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Representation Type Of One Point Extensions Of Iterated Tubular Algebras

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We dedicate this work to the memory of Sheila Brenner.

Abstract: The purpose of this work is to show that if Λ a strongly simply connected semi-regular iterated tubular algebra and M is an indecomposable Λ-module then $\Lambda[M]$ is tame if and only if $q_{\Lambda[M]}$ is weakly non negative.

Key Words: representation type e quadratic forms.

Contents

1. Introduction

Given an algebra one question of interest is to know its representation type. In particular it is not easy to know when a given algebra is of tame representation type. One possible approach to this problem is to consider the Tits form associated with the ordinary quiver of the algebra. It is known that, given a tame algebra Λ, the Tits form q_Λ is weakly non negative. The converse has been shown for some families of algebras, as for instance tilted, quasi-tilted or iterated tubular algebras, but it is not true in general. More precisely, it holds that if Λ is tilted, quasi-tilted or iterated tubular algebra, then Λ is tame if and only if the Tits quadratic form is weakly non negative.

In order to investigate the representation type of a given triangular algebra Λ we assume that $HH^1(\Lambda) = 0$, i.e. the first Hochschild Cohomology group vanishes. Then, up to duality, we get $\Lambda = B[M]$ with B a connected algebra and M an indecomposable module. Under these hypotheses we look at the following problem: if B has the property that the Tits form determines the tameness of its

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representation type, when the same property holds for Λ. In [16] de la Peña have shown that the result holds in case that B is a tame concealed algebra not of type $\tilde{\mathrm{A}}_n$. A similar result was obtained by Chalom and Merklen in [8] in case that B is a tilted algebra of euclidian type not of type $\tilde{\mathrm{A}}_n$. We consider the case that B is a strongly simply connected tame quasi-tilted algebra of canonical type, and the more general situation when B is a strongly simply connected iterated tubular algebra. Our main result in this note is the following:

Theorem 1.1 *Let B be strongly simply connected semi-regular iterated tubular algebra and M an indecomposable B-module, then the one point extension $\Lambda = B[M]$ is tame if and only if the Tits form q_Λ is weakly non negative.*

Note that tubular and tame quasi-tilted algebras are particular cases of iterated tubular algebras.

For several years have been stated the following well-known conjecture: If Λ is strongly simply connected then the Tits form determines the tameness of its representation type. Recently Brüstle and Skowroński had announced that this conjecture holds true. In [3] Assem and Castonguay gave necessary and sufficient conditions for the one point extension of a tree hereditary algebra to be again a strongly simply connected algebra. Observe that in our case, if B is a strongly simply connected algebra and M is an indecomposable module then $\Lambda = B[M]$ is a simply connected algebra with $HH^1(\Lambda) = 0$, but Λ is not necessarily a strongly simply connected algebra.

We start in section 1 with preliminary results and useful definitions. In section 2, the case of one-point extensions of strongly simply connected tubular algebras by indecomposable modules is studied. In section 3, we prove the theorem for the case of one-point extensions of strongly simply connected tame quasi-tilted algebras of canonical type by indecomposable modules, and in section 4, the case of one-point extensions of strongly simply connected semi-regular iterated tubular algebras by indecomposable modules.

2. Preliminaries

Through the whole work, k denotes an algebraically closed field. By an algebra Λ we mean a finite-dimensional, basic and connected k-algebra of the form $\Lambda \cong kQ/I$ where Q is a finite quiver and I an admissible ideal. Let Λ-mod denote the category of finite-dimensional left Λ-modules, and Λ-ind a full subcategory of Λ-mod consisting of a complete set of non-isomorphic indecomposable objects of Λ-mod.
For each $i \in Q_0$ we denote by S_i (resp. P_i, I_i) the corresponding simple Λ-module (resp. the projective cover, injective envelope of S_i).

We shall use freely the known properties of the Auslander-Reiten translations, τ and τ^{-1}, and the Auslander-Reiten quiver of Λ-mod, Γ_Λ . For basic notions we refer to [20] and [5]. See also [1] and [7].

We begin now recalling the concepts and results that form a background for our work. A *vector-space category* $(\mathbb{K}, |\ |)$ is a pair given by a *Krull-Schmidt k-category*

$\mathbb{K}$ and a faithful functor $|\;|\,:\mathbb{K} \to$ mod k (see [19]). Given a vector-space category $(\mathbb{K}, |\;|)$, its objects (resp. the morphisms) are usually considered to be the objects (resp. the morphisms) of the image of $|\;|$, and its subspace category $\mathcal{U}(\mathbb{K})$ is defined as follows: the objects are triples (X, U, φ) with $X \in \mathrm{Obj}\mathbb{K}$, U a k-vector space and φ: $U \to |X|$, k-linear. The morphisms $(X, U, \varphi) \to (X', U', \varphi')$ are the pairs (α, β) with β: $X \to X'$ in $\mathbb{K}$, $\alpha : U \to U'$ k-linear and such that $|\beta|\varphi = \varphi'\alpha$. Clearly, any object of $\mathcal{U}(\mathbb{K})$ is isomorphic to a direct sum of a triple (X, U, φ) with $\varphi : U \to |X|$ injective and copies of $(0, k, 0)$. Modules over a one point extension $B[M]$ can be identified with triples (X, U, φ) where $X \in B$-mod, U is a k-vector-space and $\varphi : U \to Hom(M, X)$ is k-linear. It is known that the representation type of $B[M]$ depends on the representation type of B and of $\mathcal{U}(\mathrm{Hom}(M, B-\mathrm{mod})$, see [19] for other notions and notations related to vector-space categories.

