m v||_{L^2(\mathbb{R})}^2. \tag{4.23}$$

Thus, replacing (4.22) and (4.23) in (4.16)

$$\frac{d}{dt}\;\Big[||J^m u||_{L^2(\mathbb{R})}^2+||J^m v||_{L^2(\mathbb{R})}^2\Big]\le c_m\;\Big[||J^m u||_{L^2(\mathbb{R})}^2+||J^m v||_{L^2(\mathbb{R})}^2\Big], \tag{4.24}$$

integrating over $t\in[0,\,T]$ and using Gronwall's inequality we obtain

$$||J^m u||_{L^2(\mathbb{R})}^2+||\,J^m v\,||_{L^2(\mathbb{R})}^2\le c_m\;\Big[||x^m u_0||_{L^2(\mathbb{R})}^2+||x^m v_0||_{L^2(\mathbb{R})}^2\Big]\;e^t. \tag{4.25}$$

The result then follows. □

5. Main Theorem

In this section we state and prove our theorem, which states that all solutions of finite energy to (1.1)-(1.4) are smooth for $t \neq 0$ provided that the initial functions in $H^1(\mathbb{R})$ decay rapidly enough as $|x| \to \infty$.

Lemma 5.1. *$\{(u^k,\, v^k)\}$ is a Cauchy sequence in $C([0,T] : H^1(\mathbb{R})) \times C([0,T] : H^1(\mathbb{R}))$ for any $T > 0$. Moreover*

$$||u^k - u^j||^2_{H^1(\mathbb{R})} + ||v^k - v^j||^2_{H^1(\mathbb{R})} \leq c(T)\, \Big[\, ||u^k_0 - u^j_0||^2_{H^1(\mathbb{R})} + ||v^k_0 - v^j_0||^2_{H^1(\mathbb{R})}\Big] \quad (5.1)$$

for any $T > 0$, where T is a positive constant independent of k and j.

Proof. Let $(u^k,\, v^k)$ be the solution of (1.1)-(1.4), then

$$i\, u^k_t + u^k_{xx} + |u^k|^2 u^k + \beta\, |v^k|^2 u^k = 0 \quad (5.2)$$
$$i\, v^k_t + v^k_{xx} + |v^k|^2 v^k + \beta\, |u^k|^2 v^k = 0 \quad (5.3)$$
$$i\, u^j_t + u^j_{xx} + |u^j|^2 u^j + \beta\, |v^j|^2 u^j = 0 \quad (5.4)$$
$$i\, v^j_t + v^j_{xx} + |v^j|^2 v^j + \beta\, |u^j|^2 v^j = 0. \quad (5.5)$$

Subtracting (5.2) with (5.4) we have

$$i\, (u^k - u^j)_t + (u^k - u^j)_{xx} + |u^k|^2 u^k - |u^j|^2 u^j + \beta\, (|v^k|^2 u^k - |v^j|^2 u^j) = 0. \quad (5.6)$$

Multiplying (5.6) by $(\overline{u^k - u^j})$ we obtain the following equation and the conjugate respectively:

$$\begin{aligned} & i\, (\overline{u^k - u^j})\, (u^k - u^j)_t + (\overline{u^k - u^j})\, (u^k - u^j)_{xx} \\ & + (|u^k|^2\, u^k - |u^j|^2\, u^j)\, (\overline{u^k - u^j}) + \beta\, (|v^k|^2\, u^k - |v^j|^2\, u^j)\, (\overline{u^k - u^j}) = 0 \end{aligned}$$

$$\begin{aligned} & -\, i\, (u^k - u^j)\, (\overline{u^k - u^j})_t + (u^k - u^j)\, (\overline{u^k - u^j})_{xx} \\ & + \overline{(|u^k|^2\, u^k - |u^j|^2\, u^j)\, (\overline{u^k - u^j})} + \beta\, \overline{(|v^k|^2\, u^k - |v^j|^2\, u^j)\, (\overline{u^k - u^j})} = 0. \end{aligned}$$

Subtracting and integrating over $x \in \mathbb{R}$ we have

$$\begin{aligned} \frac{d}{dt}\, ||u^k - u^j||^2_{L^2(\mathbb{R})} \;\; = \;\; & -2\, Im \int_{\mathbb{R}} (|u^k|^2 u^k - |u^j|^2 u^j)\, (\overline{u^k - u^j})\, dx \\ & -\, 2\, \beta\, Im \int_{\mathbb{R}} (|v^k|^2 u^k - |v^j|^2 u^j)\, (\overline{u^k - u^j})\, dx. \end{aligned} \quad (5.7)$$

In a similar way and performing similar calculations we obtain that

$$\begin{aligned} \frac{d}{dt}\, ||v^k - v^j||^2_{L^2(\mathbb{R})} \;\; = \;\; & -2\, Im \int_{\mathbb{R}} (|v^k|^2 v^k - |v^j|^2 v^j)\, (\overline{v^k - v^j})\, dx \\ & -\, 2\, \beta\, Im \int_{\mathbb{R}} (|u^k|^2 v^k - |u^j|^2 v^j)\, (\overline{v^k - v^j})\, dx. \end{aligned} \quad (5.8)$$

