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Interior gradient estimate for 1-D anisotropic curvature flow

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ABSTRACT: We establish the interior gradient estimate for general 1-D anisotropic curvature flow. The estimate depends only on the height of the graph and not on the gradient at initial time. The proof relies on the monotonicity property of the number of zeros for the parabolic equation.

Key Words: curvature flow, anisotropic curvature, gradient estimate.

Contents

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$. A surface given as a graph $u : \Omega \to \mathbb{R}$ is a minimal surface when u satisfies

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{1.1}$$

For this equation, the following interior gradient estimates are well-known ([6,8,9]): Given a constant M and $\tilde{\Omega} \subset\subset \Omega$, there exists a constant C depending only on M and $\tilde{\Omega}$ such that if $\sup_{\Omega} |u| \leq M$, then $\sup_{\tilde{\Omega}} |\nabla u| \leq C$. The standard elliptic theory ([5]) subsequently gives all the interior $C^{k,\alpha}(\tilde{\Omega})$ estimates of the graph u which depends only on M and $\tilde{\Omega}$. The similar estimates are also known for the mean curvature flow equation ([3]). That is, if $u : \Omega \times (0, T) \to \mathbb{R}$ satisfies

$$\frac{u_t}{\sqrt{1+|\nabla u|^2}} = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \tag{1.2}$$

and $\sup_{\Omega\times[0,T]} |u| \leq M$, $\tilde{\Omega} \subset\subset \Omega$, $0 < T_0 < T$, then there exists C such that $\sup_{\tilde{\Omega}\times[T_0,T]} |\nabla u| \leq C$. Again, C is a constant depending only on M, $\tilde{\Omega}$ and T_0. Note that C is independent of the gradient at $t = 0$.

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One direction to extend those results are to consider general anisotropic curvature problem, namely, to consider the variational problem corresponding to the energy functional

$$F(u) = \int_\Omega a(\nu)\sqrt{1+|\nabla u|^2},$$

where $\nu = (\nabla u, -1)/\sqrt{1+|\nabla u|^2}$ is the unit normal vector to the graph of u and the function $a : \mathbf{S}^{n-1} \to \mathbf{R}^+$ is the surface energy density and should satisfy certain convexity property. The Euler-Lagrange equation is

$$\mathrm{div}_p a_p(\nu) = 0, \tag{1.3}$$

and the curvature flow equation is

$$\frac{u_t}{\sqrt{1+|\nabla u|^2}} = \mathrm{div}_p a_p(\nu). \tag{1.4}$$

For both elliptic and parabolic problems in general dimensions, it is not known if the similar interior estimates for (1.1) and (1.2) hold equally for the anisotropic equations (1.3) and (1.4) so far. The main reason for the difficulty to extend the results is the lack of monotonicity formula for the mass ratio, which plays an important role in the measure-theoretic treatment of minimal submanifolds. On the other hand, if one allows the interior gradient estimate to depend on the gradient at $t = 0$, then the argument of [7] gives the interior gradient estimate.

In this paper, we show the interior gradient estimates for general anisotropic curvature flow for one-dimensional case which is independent of the initial time gradient. The proof utilizes the result of Angenent ([2]), which says that the number of zeros of the solution of parabolic equations is nonincreasing as time increases. We compare the solution to those for a suitable heat equation and use this result. We utilize the fact that the equation is invariant under the rotation here. We remark that the method we use is valid only for 1 dimensional case.

2. Theorem

Let $r > 0$ be given. The graph $u : [-r, r] \times [0, T] \to \mathbb{R}$ is said to be an anisotropic curvature flow if smooth function u satisfies

$$\frac{u_t}{\sqrt{1+u_x^2}} = (a_p(u_x, -1))_x. \tag{2.1}$$

where $a : \mathbf{R}^2 \to [0, \infty)$ is an anisotropic surface energy density function satisfying the following assumptions:

(a) $a(tp, tq) = t\,a(p, q)$ for all $t > 0$,

(b) a is a convex function,

(c) there exists $\delta_0 > 0$ such that $a(p, q) - \delta_0|(p, q)|$ is a convex function,

(d) a is smooth except at $(0, 0)$.