In section 2 and section 3 we consider one point extensions of tubular algebras (see [20]) and one point extensions of tame quasi-tilted algebras, (see [13] and [21]). So, in any case $gldimB \leq 2$, and consequently $gldimB[M] \leq 3$.

Associated to an algebra Λ of finite global dimension, there exist in the Grothendieck group of Λ, two quadratic forms. These forms are very important tools in the study of tame algebras.

Let C_B be the Cartan matrix of B and let x and y vectors in $K_0(B)$. Then we have a bilinear form

$$< x, y >= xC_B^{-T}y^T$$

where the corresponding quadratic form

$$\chi_B(x) =< x, x >$$

is called *the Euler form* of B, see [20].

By other hand, the *Tits quadratic form* is given by:

$$q_B(x_1, x_2, ..., x_l) = \sum_{i \in Q_0} x_i^2 - \sum_{i,j \in Q_0} x_i.x_j.dim_k Ext_B^1(S_i, S_j)$$
$$+ \sum_{i,j \in Q_0} x_i.x_j.dim_k Ext_B^2(S_i, S_j)$$

see [6].

If B is such that $gldimB \leq 2$ then for any B-module it follows that $gldimB[M] \leq 3$. Hence, using Bongartz result (see [6]) that is, if $gldimB \leq 2$ then $\chi_B = q_B$, it is possible to relate the Euler and the Tits form for $\Lambda = B[M]$.

Let $X = (Y, k^n, f)$ be a Λ-module and let:

$$\underline{dim}_\Lambda(X) = \underline{dim}_B(Y) + n.\underline{dim}_\Lambda(S_e)$$

where e is the extension vertex.

Comparing this two quadratic forms we get the following relation:

Proposition 2.1 *With the above notation:*

$$\chi_\Lambda(\underline{dim}X) = q_\Lambda(\underline{dim}X) - n.dim_k Ext_B^2(M, Y).$$

Proof: We give here a short argument:

$\chi_A(\underline{dim}X) =$

$\chi_B(\underline{dim}Y) + n^2 - n(dim_k Hom_B(M,Y) - dim_k Ext^1_B(M,Y) + dim_k Ext^2_B(M,Y))$

because $gldimB \leq 2$, and, by the other hand

$q_A(x_1, x_2, ..., x_l, x_e) = \sum_{i \in Q_0} x_i^2 - \sum_{i,j \in Q_0} x_i.x_j.dim_k Ext^1_A(S_i, S_j) + \sum_{i,j \in Q_0} x_i.x_j.dim_k Ext^2_A(S_i, S_j) =$

$q_B(x_1, x_2, ..., x_l) + x_e^2 - \sum_{j \in Q_0} x_e x_j(dim_k Ext^1_A(S_e, S_j) + dim_k Ext^1_A(S_j, S_e)) + \sum_{j \in Q_0} x_e x_j(dim_k Ext^2_A(S_e, S_j) + dim_k Ext^2_A(S_j, S_e))$.

Comparing the dimensions of the Exts, we get the desired result. □

We recall now some preliminary concepts that will be very useful in order to state our results.

Let Λ be a basic finite dimensional associative algebra (with unit) over an algebraically closed field k. Then $\Lambda \simeq kQ/I$ for some finite quiver Q and some admissible ideal I of the path algebra kQ, and the pair (Q, I) is called a presentation for Λ.

Let (Q, I) be a connected bound quiver. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ is minimal if $m > 1$ and, for any non empty proper subset $J \subset \{1, 2, ..., m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$. A walk in Q from x to y is a path of the quiver formed by Q and the formal inverses α^{-1} of the arrows $\alpha \in Q$. That is, it is a composition $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} ... \alpha_t^{\varepsilon_t}$ where α_i are arrows in Q and $\varepsilon_i \in \{1, -1\}$ for all i, with source x and target y. We denote by e_x the trivial path at x. Let $\sim$ be the least equivalence relation on the set of all walks in Q such that:

(a) If $\alpha : X \to Y$ is an arrow, then $\alpha^{-1}\alpha \sim e_x$ and $\alpha\alpha^{-1} \sim e_y$.

(b) If $\rho = \sum_{i=1}^{m} \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all i, j.

(c) If $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions make sense.