We estimate in (5.7) the term

$$
\begin{aligned}
[\,|u^k|^2\,u^k - |u^j|^2\,u^j\,]\,(\overline{u^k - u^j}) \;&=\; [\,(\,|u^k|^2 - |u^j|^2\,)\,u^k + |u^j|^2\,(u^k - u^j)\,]\,(\overline{u^k - u^j})\\
&=\; (\,|u^k|^2 - |u^j|^2\,)\,u^k\,(\overline{u^k - u^j}) + |u^j|^2\,|u^k - u^j|^2
\end{aligned}
$$

then, using Hölder's inequality and $|\,|a| - |b|\,| \leq |a - b|$ we obtain

$$
\begin{aligned}
& Im \int_{\mathbb{R}} [\,|u^k|^2\,u^k - |u^j|^2\,u^j\,]\,(\overline{u^k - u^j})\,dx\\
&= Im \int_{\mathbb{R}} (\,|u^k|^2 - |u^j|^2\,)\,u^k\,(\overline{u^k - u^j})\,dx\\
&= Im \int_{\mathbb{R}} (\,|u^k| - |u^j|\,)\,(\,|u^k| + |u^j|\,)\,u^k\,(\overline{u^k - u^j})\,dx\\
&\leq \int_{\mathbb{R}} |\,|u^k| - |u^j|\,|\,(\,|u^k| + |u^j|\,)\,|u^k|\,|u^k - u^j|\,dx\\
&\leq \int_{\mathbb{R}} |u^k - u^j|\,(\,|u^k| + |u^j|\,)\,|u^k|\,|u^k - u^j|\,dx\\
&\leq ||u^k||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})} + ||u^j||_{L^\infty(\mathbb{R})}\,) \int_{\mathbb{R}} |u^k - u^j|^2\,dx\\
&= ||u^k||_{L^\infty(\mathbb{R})}\,\big(\,||u^k||_{L^\infty(\mathbb{R})} + ||u^j||_{L^\infty(\mathbb{R})}\,\big)\,||u^k - u^j||^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
& 2\,Im \int_{\mathbb{R}} [\,|u^k|^2\,u^k - |u^j|^2\,u^j\,]\,(\overline{u^k - u^j})\,dx\\
&\leq 2\,||u^k||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})} + ||u^j||_{L^\infty(\mathbb{R})}\,)\,||u^k - u^j||^2_{L^2(\mathbb{R})}. \qquad (5.9)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
& [\,|v^k|^2\,u^k - |v^j|^2\,u^j\,]\,(\overline{u^k - u^j}) = [\,|v^k|^2\,(u^k - u^j) + u^j\,(|v^k|^2 - |v^j|^2)\,]\,(\overline{u^k - u^j})\\
&= |v^k|^2\,|u^k - u^j|^2 + u^j\,(\,|v^k| + |v^j|\,)\,(\,|v^k| - |v^j|\,)\,(\overline{u^k - u^j})
\end{aligned}
$$

then using Hölder's inequality and $|\,|a| - |b|\,| \leq |a - b|$ we obtain

$$
\begin{aligned}
& Im \int_{\mathbb{R}} [\,|v^k|^2\,u^k - |v^j|^2\,u^j\,]\,(\overline{u^k - u^j})\,dx\\
&= Im \int_{\mathbb{R}} u^j\,(\,|v^k| + |v^j|\,)\,(\,|v^k| - |v^j|\,)\,(\overline{u^k - u^j})\,dx\\
&\leq \int_{\mathbb{R}} |u^j|\,(\,|v^k| + |v^j|\,)\,|\,|v^k| - |v^j|\,|\,|u^k - u^j|\,dx\\
&\leq ||u^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})} + ||v^j||_{L^\infty(\mathbb{R})}\,) \int_{\mathbb{R}} |v^k - v^j|\,|u^k - u^j|\,dx\\
&\leq ||u^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})} + ||v^j||_{L^\infty(\mathbb{R})}\,)\,||u^k - u^j||_{L^2(\mathbb{R})}\,||v^k - v^j||_{L^2(\mathbb{R})}
\end{aligned}
$$

then

$$2\,Im\int_{\mathbb{R}}[\,|v^k|^2\,u^k-|v^j|^2\,u^j\,]\,(\overline{u^k-u^j})\,dx$$
$$\leq\quad ||u^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,(\,||u^k-u^j||^2_{L^2(\mathbb{R})}+||v^k-v^j||^2_{L^2(\mathbb{R})}\,). \tag{5.10}$$

Hence, by replacing (5.9) and (5.10) in (5.7) we obtain

$$\frac{d}{dt}\,||u^k-u^j||^2_{L^2(\mathbb{R})}$$
$$\leq\quad 2\,||u^k||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})}+||u^j||_{L^\infty(\mathbb{R})}\,)\,||u^k-u^j||^2_{L^2(\mathbb{R})}$$
$$+\,\beta\,||u^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,(\,||u^k-u^j||^2_{L^2(\mathbb{R})}+||v^k-v^j||^2_{L^2(\mathbb{R})}\,). \tag{5.11}$$

In a similar way, we estimate (5.8)

$$2\,Im\int_{\mathbb{R}}[\,|v^k|^2\,v^k-|v^j|^2\,v^j\,]\,(\overline{v^k-v^j})\,dx$$
$$\leq\quad 2\,||v^k||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,||v^k-v^j||^2_{L^2(\mathbb{R})}, \tag{5.12}$$

and

$$2\,Im\int_{\mathbb{R}}[\,|u^k|^2\,v^k-|u^j|^2\,v^j\,]\,(\overline{v^k-v^j})\,dx$$
$$\leq\quad ||v^j||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})}+||u^j||_{L^\infty(\mathbb{R})}\,)\,(\,||u^k-u^j||^2_{L^2(\mathbb{R})}+||v^k-v^j||^2_{L^2(\mathbb{R})}\,). \tag{5.13}$$

Replacing (5.12) and (5.13) in (5.8) we obtain

$$\frac{d}{dt}\,||\,v^k-v^j\,||^2_{L^2(\mathbb{R})}$$
$$\leq\quad 2\,||v^k||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,||v^k-v^j||^2_{L^2(\mathbb{R})}$$
$$+\,||v^j||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})}+||u^j||_{L^\infty(\mathbb{R})}\,)\,(\,||u^k-u^j||^2_{L^2(\mathbb{R})}+||v^k-v^j||^2_{L^2(\mathbb{R})}\,). \tag{5.14}$$