The left-hand side of the equation corresponds to the normal velocity of the curve $(x, u(x,\cdot))$ while the right-hand side is the weighted anisotropic curvature. This is a gradient flow of the anisotropic surface energy functional

$$\int_{-r}^{r} a(\nu)\, ds,$$

where $ds = \sqrt{1+u_x^2}dx$ and $\nu = (-u_x, 1)/\sqrt{1+u_x^2}$ with homogeneous Dirichlet $(u = 0)$ or Neumann $(a_p(-u_x, 1) = 0)$ boundary conditions, since

$$\frac{d}{dt}\int_{-r}^{r} a(\nu)\, ds = \int_{-r}^{r} a_p(-u_x, 1)u_{xt}\, dx = -\int_{-r}^{r} |(a_p(-u_x, 1))_x|^2\, ds.$$

Under these assumptions, we show

Theorem 2.1 *Suppose u is a smooth solution of* (2.1) *on* $[-r, r] \times [0, T]$ *satisfying*

$$\sup_{[-r,r]\times[0,T]} |u| \leq M.$$

Given $0 < s < r$ *and* $0 < t_0 < T$*, there exists a constant* $C > 0$ *depending only on* δ_0, M, t_0, s, r *such that*

$$\sup_{[-(r-s),r-s]\times[t_0,T]} |u_x| \leq C.$$

Note that the estimate is independent the gradient of the initial data. Also we point out that the dependence of C on a is only through the lower bound of the uniform convexity δ_0, but not on the upper bound (such as C^1 bound). Thus, the result in this paper can be extended equally to the non-smooth anisotropic curvature flow problem [4] by approximations. Before the proof, we cite the following theorem due to Angenent [2] which is crucial in the proof:

Lemma 2.1 *Suppose* $u \in C^\infty([x_1, x_2] \times [0, T])$ *satisfies the equation*

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u \tag{2.2}$$

on $[x_1, x_2] \times [0, T]$ *and*

$$u(x_j, t) \neq 0 \text{ for } t \in [0, T] \qquad j = 1, 2.$$

Here, a, b, c *are smooth functions of* (x, t) *and* $a > 0$*. Then for all* $t \in (0, T]$*, the zero set of* $x \to u(x, t)$ *will be finite, even when counted with multiplicity. The number of zeros of* $x \to u(x, t)$ *counted with multiplicity is nonincreasing function of* t*.*

Proof of Theorem. Given $0 < s < r$ and $0 < t_0 < T$, we construct a solution v for (2.1) on $[-s, s] \times (0, T]$ with the following properties:

(a) $v(-s, t) = -M - 1$ and $v(s, t) = M + 1$ for $0 < t \leq T$,

(b) $v_x > 0$ on $[-s, s] \times (0, T]$,

(c) for any $-s < x \le s$, $\lim_{t\to 0} v(x, t) > M$.

The property (c) means that v has an initial data which is vertical at $x = -s$. We show that the function v has a gradient bound $0 < v_x \le C$ on $[-s, s] \times [t_0, T]$, where C depends only on M, δ_0, s, t_0. We show the existence of such v later in the proof. Assuming such v exists for now, we then prove that any solution with $\sup_{[-r,r]\times[0,T]} |u| \le M$ satisfies $\sup_{[-(r-s),r-s]\times[t_0,T]} u_x \le C$. The same argument using $-u$ will show $\sup_{[-(r-s),r-s]\times[t_0,T]} |u_x| \le C$. For a contradiction, assume that there exists a point $(\bar{x}, \bar{t}) \in [-(r-s), r-s] \times [t_0, T]$ with $u_x(\bar{x}, \bar{t}) > C$. Since $\sup |u| \le M$ and by (a), we may choose λ so that $|\bar{x}-\lambda| < s$ and $v(\bar{x}-\lambda, \bar{t}) = u(\bar{x}, \bar{t})$. With this λ, define $v_\lambda(x, t) = v(x - \lambda, t)$. Since $u_x(\bar{x}, \bar{t}) > C \ge (v_\lambda)_x(\bar{x}, \bar{t})$ and $v_\lambda(\lambda + s, \bar{t}) = v(s, \bar{t}) = M + 1 > u(\lambda + s, \bar{t})$, there has to be at least another point $\bar{x} < \tilde{x} < \lambda + s$ such that $u(\tilde{x}, \bar{t}) = v_\lambda(\tilde{x}, \bar{t})$. Thus $u - v_\lambda$ has at least two zeros at $t = \bar{t}$ on $\lambda - s < x < \lambda + s$. Function $u - v_\lambda$ satisfies the equation of the type (2.2) on $[\lambda - s, \lambda + s] \times (0, T]$, with non-zero boundary values for all $t > 0$ due to $\sup |u| \le M$ and (a). Thus we may use Lemma 1 and conclude that $u - v_\lambda$ has at least two zeros in x variable for all $\bar{t} > t > 0$. Since $v_\lambda > M$ for x away from $\lambda - s$ and all small t, and since we assume that u is a smooth function up to $t = 0$, this is impossible to satisfy for all small enough t.

