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A Note on Indefinite Ternary Quadratic Forms Representing All Odd Integers

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ABSTRACT: In this paper we determine, up to equivalence, all the indefinite ternary quadratic forms over $\mathbb{Z}$ that represent all odd integers.

Key Words: Quadratic Forms, indefinite ternary quadratic.

Contents

1. Introduction

The problem of representing integers by ternary quadratic forms has a long and venerable history. The best known of these problems is determining which integers can be written in the form $x^2 + y^2 + z^2$, which was solved by Gauss [8, p. 79].

Another interesting classical example is the form $x^2 + y^2 + 2z^2$, studied by Fermat, Legendre and Torelli [3, p. 282], [4, p. 224], which represents all odd positive integers. I. Kaplansky [5] found all positive-definite ternary quadratic forms that share this property. In this note, we deal with the equivalent question for *indefinite* forms, which was also raised by Kaplansky in an unpublished note [6].

The classification of the indefinite ternary forms that represent all odd integers turns out to be simpler than in the definite situation, as these forms happen to have always class number one, as shown below, and hence determined by the "obvious" local conditions. As particular cases, we find the so-called *universal* forms, i.e. forms that represent all integers, that were already described by Dickson in the 1930's.

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2. Notation and terminology

Let B be a symmetric $n \times n$ matrix with coefficients in $\mathbb{Q}$. The associated quadratic form on $\mathbb{Z}^n$ is

$$q(\mathbf{x}) = \mathbf{x}^t B\mathbf{x}.$$

We say that q is *integral* if it takes values in $\mathbb{Z}$. This is equivalent to $2B$ being an integer matrix with even coefficients on the diagonal. We say that q is *classically integral* if B has integer coefficients.

The *symmetric bilinear form* of q is defined by

$$\beta(\mathbf{x}, \mathbf{y}) = q(\mathbf{x} + \mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y}) \tag{1}$$

and is related to B by

$$\beta(\mathbf{x}, \mathbf{y}) = 2\mathbf{x}^t B\mathbf{y}.$$

The *determinant* of q is defined by $d(q) = \det B$. We say that q is *nondegenerate* if $d(q) \neq 0$. All the quadratic forms in this paper will be assumed nondegenerate.

Let R be a commutative ring containing $\mathbb{Z}$. Two quadratic forms q and q' as above are *R-equivalent* if there exists $U \in \mathbf{GL}_n(R)$ such that $q(U\mathbf{x}) = q'(\mathbf{x})$. We shall denote this equivalence by $q \simeq_R q'$. The rings R relevant for this paper are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{R}$.

Recall that q and q' are said to be in the *same genus* if $q \simeq_{\mathbb{Z}_p} q'$ for all prime numbers p and $q \simeq_{\mathbb{R}} q'$. The determinant $d(q)$ is an invariant of the genus of q. It is a standard fact that the genus of a form q contains finitely many $\mathbb{Z}$-equivalence classes (see e.g. [7, Theorem 6.1.2]); the number of these classes is called *the class number of q*.

3. Sufficiency of Local Conditions

Recall that an integral quadratic form q is *indefinite* if it takes both positive and negative values.

Theorem 3.1 *Let q be a nondegenerate integral indefinite ternary quadratic form representing all odd integers, then the genus of q contains exactly one class.*

Proof: Let q be first any nondegenerate quadratic form on a $\mathbb{Z}$-lattice L of rank ≥ 3 and let $V = L \otimes \mathbb{Q}$. Let $\theta_p : \mathbf{SO}(V_p, q) \to \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ be the spinor norm.