Differentiating (5.6) with respect to the x-variable we have the following equation:

$$\begin{aligned} i\,(u^k-u^j)_{xt}\quad &+\quad (u^k-u^j)_{xxx}+(\,|u^k|^2\,u^k-|u^j|^2\,u^j\,)_x \\ &+\quad \beta\,(\,|v^k|^2\,u^k-|v^j|^2\,u^j\,)_x\;=\;0. \end{aligned} \tag{5.15}$$

Multiplying (5.15) by $(\overline{u^k-u^j})_x$ we obtain the following identity and its conjugate:

$$i\,(\overline{u^k-u^j})_x\,(u^k-u^j)_{xt}+(\overline{u^k-u^j})_x\,(u^k-u^j)_{xxx}$$
$$+\,(\,|u^k|^2\,u^k-|u^j|^2\,u^j\,)_x\,(\overline{u^k-u^j})_x+\beta\,(\,|v^k|^2\,u^k-|v^j|^2\,u^j\,)_x\,(\overline{u^k-u^j})_x=0$$

$$-\,i\,(u^k-u^j)_x\,(\overline{u^k-u^j})_{xt}+(u^k-u^j)_x\,(\overline{u^k-u^j})_{xxx}$$
$$+\,\overline{(\,|u^k|^2\,u^k-|u^j|^2\,u^j\,)_x\,(\overline{u^k-u^j})_x}+\beta\,\overline{(\,|v^k|^2\,u^k-|v^j|^2\,u^j\,)_x\,(\overline{u^k-u^j})_x}=0.$$

Subtracting and integrating over $x \in \mathbb{R}$ we have

$$\frac{d}{dt}\,||\,(u^k - u^j)_x\,||^2_{L^2(\mathbb{R})} = -\,2\,Im\int_{\mathbb{R}}(\,|u^k|^2\,u^k - |u^j|^2\,u^j\,)_x\,(\overline{u^k - u^j})_x\,dx$$
$$-\,2\,\beta\,Im\int_{\mathbb{R}}(\,|v^k|^2\,u^k - |v^j|^2\,u^j\,)_x\,(\overline{u^k - u^j})_x\,dx = 0. \tag{5.16}$$

In a similar way and performing similar calculations we obtain

$$\frac{d}{dt}\,||\,(v^k - v^j)_x\,||^2_{L^2(\mathbb{R})} = -\,2\,Im\int_{\mathbb{R}}(\,|v^k|^2\,v^k - |v^j|^2\,v^j\,)_x\,(\overline{v^k - v^j})_x\,dx$$
$$-\,2\,\beta\,Im\int_{\mathbb{R}}(\,|u^k|^2\,v^k - |u^j|^2\,v^j\,)_x\,(\overline{v^k - v^j})_x\,dx. \tag{5.17}$$

In order to estimate (5.16), we have first that

$$\begin{aligned}
&(\,|u^k|^2\,u^k - |u^j|^2\,u^j\,)_x\,(\overline{u^k - u^j})_x\\
&\quad= \quad |u^k|^2\,|(u^k - u^j)_x|^2 + (\,|u^k|^2 - |u^j|^2\,)\,u^j_x\,(\overline{u^k - u^j})_x\\
&\qquad\quad + u^k\,[\,2\,Re(u^k_x\,\overline{u}^k) - 2\,Re(u^j_x\,\overline{u}^j)\,]\,(\overline{u^k - u^j})_x\\
&\qquad\quad + 2\,Re(u^j_x\,\overline{u}^j)\,(u^k - u^j)\,(\overline{u^k - u^j})_x\\
&\quad= \quad |u^k|^2\,|(u^k - u^j)_x|^2 + (\,|u^k| - |u^j|\,)\,(\,|u^k| + |u^j|\,)\,u^j_x\,(\overline{u^k - u^j})_x\\
&\qquad\quad + u^k\,[\,2\,Re(u^k_x\,\overline{u}^k) - 2\,Re(u^j_x\,\overline{u}^j)\,]\,(\overline{u^k - u^j})_x\\
&\qquad\quad + 2\,Re(u^j_x\,\overline{u}^j)\,(u^k - u^j)\,(\overline{u^k - u^j})_x,
\end{aligned}$$

then integrating and taking imaginary part in this last identity, we can estimate

$$\begin{aligned}
&Im\int_{\mathbb{R}}(\,|u^k|^2\,u^k - |u^j|^2\,u^j\,)_x\,(\overline{u^k - u^j})_x\,dx\\
&\quad\leq \int_{\mathbb{R}}|\,|u^k| - |u^j|\,|\,|\,|u^k| + |u^j|\,|\,|u^j_x|\,|(u^k - u^j)_x|\,dx\\
&\qquad + 2\int_{\mathbb{R}}|u^k|\,[\,|u^k_x|\,|u^k| - |u^j_x|\,|u^j|\,]\,|(u^k - u^j)_x|\,dx\\
&\qquad + 2\int_{\mathbb{R}}|u^j_x|\,|u^j|\,|u^k - u^j|\,|(u^k - u^j)_x|\,dx,
\end{aligned}$$