Thus it remains to prove the existence of such v. To do this, we invert the role of independent variable x and dependent variable $y = v(x, t)$. Let $y = w(x, t)$ be the inverse function of v with respect to the space variables, i.e., w satisfies $y = v(w(y, t), t)$ identically. Since the equation is geometric, w should satisfy the similar equation to (2.1) on $[-M - 1, M + 1] \times (0, T]$ with the role of y and x exchanged. Now, the conditions on v in terms of w are

(a') $w(-M - 1, t) = -s$ and $w(M + 1, t) = s$ for $0 < t \le T$,

(b') $w_x > 0$ on $[-M - 1, M + 1] \times (0, T]$,

(c') for any $-M - 1 \le x \le M$, $\lim_{t\to 0} w(x, t) = -s$.

Furthermore, on $[-M - 1, M + 1] \times (0, T]$, w should satisfy

$$\frac{w_t}{\sqrt{1 + w_x^2}} = (a_q(1, w_x))_x. \tag{2.3}$$

Since $\frac{\partial y}{\partial x} = 1/\frac{\partial x}{\partial y}$, we need to show that there exists a constant $C > 0$ such that $w_x > C$ on $[-M, M] \times [t_0, T]$. We solve (2.3) with the following convex initial data. Let $\Gamma \in C^\infty([-M - 1, M + 1])$ be

- $\Gamma(x) = -s$ for $x \in [-M - 1, M]$,
- $\Gamma(M + 1) = s$, $\Gamma''(M + 1) = 0$,
- $\Gamma(x) \ge -s$, $\Gamma'(x) \le 3s$, $\Gamma''(x) \ge 0$ for $x \in [M, M + 1]$.

Let w be the unique smooth solution of (2.3) with the initial data Γ and the boundary data (a'). Since any functions $c_1 + c_2 x$ are solutions of (2.3), one obtains the gradient estimate

$$0 \leq w_x \leq 3s \tag{2.4}$$

on $[-M-1, M+1] \times [0,T]$, by using these functions as barriers and the standard maximum principle applied to w_x. Also, note that the convexity of w is preserved, i.e., $w_{xx} \geq 0$. This is seen by differentiating the equation with respect to t and then applying the maximum principle to w_t. $w_t = 0$ on the boundary and $w_t = a_{qq} w_{xx} \geq 0$ for $t = 0$ imply $w_t \geq 0$. The equation then yields $w_{xx} \geq 0$ on $[-M-1, M+1] \times [0,T]$.

Now, (2.4) implies that $a_{qq}(-1, w_x) \geq c(s, \delta_0)$(call this δ)> 0 by assumption (c). We claim that the solution of

$$\begin{cases} z_t = \delta z_{xx} & [-M-1, M+1] \times [0,T], \\ z(\pm(M+1), t) = \pm s & t \in [0,T], \\ z(x,0) = \Gamma(x) & x \in [-M-1, M+1] \end{cases}$$

satisfies $w \geq z$ on $[-M-1, M+1] \times [0,T]$. This is because of the following combined with the standard maximum principle:

$$\begin{aligned}(w-z)_t = a_{qq}(-1, w_x) w_{xx} - \delta z_{xx} &= a_{qq}(-1, w_x)(w-z)_{xx} + (a_{qq}(-1, w_x) - \delta) z_{xx} \\ &\geq a_{qq}(-1, w_x)(w-z)_{xx}.\end{aligned}$$

In the last line, we used $z_{xx} \geq 0$, which follows by the same reason for $w_{xx} \geq 0$ before, and $a_{qq}(-1, w_x) \geq \delta$. We next claim that for $t_0 \leq t$, there exists $c = c(t_0, s, \delta) > 0$ such that $z_x \geq c$ on $[-M-1, M+1] \times [t_0, T]$. z_x satisfies again the heat equation with non-negative initial data and the homogeneous Neumann data, and thus by the strong maximum principle (or extending the solution to $\mathbb{R}$ by a suitable reflection argument and then using the representation formula with the heat kernel) we have such c. Since $w_{xx} \geq 0$, for (x,t) with $t \geq t_0$, we have

$$w_x(x,t) \geq w_x(-M-1, t) \geq z_x(-M-1, t) \geq c$$

as the result. Note that we are using $w \geq z$ and $w = z$ on the boundary $x = -M-1$. This completes the proof.

Remark 2.1 *After completing the manuscript, we learned that Julie Clutterbuck at Australian National University has recently considered the problem and obtained the similar interior estimates.*

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