Let P be a finite set of prime numbers containing 2 and such that $\theta_p(\mathbf{SO}(L_p, q)) = \mathbb{Z}_p^\times/\mathbb{Z}_p^{\times 2}$ for all $p \notin P$ and define

$$R_P = \prod_{p \in P} \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} \quad ; \quad S_P = \prod_{p \in P} \theta_p(\mathbf{SO}(L_p, q)) \subset R_P. \tag{2}$$

Define also $T'_P = \{x \in \mathbb{Q}^\times : |x|_p = 1 \text{ for all } p \notin P\}$ and denote by T_P the image of T'_P in R_P under the "diagonal" map $T'_P \longrightarrow R_P$. It is known (see Cassels [1, Theorem 3.1, Capter 11]) that the number of spinor genera in the genus of q is equal to the index $[R_P : S_P T_P]$ for any set of primes P as above. If in addition q

is *indefinite*, then by strong approximation [7, Theorem 6.3.2], spinor genera and classes coincide, so in this case the index $[R_P : S_P T_P]$ is actually equal to the class number.

Let now q be a nondegenerate integral indefinite ternary quadratic form representing all odd integers. If p is an odd prime, the form q represents all of $\mathbb{Z}_p$ since the odd numbers are dense in $\mathbb{Z}_p$. If $p = 2$, q represents the units of $\mathbb{Z}_2$. Let $u \in \mathbb{Z}_p^\times$ and choose $\mathbf{e}_1, \mathbf{e}_2 \in L_p$ such that $q(\mathbf{e}_1) = u$ and $q(\mathbf{e}_2) = 1$. The reflections $\tau_{\mathbf{e}_i}$, defined by $\tau_{\mathbf{e}_i}(\mathbf{x}) = \mathbf{x} - \beta(\mathbf{x}, \mathbf{e}_i) q(\mathbf{e}_i)^{-1} \mathbf{e}_i$, preserve L_p and hence $\tau_{\mathbf{e}_1}\tau_{\mathbf{e}_2} \in \mathbf{SO}(L_p, q)$. It follows that $u = \theta_p(\tau_{\mathbf{e}_1}\tau_{\mathbf{e}_2}) \in \theta_p(\mathbf{SO}(L_p, q))$ and therefore $\theta_p(\mathbf{SO}(L_p, q)) \supseteq \mathbb{Z}_p^\times/\mathbb{Z}_p^{\times 2}$. Since $\mathbb{Z}_p^\times/\mathbb{Z}_p^{\times 2}$ has index two in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$, there are only two possibilities for every prime p: either $\theta_p(\mathbf{SO}(L_p, q)) = \mathbb{Z}_p^\times/\mathbb{Z}_p^{\times 2}$ or $\theta_p(\mathbf{SO}(L_p, q)) = \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$.

Now let

$$P = \{2\} \cup \left\{p \text{ odd} \,:\, \theta_p(\mathbf{SO}(L_p, q)) = \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}\right\}.$$

By [1, Lemma 3.5 Chapter 11], the odd primes in P are divisors of $d(q)$, so P is s finite set. Therefore we can use the groups in (2) to compute the class number of q. In our situation

$$S_P = \theta_2(\mathbf{SO}(L_2, q)) \times \prod_{p \in P \setminus \{2\}} \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2},$$

thus $R_P/S_P = (\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2})/\theta_2(\mathbf{SO}(L_2, q))$ is a group of order at most 2. If $[R_P : S_P] = 2$, the nontrivial element is represented by 2, which is obviously in the subgroup T_P, so $[R_P : S_P T_P] = 1$. Therefore q has class number one as claimed. □

In the next two sections we shall determine the genera of nondegenerate integral indefinite ternary quadratic forms that represent all odd integers.

4. The Case $p \neq 2$

In the remaining of this paper, we shall denote by $\langle a_1; \cdots; a_n\rangle$ the diagonal quadratic form $a_1 x_1^2 + \cdots + a_n x_n^2$.