then using Hölder's inequality and $||a|-|b|| \leq |a-b|$ we have

$$
\begin{aligned}
& Im \int_{\mathbb{R}} (\, |u^k|^2\, u^k - |u^j|^2\, u^j\,)_x\, (\overline{u^k - u^j})_x\, dx \\
\leq\ & \int_{\mathbb{R}} |\, u^k - u^j\, |\, |\, |u^k| + |u^j|\, |\, |u^j_x|\, |(u^k - u^j)_x|\, dx \\
& + 2\, ||u^k||_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} [\, |u^k_x|\, |u^k| - |u^j_x|\, |u^j|\,]\, |(u^k - u^j)_x|\, dx \\
& + 2\, ||u^j||_{L^\infty(\mathbb{R})}\, ||u^k - u^j||_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |u^j_x|\, |(u^k - u^j)_x|\, dx \\
\leq\ & ||u^k - u^j||_{L^\infty(\mathbb{R})}\, (\, ||u^k||_{L^\infty(\mathbb{R})} + ||u^j||_{L^\infty(\mathbb{R})}\,)\, ||u^j_x||_{L^2(\mathbb{R})}\, ||(u^k - u^j)_x||_{L^2(\mathbb{R})} \\
& + 2\, ||u^k||_{L^\infty(\mathbb{R})}\, ||u^k||_{L^\infty(\mathbb{R})}\, ||u^k_x||_{L^2(\mathbb{R})}\, ||(u^k - u^j)_x||_{L^2(\mathbb{R})} \\
& + 2\, ||u^k||_{L^\infty(\mathbb{R})}\, ||u^j||_{L^\infty(\mathbb{R})}\, ||u^j_x||_{L^2(\mathbb{R})}\, ||(u^k - u^j)_x||_{L^2(\mathbb{R})} \\
& + 2\, ||u^j||_{L^\infty(\mathbb{R})}\, ||u^k - u^j||_{L^\infty(\mathbb{R})}\, ||u^j_x||_{L^2(\mathbb{R})}\, ||(u^k - u^j)_x||_{L^2(\mathbb{R})},
\end{aligned}
$$

then using $2ab \leq a^2 + b^2$, we have

$$
\begin{aligned}
& 2\, Im \int_{\mathbb{R}} (\, |u^k|^2\, u^k - |u^j|^2\, u^j\,)_x\, (\overline{u^k - u^j})_x\, dx \\
\leq\ & ||u^k - u^j||_{L^\infty(\mathbb{R})}\, (\, ||u^k||_{L^\infty(\mathbb{R})} + ||u^j||_{L^\infty(\mathbb{R})}\,)\, \left(||u^j_x||^2_{L^2(\mathbb{R})} + ||(u^k - u^j)_x||^2_{L^2(\mathbb{R})} \right) \\
& + 2\, ||u^k||_{L^\infty(\mathbb{R})}\, ||u^k||_{L^\infty(\mathbb{R})}\, \left(||u^k_x||^2_{L^2(\mathbb{R})} + ||(u^k - u^j)_x||^2_{L^2(\mathbb{R})} \right) \\
& + 2\, ||u^k||_{L^\infty(\mathbb{R})}\, ||u^j||_{L^\infty(\mathbb{R})}\, \left(||u^j_x||^2_{L^2(\mathbb{R})} + ||(u^k - u^j)_x||^2_{L^2(\mathbb{R})} \right) \\
& + 2\, ||u^j||_{L^\infty(\mathbb{R})}\, ||u^k - u^j||_{L^\infty(\mathbb{R})}\, \left(||u^j_x||^2_{L^2(\mathbb{R})} + ||(u^k - u^j)_x||^2_{L^2(\mathbb{R})} \right). \qquad (5.18)
\end{aligned}
$$

Continuing with the estimate of (5.16), we have

$$
\begin{aligned}
& (\, |v^k|^2\, u^k - |v^j|^2\, u^j\,)_x\, (\overline{u^k - u^j})_x \\
=\ & |v^k|^2\, |(u^k - u^j)_x|^2 + (\, |v^k|^2 - |v^j|^2\,)\, u^j_x\, (\overline{u^k - u^j})_x \\
& + u^k\, [\, 2\, Re(v^k_x\, \overline{v}^k) - 2\, Re(v^j_x\, \overline{v}^j)\,]\, (\overline{u^k - u^j})_x \\
& + 2\, Re(v^j_x\, \overline{v}^j)\, (u^k - u^j)\, (\overline{u^k - u^j})_x \\
=\ & |v^k|^2\, |(u^k - u^j)_x|^2 + (\, |v^k| - |v^j|\,)\, (\, |v^k| + |v^j|\,)\, u^j_x\, (\overline{u^k - u^j})_x \\
& + u^k\, [\, 2\, Re(v^k_x\, \overline{v}^k) - 2\, Re(v^j_x\, \overline{v}^j)\,]\, (\overline{u^k - u^j})_x \\
& + 2\, Re(v^j_x\, \overline{v}^j)\, (u^k - u^j)\, (\overline{u^k - u^j})_x
\end{aligned}
$$

then integrating and taking the imaginary part in this last identity, we can estimate

$$
\begin{aligned}
& Im \int_{\mathbb{R}} (\,|v^k|^2\, u^k - |v^j|^2\, u^j\,)_x\, (\overline{u^k - u^j})_x\, dx \\
\leq & \int_{\mathbb{R}} |\,|v^k| - |v^j|\,|\,|\,|v^k| + |v^j|\,|\,|u^j_x|\,|(u^k - u^j)_x|\, dx \\
& + \int_{\mathbb{R}} |u^k|\,[\,2\,|v^k_x|\,|v^k| - 2\,|v^j_x|\,|v^j|\,]\,|(u^k - u^j)_x|\, dx \\
& + 2\int_{\mathbb{R}} |v^j_x|\,|v^j|\,|u^k - u^j|\,|(u^k - u^j)_x|\, dx,
\end{aligned}
$$