Proposition 4.1 *Let q be a ternary form over $\mathbb{Z}$ of determinant d that represents all odd integers. Then $q \simeq_{\mathbb{Z}_p} \langle 1; -1; -d\rangle$ for all odd primes p.*

Proof: We diagonalize q to find $q \simeq_{\mathbb{Z}_p} \langle a_1; a_2; a_3\rangle$. One, at least, of the a_i's (say a_1) is a p-adic unit; in fact we can assume $a_1 = 1$ since q represents 1. So we can write $q = \langle 1; p^a u; p^b v\rangle$ where $u, v \in \mathbb{Z}_p^\times$, and $a, b \in \mathbb{Z}$, $a \leq b$. On the other hand, $1 + 2\mathbb{Z}$ is dense in $\mathbb{Z}_p$ and consequently $q(\mathbb{Z}_p) = \mathbb{Z}_p$. If $a > 0$, then reducing (mod p) we would get $q(\mathbf{x}) = x^2$ (mod p), which is clearly impossible. It follows that $a = 0$ and thus $q \simeq \langle 1; u; p^b v\rangle$

If $b = 0$, i.e. $d \in \mathbb{Z}_p^\times$, the equivalence class of q over $\mathbb{Z}_p$ is determined by d, so $q \simeq \mathbb{Z}_p\langle 1; -1; -d\rangle$.

Suppose $b > 0$. The equation

$$x^2 + uy^2 + vp^b z^2 = pw \tag{3}$$

has a solution for any $w \in \mathbb{Z}_p^\times$. If $-u \in \mathbb{Z}_p^{\times 2}$ then obviously $q \simeq \langle 1; -1; -d \rangle$. If not, $x^2 + uy^2$ does not represent 0 non trivially $\pmod p$. In particular, assuming (x, y, z) is a solution to (3), it follows $x \equiv y \equiv 0 \pmod p$.

Let $s = \min(\mathrm{ord}_p(x), \mathrm{ord}_p(y), \mathrm{ord}_p(z))$. Clearly p^{2s} must divide pw and thus we get $s = 0$. So z is a unit. Finally,

$$x^2 + uy^2 = p(w - vp^{b-1} z^2)$$

but $p^2 | x^2 + ay^2$ and therefore $p | w - bp^{b-1} z^2$. If $b > 1$ this relation is clearly impossible. If $b = 1$ we just choose w so that $w \neq v \pmod{\mathbb{Z}_p^{\times 2}}$ to get a contradiction. □

Corollary 4.1A *Let q be an indefinite nondegenerate ternary form over $\mathbb{Z}$ that represents all odd integers. Then q is isotropic.*

Proof: By Proposition 4.1, q is isotropic over $\mathbb{Q}_p$ for all odd primes and also over $\mathbb{R}$ since it is indefinite. By reciprocity, q is isotropic over $\mathbb{Q}_2$ as well and therefore is isotropic over $\mathbb{Q}$ by Hasse-Minkowski. □

5. The case $p = 2$

We shall describe below all the nondegenerate ternary integral quadratic forms over $\mathbb{Z}_2$ that satisfy the following properties:

$$\begin{array}{ll} \text{(a)} & q \text{ is isotropic} \\ \text{(b)} & q \text{ represents all dyadic units} \end{array} \tag{4}$$

The form $\langle 1; -1; -d \rangle$ satisfies trivially (4) for any $d \in \mathbb{Z}_2$. We shall call it the *trivial* form.

Proposition 5.1 *Let $d = 2^k m$ (m odd) and let q be a classically integral quadratic form over $\mathbb{Z}_2$ of determinant d satisfying (4). Then*

(i) If $k = 3$, then either $q \simeq_{\mathbb{Z}_2} \langle 1; -1; -8m \rangle$ or $q \simeq_{\mathbb{Z}_2} \langle 1; -2m; -4 \rangle$

(ii) If $k \geq 4$ and is even, then either $q \simeq_{\mathbb{Z}_2} \langle 1; -1; -d \rangle$ or $q \simeq_{\mathbb{Z}_2} \langle 1; -5; -5d \rangle$

(iii) For all other values of k, $q \simeq_{\mathbb{Z}_2} \langle 1; -1; -d \rangle$

Proof: Since q represents 1 and is classically integral, it can be written in the form $x^2 + 2^t r(y, z)$, where r is a primitive binary form and $t \geq 0$. Note that $t \geq 2$ is impossible since q represents all odd integers, so $t = 0, 1$.