then using Hölder's inequality and $||a| - |b|| \leq |a - b|$ we have

$$
\begin{aligned}
& Im \int_{\mathbb{R}} (\,|v^k|^2\, u^k - |v^j|^2\, u^j\,)_x\, (\overline{u^k - u^j})_x\, dx \\
\leq & \int_{\mathbb{R}} |\,v^k - v^j\,|\,(\,|v^k| + |v^j|\,)\,|u^j_x|\,|(u^k - u^j)_x|\, dx \\
& + 2\,||u^k||_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} [\,|v^k_x|\,|v^k| - |v^j_x|\,|v^j|\,]\,|(u^k - u^j)_x|\, dx \\
& + 2\,||v^j||_{L^\infty(\mathbb{R})}\,||u^k - u^j||_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |v^j_x|\,|(u^k - u^j)_x|\, dx \\
\leq & \; ||v^k - v^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})} + ||v^j||_{L^\infty(\mathbb{R})}\,)\,||u^j_x||_{L^2(\mathbb{R})}\,||(u^k - u^j)_x||_{L^2(\mathbb{R})} \\
& + 2\,||u^k||_{L^\infty(\mathbb{R})}\,||v^k||_{L^\infty(\mathbb{R})}\,||v^k_x||_{L^2(\mathbb{R})}\,||(u^k - u^j)_x||_{L^2(\mathbb{R})} \\
& + 2\,||u^k||_{L^\infty(\mathbb{R})}\,||v^j||_{L^\infty(\mathbb{R})}\,||v^j_x||_{L^2(\mathbb{R})}\,||(u^k - u^j)_x||_{L^2(\mathbb{R})} \\
& + 2\,||v^j||_{L^\infty(\mathbb{R})}\,||u^k - u^j||_{L^\infty(\mathbb{R})}\,||v^j_x|||_{L^2(\mathbb{R})}\,||(u^k - u^j)_x||_{L^2(\mathbb{R})},
\end{aligned}
$$

then using $2ab \leq a^2 + b^2$, we have

$$
\begin{aligned}
& 2\, Im \int_{\mathbb{R}} (\,|v^k|^2\, u^k - |v^j|^2\, u^j\,)_x\, (\overline{u^k - u^j})_x\, dx \\
\leq & \; ||v^k - v^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})} + ||v^j||_{L^\infty(\mathbb{R})}\,)\left(||u^j_x||^2_{L^2(\mathbb{R})} + ||(u^k - u^j)_x||^2_{L^2(\mathbb{R})}\right) \\
& + 2\,||u^k||_{L^\infty(\mathbb{R})}\,||v^k||_{L^\infty(\mathbb{R})}\left(||v^k_x||^2_{L^2(\mathbb{R})} + ||(u^k - u^j)_x||^2_{L^2(\mathbb{R})}\right) \\
& + 2\,||u^k||_{L^\infty(\mathbb{R})}\,||v^j||_{L^\infty(\mathbb{R})}\left(||v^j_x||^2_{L^2(\mathbb{R})} + ||(u^k - u^j)_x||^2_{L^2(\mathbb{R})}\right) \\
& + 2\,||v^j||_{L^\infty(\mathbb{R})}\,||u^k - u^j||_{L^\infty(\mathbb{R})}\left(||v^j_x|||^2_{L^2(\mathbb{R})} + ||(u^k - u^j)_x||^2_{L^2(\mathbb{R})}\right). \qquad (5.19)
\end{aligned}
$$

Replacing (5.18) and (5.19) in (5.16) we have

$$
\begin{aligned}
&\frac{d}{dt}\,||\,(u^k-u^j)_x\,||^2_{L^2(\mathbb{R})}\\
\leq\;& ||u^k-u^j||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})}+||u^j||_{L^\infty(\mathbb{R})}\,)\,\Big(\,||u^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\,\Big)\\
+\;& 2\,||u^k||_{L^\infty(\mathbb{R})}\,||u^k||_{L^\infty(\mathbb{R})}\,\Big(||u^k_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& 2\,||u^k||_{L^\infty(\mathbb{R})}\,||u^j||_{L^\infty(\mathbb{R})}\,\Big(||u^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& 2\,||u^j||_{L^\infty(\mathbb{R})}\,||u^k-u^j||_{L^\infty(\mathbb{R})}\,\Big(||u^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& \beta\,||v^k-v^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,\Big(||u^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& 2\,\beta\,||u^k||_{L^\infty(\mathbb{R})}\,||v^k||_{L^\infty(\mathbb{R})}\,\Big(||v^k_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& 2\,\beta\,||u^k||_{L^\infty(\mathbb{R})}\,||v^j||_{L^\infty(\mathbb{R})}\,\Big(||v^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
&+2\,\beta\,||v^j||_{L^\infty(\mathbb{R})}\,||u^k-u^j||_{L^\infty(\mathbb{R})}\,\Big(||v^j_x|||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\Big)\,. \qquad (5.20)
\end{aligned}
$$

In a similar way and performing similar calculations we obtain in (5.17) that

$$
\begin{aligned}
&\frac{d}{dt}\,||\,(v^k-v^j)_x\,||^2_{L^2(\mathbb{R})}\\
\leq\;& ||v^k-v^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,\Big(\,||v^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\,\Big)\\
+\;& 2\,||v^k||_{L^\infty(\mathbb{R})}\,||v^k||_{L^\infty(\mathbb{R})}\,\Big(||v^k_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& 2\,||v^k||_{L^\infty(\mathbb{R})}\,||v^j||_{L^\infty(\mathbb{R})}\,\Big(||v^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& 2\,||v^j||_{L^\infty(\mathbb{R})}\,||v^k-v^j||_{L^\infty(\mathbb{R})}\,\Big(||v^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& \beta\,||u^k-u^j||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})}+||u^j||_{L^\infty(\mathbb{R})}\,)\,\Big(||v^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& 2\,\beta\,||v^k||_{L^\infty(\mathbb{R})}\,||u^k||_{L^\infty(\mathbb{R})}\,\Big(||u^k_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& 2\,\beta\,||v^k||_{L^\infty(\mathbb{R})}\,||u^j||_{L^\infty(\mathbb{R})}\,\Big(||u^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\Big)\\
+\;& 2\,\beta\,||u^j||_{L^\infty(\mathbb{R})}\,||v^k-v^j||_{L^\infty(\mathbb{R})}\,\Big(||u^j_x|||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\Big)\,. \qquad (5.21)
\end{aligned}
$$