If $t = 0$, r is classically integral, so q can be diagonalized over $\mathbb{Z}_2$ in the form $q \simeq_{\mathbb{Z}_2} \langle 1; -u; -ud\rangle$ for some unit $u \in \mathbb{Z}_2^\times$. An easy computation on the Hasse symbol shows that q is isotropic if and only if $(u, -d)_2 = 1$. If $k \geq 3$, then $u \equiv 1 \pmod 4$ since q must represent all odd integers mod 8. If in addition k is odd, the condition $(u, -d)_2 = 1$ implies $u \equiv 1 \pmod 8$, so u is a square and $q \simeq_{\mathbb{Z}_2} \langle 1; -1; -d\rangle$. If k is even and ≥ 4, the condition on the Hasse symbol is automatically satisfied, so in this cases we get two inequivalent forms $\langle 1; -u; -ud\rangle$ for $u \equiv 1, 5 \pmod 8$. For $k \leq 2$, we see easily case by case that the set of $u \in \mathbb{Z}_2^\times$ such that $\langle 1, -u, -ud\rangle$ satisfies condition (4) coincides with the set of units of the form $a^2 + db^2$ with $a, b \in \mathbb{Z}_2$ (it is enough to check this modulo 8), so these forms are all equivalent to the "trivial form" $\langle 1; -1; -d\rangle$.

If $t = 1$ and r is not classically integral, then $r \simeq_{\mathbb{Z}_2} yz$ or $r \simeq_{\mathbb{Z}_2} y^2 + yz + z^2$. In both cases $q = x^2 - 2q$ can be diagonalized (see [1, Lemma 4.1, Chapter 8]) and we are back in the previous case where $t = 0$. If r is classically integral, then $k \geq 2$ and $q \simeq_{\mathbb{Z}_2} \langle 1; -2u; -2^{k-1}mu\rangle$. We see easily that if $k > 3$ or $k = 2$, q fails to represent all odd integers mod 8, so we are left with $k = 3$. A straightforward computation with Hasse symbols shows that $\langle 1; -2u; -4mu\rangle$ is isotropic if and only if $u \equiv m \pmod 8$ or $u \equiv m + 2 \pmod 8$. Since $m(m + 2)$ is obviously represented by the binary form $\langle 1; 2m\rangle$, both choices of u lead to equivalent forms. One verifies easily that that $\langle 1; -2m; -4\rangle$ represents all odd integers modulo 8. □

Corollary 5.1A *A classically integral form q as in Proposition 5.1 is universal if and only if $q \simeq_{\mathbb{Z}_2} \langle 1; -1; -d\rangle$ with $k = \mathrm{ord}_2(d) \leq 1$.*

Proof: If $k \geq 2$, the forms $\langle 1; -1; -d\rangle$ and $\langle 1; -5; -5d\rangle$ fail to represent $2 \pmod 4$. The form $\langle 1; -2m; -4\rangle$ fails to represent $6m \pmod{16}$, so we are left with $q = \langle 1; -1; -d\rangle$ and $k \leq 1$. It is enough to check that q represents all dyadic integers of the form $2u$, where u is a unit. By Hensel's Lemma, it is sufficient to verify this condition modulo 32 and this is done by direct computation. □

We shall now give the corresponding statements for integral, but not classically integral, forms satisfying (4). In this situation, the "trivial form" is $xy - 4dz^2$.

Proposition 5.2 *Let $d = 2^k m$ (m odd and $k \geq -2$) and let q be an integral but not classically integral quadratic form over $\mathbb{Z}_2$ of determinant d satisfying (4). Let $D = 4d$. Then*

(i) If k is even and $k \geq 0$, either $q \simeq_{\mathbb{Z}_2} xy - Dz^2$ or $q \simeq_{\mathbb{Z}_2} x^2 + xy + y^2 + 3Dz^2$

(ii) If k is odd or $k = -2$, then $q \simeq_{\mathbb{Z}_2} xy - Dz^2$.

Proof: By [1, Lemma 4.1, Chapter 8], q must be of one of the types $xy - Dz^2$ or $x^2 + xy + y^2 + 3Dz^2$. An easy computation with the Hasse symbol shows that the latter is isotropic only when k is even. When $k = -2$ (and only in this case), the forms $xy - Dz^2$ and $x^2 + xy + y^2 + 3Dz^2$ are equivalent over $\mathbb{Z}_2$. It is trivial that the latter represents all dyadic units since $x^2 + xy + y^2$ does. □

Corollary 5.1B *Let q be a quadratic form as in Proposition 5.2. Then q is universal if and only if $q \simeq_{\mathbb{Z}_2} xy - Dz^2$.*

Proof: The form $x^2 + xy + y^2 + 3Dz^2$, with $\mathrm{ord}_2(D) \geq 2$, does not represent 2 (mod 4). It is trivial that $xy - Dz^2$ represents all dyadic integers. □

6. Global forms representing all odd integers

We can now answer completely the initial question over $\mathbb{Z}$. By Theorem 3.1, the indefinite ternary quadratic forms over $\mathbb{Z}$ that represent all integers are determined completely by their local data.

Theorem 6.1 *Let q be a classically integral indefinite quadratic form of determinant $d = 2^k m$ ($m \geq 1$ odd) that represents all odd integers.*

(i) If $k = 3$, then either $q \simeq_{\mathbb{Z}} \langle 1; -1; -d \rangle$ or $q \simeq_{\mathbb{Z}} \langle 1; -2m; -4 \rangle$.

(ii) If $k \geq 4$ and is even, then either $q \simeq_{\mathbb{Z}} \langle 1; -1; -d \rangle$ or $q \simeq_{\mathbb{Z}} x^2 - r(y, z)$, where r is a positive-definite binary form satisfying $r \simeq_{\mathbb{Z}_2} \langle 5, 5d \rangle$ and $r \simeq_{\mathbb{Z}_p} \langle 1, d \rangle$ for $p \neq 2$.

(iii) For all other values of k, $q \simeq_{\mathbb{Z}} \langle 1; -1; -d \rangle$.

Proof: The theorem follows immediately from Proposition 5.1 since q has class number one by Theorem 3.1. Note that the form r of (ii) is in general not unique, but the equivalence class of the associated ternary form $x^2 - r(y, z)$ is independent of this choice by Theorem 3.1. □

Remark. A binary form r satisfying the property of (ii) in Theorem 6.1 can be easily constructed by choosing a prime number ℓ satisfying the congruences $\ell \equiv 5$ (mod 8) and $\ell \equiv 1$ (mod m) and letting $r = \ell y^2 + 2byz + cz^2$, where $b^2 \equiv -d$ (mod ℓ) and $c = (b^2 + d)/\ell$. Note that the Legendre symbol $(-d/\ell)$ is trivial by quadratic reciprocity.

We now consider the case of integral but not classically integral forms. Let $d = 2^k m$, with $m \geq 1$ odd and let $D = 4d$. Let r be a positive definite binary form of determinant D satisfying $r \simeq_{\mathbb{Z}_2} \langle 5; 5D \rangle$ and and $r \simeq_{\mathbb{Z}_p} \langle 1, D \rangle$ for $p \neq 2$. We can assume without loss of generality that $r(1, 0) \equiv 5$ (mod 8) and $r(1, 0) \equiv 1$ (mod m)

Theorem 6.2 *Let q be an integral but not classically integral indefinite quadratic form of determinant $d = 2^k m$ ($m \geq 1$ odd, $k \geq -2$) that represents all odd integers. With the notation above, we have*

(i) If $k \geq 0$ is even, then either $q \simeq_{\mathbb{Z}} xy - Dz^2$ or $q \simeq_{\mathbb{Z}} (2x + y)^2/4 - r(y/2, z)$

(ii) If k is odd or $k = -2$, then $q \simeq_{\mathbb{Z}} xy - Dz^2$.