Finally, using (5.11), (5.14), (5.20) and (5.21) we have

$$
\begin{aligned}
&\frac{d}{dt}\left[\,||\,u^k-u^j\,||^2_{H^1(\mathbb{R})}+||\,v^k-v^j\,||^2_{H^1(\mathbb{R})}\right]\\
\leq\;& 2\,||u^k||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})}+||u^j||_{L^\infty(\mathbb{R})}\,)\,||u^k-u^j||^2_{L^2(\mathbb{R})}\\
+\;& 2\,||v^k||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,||v^k-v^j||^2_{L^2(\mathbb{R})}\\
+\;& ||v^j||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})}+||u^j||_{L^\infty(\mathbb{R})}\,)\,(\,||u^k-u^j||^2_{L^2(\mathbb{R})}+||v^k-v^j||^2_{L^2(\mathbb{R})}\,)\\
+\;& ||u^k-u^j||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})}+||u^j||_{L^\infty(\mathbb{R})}\,)\,\left(\,||u^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\,\right)\\
+\;& 2\,||u^k||_{L^\infty(\mathbb{R})}\,||u^k||_{L^\infty(\mathbb{R})}\,\left(||u^k_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,||u^k||_{L^\infty(\mathbb{R})}\,||u^j||_{L^\infty(\mathbb{R})}\,\left(||u^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,||u^j||_{L^\infty(\mathbb{R})}\,||u^k-u^j||_{L^\infty(\mathbb{R})}\,\left(||u^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& ||v^k-v^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,\left(\,||v^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\,\right)\\
+\;& 2\,||v^k||_{L^\infty(\mathbb{R})}\,||v^k||_{L^\infty(\mathbb{R})}\,\left(||v^k_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,||v^k||_{L^\infty(\mathbb{R})}\,||v^j||_{L^\infty(\mathbb{R})}\,\left(||v^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,||v^j||_{L^\infty(\mathbb{R})}\,||v^k-v^j||_{L^\infty(\mathbb{R})}\,\left(||v^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& \beta\,||u^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,(\,||u^k-u^j||^2_{L^2(\mathbb{R})}+||v^k-v^j||^2_{L^2(\mathbb{R})}\,)\\
+\;& \beta\,||v^k-v^j||_{L^\infty(\mathbb{R})}\,(\,||v^k||_{L^\infty(\mathbb{R})}+||v^j||_{L^\infty(\mathbb{R})}\,)\,\left(||u^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,\beta\,||u^k||_{L^\infty(\mathbb{R})}\,||v^k||_{L^\infty(\mathbb{R})}\,\left(||v^k_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,\beta\,||u^k||_{L^\infty(\mathbb{R})}\,||v^j||_{L^\infty(\mathbb{R})}\,\left(||v^j_x||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,\beta\,||v^j||_{L^\infty(\mathbb{R})}\,||u^k-u^j||_{L^\infty(\mathbb{R})}\,\left(||v^j_x|||^2_{L^2(\mathbb{R})}+||(u^k-u^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& \beta\,||u^k-u^j||_{L^\infty(\mathbb{R})}\,(\,||u^k||_{L^\infty(\mathbb{R})}+||u^j||_{L^\infty(\mathbb{R})}\,)\,\left(||v^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,\beta\,||v^k||_{L^\infty(\mathbb{R})}\,||u^k||_{L^\infty(\mathbb{R})}\,\left(||u^k_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,\beta\,||v^k||_{L^\infty(\mathbb{R})}\,||u^j||_{L^\infty(\mathbb{R})}\,\left(||u^j_x||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\right)\\
+\;& 2\,\beta\,||u^j||_{L^\infty(\mathbb{R})}\,||v^k-v^j||_{L^\infty(\mathbb{R})}\,\left(||u^j_x|||^2_{L^2(\mathbb{R})}+||(v^k-v^j)_x||^2_{L^2(\mathbb{R})}\right)
\end{aligned}
\tag{5.22}
$$

hence,

$$
\frac{d}{dt}\left[\,||u^k-u^j||^2_{H^1(\mathbb{R})}+||v^k-v^j||^2_{H^1(\mathbb{R})}\right]\leq c_0+c_1\left[\,||u^k-u^j||^2_{H^1(\mathbb{R})}+||v^k-v^j||^2_{H^1(\mathbb{R})}\right].
$$

Now, using Gronwall's inequality, the result follows. □

Lemma 5.2. *For* $m=1,\,2,\,3,\,\ldots,\,n$, $\{(J^m u^k,\,J^m v^k)\}$ *is a Cauchy sequence in*

$C([0,\,T]:\,L^2(\mathbb{R}))\times C([0,\,T]:\,L^2(\mathbb{R}))$

$$\begin{aligned}
&||\,J^m u^k - J^m u^j\,||^2_{L^2(\mathbb{R})} + ||\,J^m v^k - J^m v^j\,||^2_{L^2(\mathbb{R})}\\
&\le\quad c(T)\,\Big[\,||\,x^m\,u_0^k - x^m\,u_0^j\,||^2_{L^2(\mathbb{R})} + ||\,x^m\,v_0^k - x^m\,v_0^j\,||^2_{L^2(\mathbb{R})}\Big] \qquad (5.23)
\end{aligned}$$

for any $T>0$, where $c(T)$ is a positive constant independent of k and j.
Proof. Let $(u^k,\,v^k)$ be the solution of (1.1)-(1.4), then applying J^m to (5.2)-(5.5) we have

$$i\,(J^m u^k)_t + (J^m u^k)_{xx} + J^m(|u^k|^2\,u^k) + \beta\,J^m(|v^k|^2 u^k) = 0 \qquad (5.24)$$
$$i\,(J^m v^k)_t + (J^m v^k)_{xx} + J^m(|v^k|^2 v^k) + \beta\,J^m(|u^k|^2 v^k) = 0 \qquad (5.25)$$
$$i\,(J^m u^j)_t + (J^m u^j)_{xx} + J^m(|u^j|^2 u^j) + \beta\,J^m(|v^j|^2 u^j) = 0 \qquad (5.26)$$
$$i\,(J^m v^j)_t + (J^m v^j)_{xx} + J^m(|v^j|^2 v^j) + \beta\,J^m(|u^j|^2 v^j) = 0. \qquad (5.27)$$

Subtracting (5.24) with (5.26) we have

$$\begin{aligned}
&i\,(J^m u^k - J^m u^j)_t + (J^m u^k - J^m u^j)_{xx}\\
&\quad + J^m(|u^k|^2 u^k) - J^m(|u^j|^2 u^j) + \beta\,J^m(|v^k|^2 u^k) - \beta\,J^m(|v^j|^2 u^j) = 0 \qquad (5.28)
\end{aligned}$$

Multiplying (5.28) by $(\overline{J^m u^k - J^m u^j})$ we have

$$\begin{aligned}
&i\,(\overline{J^m u^k - J^m u^j})\,(J^m u^k - J^m u^j)_t + (\overline{J^m u^k - J^m u^j})\,(J^m u^k - J^m u^j)_{xx}\\
&\quad + (\overline{J^m u^k - J^m u^j})\,[J^m(|u^k|^2 u^k) - J^m(|u^j|^2\,u^j)\,]\\
&\quad + \beta\,(\overline{J^m u^k - J^m u^j})\,[J^m(|v^k|^2 u^k) -\ J^m(|v^j|^2 u^j)] = 0
\end{aligned}$$

$$\begin{aligned}
&-\,i\,(J^m u^k - J^m u^j)\,(\overline{J^m u^k - J^m u^j})_t + (J^m u^k - J^m u^j)\,(\overline{J^m u^k - J^m u^j})_{xx}\\
&\quad + \overline{(\overline{J^m u^k - J^m u^j})\,[J^m(|u^k|^2 u^k) - J^m(|u^j|^2 u^j)]}\\
&\quad + \overline{\beta\,(\overline{J^m u^k - J^m u^j})\,[J^m(|v^k|^2\,u^k) -\ J^m(|v^j|^2 u^j)]} = 0. \qquad \text{(applying conjugate)}
\end{aligned}$$

Subtracting and integrating over $x\in\mathbb{R}$ we have

$$\begin{aligned}
&\frac{d}{dt}\,||J^m u^k - J^m u^j||^2_{L^2(\mathbb{R})}\\
&=\quad -\,2\,Im\int_{\mathbb{R}}[J^m(|u^k|^2 u^k) - J^m(|u^j|^2 u^j)]\,(\overline{J^m u^k - J^m u^j})\,dx\\
&\qquad -\,2\,\beta\,Im\int_{\mathbb{R}}[J^m(|v^k|^2 u^k) -\ J^m(|v^j|^2 u^j)](\overline{J^m u^k - J^m u^j})\,dx. \qquad (5.29)
\end{aligned}$$

In a similar way we obtain

$$\begin{aligned}
&\frac{d}{dt}\,||J^m v^k - J^m v^j||^2_{L^2(\mathbb{R})}\\
&=\quad -\,2\,Im\int_{\mathbb{R}}[\,J^m(|v^k|^2 v^k) - J^m(|v^j|^2 v^j)\,](\overline{J^m v^k - J^m v^j})\,dx\\
&\qquad -\,2\,\beta\,Im\int_{\mathbb{R}}[J^m(|u^k|^2 v^k) -\ J^m(|u^j|^2 v^j)\,](\overline{J^m v^k - J^m v^j})\,dx. \qquad (5.30)
\end{aligned}$$

We estimate now (5.29) using Hölder's inequality

$$
\begin{aligned}
&\frac{d}{dt}\,||J^m u^k - J^m u^j||^2_{L^2(\mathbb{R})}\\
&= \; -2\,Im\int_{\mathbb{R}}[J^m(|u^k|^2u^k) - J^m(|u^j|^2u^j)]\,(\overline{J^m u^k - J^m u^j}\,)\,dx\\
&\quad -2\,\beta\,Im\int_{\mathbb{R}}[J^m(|v^k|^2\,u^k) - \,J^m(|v^j|^2u^j)\,]\,(\overline{J^m u^k - J^m u^j})\,dx\\
&\le \; 2\,||J^m(|u^k|^2u^k) - J^m(|u^j|^2u^j)||_{L^2(\mathbb{R})}\,||J^m u^k - J^m u^j||_{L^2(\mathbb{R})}\\
&\quad + 2\,\beta\,||J^m(|v^k|^2u^k) - \,J^m(|v^j|^2\,u^j)||_{L^2(\mathbb{R})}\,||J^m u^k - J^m u^j||_{L^2(\mathbb{R})}\\
&\le \; 2\,\big(\,||J^m(|u^k|^2u^k)||_{L^2(\mathbb{R})} + ||J^m(|u^j|^2\,u^j)||_{L^2(\mathbb{R})}\,\big)\,||J^m u^k - J^m u^j||_{L^2(\mathbb{R})}\\
&\quad +2\,\beta\,\big(\,||J^m(|v^k|^2\,u^k)||_{L^2(\mathbb{R})} + ||J^m(|v^j|^2\,u^j)||_{L^2(\mathbb{R})}\,\big)\,||J^m u^k - J^m u^j||_{L^2(\mathbb{R})}.
\end{aligned}
$$