Proof: If k is odd or $k = -2$, by Proposition 5.2 and Theorem 3.1, we have $q \simeq_{\mathbb{Z}} xy - Dz^2$.

Let k be even and ≥ 0 and assume $q \simeq_{\mathbb{Z}_2} x^2 + xy + y^2 + 3Dz^2$ (otherwise $q \simeq_{\mathbb{Z}_2} xy - dz^2$ as in the previous case). Let L be the underlying $\mathbb{Z}$-lattice of q and let $L^\sharp$ be the dual lattice (with respect to the bilinear form β of (1)). Let $\mathbf{v}_1 \in L$ be such that $q(\mathbf{v}_1) = 1$. Since q takes only even values on $L \cap 2L^\sharp$ if $\operatorname{ord}_2(D) \geq 2$, we have $\mathbf{v}_1 \notin 2L^\sharp$. Define $L' = \{\mathbf{x} \in L : \beta(\mathbf{x}, \mathbf{v}_1) \equiv 0 \pmod 2\}$. Note that $L' = \mathbb{Z}\mathbf{v}_1 \oplus N$, where $N = (\mathbb{Q}\mathbf{v}_1)^\perp \cap L$. Let $\mathbf{w} \in L$ be such that $\beta(\mathbf{w}, \mathbf{v}_1) = 1$. Then $q(\mathbf{w})$ must be odd, otherwise the restriction of q to $\mathbb{Z}_2\mathbf{v}_1 \oplus \mathbb{Z}_2\mathbf{w}$ would be equivalent to xy (the two possible dyadic forms xy and x^2+xy+z^2 are distinguished by their determinant). Write $2\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}_2$, with $\mathbf{v}_2 \in N$ primitive. Then $a = 1$ and $b^2 q(\mathbf{v}_2) \equiv 3 \pmod 8$, so b is odd, say $b = 2c + 1$. Replacing $\mathbf{w}$ by $\mathbf{w} - c\mathbf{v}_2$ we can assume $b = 1$. Choose $\mathbf{v}_3 \in N$ such that $\{\mathbf{v}_2, \mathbf{v}_3\}$ is a $\mathbb{Z}$-basis of N. Then the vectors $\mathbf{w} = (\mathbf{v}_1 + \mathbf{v}_2)/2, \mathbf{v}_2, \mathbf{v}_3$ form a basis of L. In this basis, q has the desired form. Note that $r := -q|_N \simeq_{\mathbb{Z}_2} \langle -3; -3D\rangle \simeq_{\mathbb{Z}_2} \langle 5; 5D\rangle$. For $p \neq 2$ we must have $r \simeq_{\mathbb{Z}_p} \langle -1; -D\rangle$ by Proposition 4.1.

□

Example. The indefinite ternary quadratic forms of determinant $d = 16$ that represent all odd integers are $x^2 - y^2 - 16z^2$, $xy - 64z^2$ (these are the "trivial" ones), and $x^2 - 4y^2 + 4yz - 5z^2$ and $x^2 + xy - y^2 - yz - 13z^2$ (these are the "non-trivial" ones).

Corollary 6.2A (Dickson [2, Theorem 109]) *Up to $\mathbb{Z}$-equivalence, there are only two types of integral ternary universal quadratic forms:* (1) $x^2 - y^2 - dz^2$ *with* $\operatorname{ord}_2(d) \leq 1$, *and* (2) $xy - Dz^2$.

Proof: Follows immediately from Theorems 6.1 and 6.2 and Corollaries 5.1A and 5.1B. □

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