Using (4.14) and (4.21) we have

$$
\begin{aligned}
&\frac{d}{dt}\,||J^m u^k - J^m u^j||^2_{L^2(\mathbb{R})}\\
\le\;& 2\,\big(\,||J^m(|u^k|^2\,u^k)||_{L^2(\mathbb{R})} + ||J^m(|u^j|^2u^j)||_{L^2(\mathbb{R})}\,\big)\;||J^m u^k - J^m u^j||_{L^2(\mathbb{R})}\\
&+2\beta\,\big(\,||J^m(|v^k|^2u^k)||_{L^2(\mathbb{R})} + ||J^m(|v^j|^2u^j)||_{L^2(\mathbb{R})}\,\big)\,||J^m u^k - J^m u^j||_{L^2(\mathbb{R})}\\
\le\;& 2\,\big(c_1\,||J^m u^k||_{L^2(\mathbb{R})}||u^k||_{L^\infty(\mathbb{R})} + c_2\,||J^m u^j||_{L^2(\mathbb{R})}||u^j||_{L^\infty(\mathbb{R})}\big)\,||J^m u^k - J^m u^j||_{L^2(\mathbb{R})}\\
+\;& 2\beta\left(c_3\sum_{p+q+r=m}\left[\frac{(p+q)}{m}\,||J^m v^k||_{L^2(\mathbb{R})} + \frac{r}{m}\,||J^m v^k||_{L^2(\mathbb{R})}\right]||v^k||^{[\,2m-(p+q)\,]/m}_{L^\infty(\mathbb{R})}||u^k||^{(m-r)/m}_{L^\infty(\mathbb{R})}\right)\\
\times\;& ||J^m u^k - J^m u^j||_{L^2(\mathbb{R})}\\
+\;& 2\,\beta\left(c_4\sum_{p+q+r=m}\left[\frac{(p+q)}{m}\,||J^m v^j||_{L^2(\mathbb{R})} + \frac{r}{m}\,||J^m v^j||_{L^2(\mathbb{R})}\right]||v^j||^{[\,2\,m-(p+q)\,]/m}_{L^\infty(\mathbb{R})}||u^j||^{(m-r)/m}_{L^\infty(\mathbb{R})}\right)\\
\times\;& ||J^m u^k - J^m u^j||_{L^2(\mathbb{R})}.
\end{aligned}
$$

Using that $2ab \le a^2+b^2$, (4.1), (4.9) and by a straightforward calculation we obtain

$$\frac{d}{dt}\,||J^m u^k - J^m u^j||^2_{L^2(\mathbb{R})} \le c(T) + ||J^m u^k - J^m u^j||^2_{L^2(\mathbb{R})}.$$

In a similar way we obtain

$$\frac{d}{dt}\,||J^m u^k - J^m v^j||^2_{L^2(\mathbb{R})} \le c(T) + ||J^m v^k - J^m v^j||^2_{L^2(\mathbb{R})}$$

from where (5.23) follows. □

Remark. If the assumption (1.10) holds, then

$$e^{i\,x^2/4\,t}u \in C(\mathbb{R}\backslash\{0\}\,:\,H^m(\mathbb{R})) \qquad \text{and} \qquad e^{i\,x^2/4\,t}v \in C(\mathbb{R}\backslash\{0\}\,:\,H^m(\mathbb{R}))$$

Proof of the Main Theorem. From Lemma 5.1 and Lemma 5.2 we obtain that there exists $u = u(x, t)$ and $v = v(x, t)$ satisfying (1.5) and (1.6) and such that for any $T > 0$ we have

$$\begin{array}{llll} u^k & \longrightarrow & u & \text{strongly in} \quad C(\mathbb{R} : H^1(\mathbb{R})) \\ v^k & \longrightarrow & v & \text{strongly in} \quad C(\mathbb{R} : H^1(\mathbb{R})) \end{array}$$

and

$$\begin{array}{llll} J^m u^k & \longrightarrow & J^m u & \text{strongly in} \quad C(\mathbb{R} : L^2(\mathbb{R})) \\ J^m v^k & \longrightarrow & J^m v & \text{strongly in} \quad C(\mathbb{R} : L^2(\mathbb{R})). \end{array}$$

It is easily verified that (u, v) solves (1.1)-(1.4) and satisfies (1.7)-(1.10). The proof then follows. □

Corollary 5.1. *If the hypotheses in Theorem 5.3 are satisfied, then*

$$u \in \bigcap_{m=0}^{[\frac{n}{2}]} C^m(\mathbb{R}\backslash\{0\} : C^{n-2\,m-1}(\mathbb{R})) \qquad \textit{and} \qquad v \in \bigcap_{m=0}^{[\frac{n}{2}]} C^m(\mathbb{R}\backslash\{0\} : C^{n-2\,m-1}(\mathbb{R}))$$

Corollary 5.2. *If* $(x^n\, u_0,\, x^n\, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ *for all* $n \in \mathbb{N}$ *in Theorem 5.3, then the solution* (u, v) *of* (1.1)-(1.4) *is infinitely differentiable in* x *and* t *for* $t \neq 0$.

Acknowledgments

The authors are grateful to Rolci Cipolatti(IM-Universidade Federal do Rio de Janeiro) whose comments greatly improved this work.

MS has been supported by Fondecyt project # 1070694 and Fondap in Applied Mathematics (Project # 15000001). OV and MS acknowledge support by CNPq/CONICYT Project, # 490987/2005-2 (Brazil) and # 2005-075 (Chile). Finally,we want to thank Fernando Miranda for his corrections and suggestions about the spelling and grammar of English in this paper and others.

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