Let $x \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x)$ of equivalences classes $\overline{u}$ of closed paths u starting and ending at x has a group structure defined by the operation $\overline{u}.\overline{v} = \overline{u.v}$. Since Q is connected then this group does not depend on the choice of x. We denote it $\pi_1(Q, I)$ and call it the *fundamental group* of (Q, I), see [15].

A triangular algebra Λ is simply connected if, for any presentation (Q, I) of Λ, the fundamental group $\pi_1(Q, I)$ is trivial.

An algebra B is a convex subcategory of Λ if there is a full and convex subquiver Q' of Q such that $B = kQ'/(I \bigcap kQ')$. The algebra Λ is said to be *strongly simply connected* if any full convex subcategory of Λ is simply connected. (See [22]).

Given a directed component Γ of Γ_A, its *orbit graph* $O(\Gamma)$ has as points the τ-orbits $O(M)$ of the modules M in Γ. There exists an edge $O(M) - O(N)$ in $O(\Gamma)$ if there are $m, n \in \mathbb{Z}$ and an irreducible morphism $\tau^m M \to \tau^n N$ or $\tau^n N \to \tau^m M$. The number of such edges equals $\dim_k \mathrm{Irr}(\tau^m M, \tau^n N)$ or $\dim_k \mathrm{Irr}(\tau^n N, \tau^m M)$ respectively, where Irr(X,Y) denotes the space of irreducible morphisms from X to Y. A component Γ of Γ_Λ is of *tree type* if its orbit graph $O(\Gamma)$ is a tree.

It was shown in [16] that if B is a tame algebra, then q_B is weakly non negative, this important result was also obtained by Drozd for matrix problems in [12] . It is known that the converse is not true in general, see example in section 2.4 of [17], but it is true for some families of algebras, as tubular algebras [20], quasi-tilted algebras [21] and iterated tubular algebras [18]. The main motivation of our work was the result in [16] that if C is a tame concealed algebra, not of type $\tilde{A}_n$, and M an indecomposable C-module, then the one point extension $C[M]$ is tame if and only if $q_{C[M]}$ is weakly non negative. This result was extended in [8] to the case of B be a strongly simply connected tilted algebra of euclidian type, i.e, if B is a strongly simply connected tilted algebra of euclidian type and M an indecomposable B-module, then the one point extension $B[M]$ is tame if and only if $q_{B[M]}$ is weakly non negative. Our objective is to generalize this result to the case when B is a strongly simply connected tame quasi-tilted algebras of canonical type, or B is a strongly simply connected iterated tubular algebra. We start considering the case when B is a tubular algebra.

3. One Point Extensions Of Tubular Algebras

In this section, we consider the class of tubular algebras considered by Ringel in [20]. We recall that a tubular algebra B is a tubular extension of a tame concealed algebra B_0 with tubular type $(2,2,2,2)$, $(3,3,3)$, $(4,4,2)$ or $(6,3,2)$. Any tubular algebra is also co-tubular, that is $B = B_0[E_i, R_i]_{i=1}^{t} = [E'_i, R'_i]_{i=1}^{t'} B_\infty$ with B_0 and B_∞ both tame concealed, E_i and E'_i ray modules of the separating tubular families of the corresponding algebras, and R_i , R'_i branches. We begin by proving the following lemma:

Lemma 3.1 *Let B_0 be a convex subcategory of B such that B is a iterated co-extension or a branch coextension of B_0 and assume that $M_0 = M|_{B_0}$, $M_0 \neq 0$. Then $B_0[M_0]$ is a convex subcategory of $B[M]$.*

Proof: The proof is done by induction in the number of the coextensions and the length of the branch. □

We recall the structure of the Auslander-Reiten quiver of a tubular algebra, as in [20] (pag. 273). Let B be a tubular algebra, then $B = B_0[E_i, R_i]_{i=1}^{t} = [E'_i, R'_i]_{i=1}^{t'} B_\infty$ with B_0 and B_∞ both tame concealed. We have the following pairwise disjoint modulo classes: $\mathcal{P}_0, \mathcal{T}_0, \cup_{\gamma \in P^1(k)} \mathcal{T}, \mathcal{T}_\infty, \mathcal{Q}_\infty$. Note that $\mathcal{P}_0$ is the preprojective component of B_0, and $\mathcal{Q}_\infty$ is the preinjective component of B_∞ and $\mathcal{T}_0, \cup_{\gamma \in P^1(k)} \mathcal{T}, \mathcal{T}_\infty$, are tubular families. Also the indecomposable projective modules belong to $\mathcal{P}_0$ or $\mathcal{T}_0$ and the indecomposable injective modules belong to $\mathcal{T}_\infty$ or $\mathcal{Q}_\infty$.

Now, we consider the case of B be a tubular algebra with directed components of tree type and we get the following result:

Theorem 3.1 *Let B be a tubular algebra with each directed component of tree type and M be an indecomposable B-module. If $B[M]$ is wild then $q_{B[M]}$ is strongly indefinite.*

Proof: Observe that the pre-injective component and the pre-projective component of Γ_B are of tree type. Then B_0 and B_∞ are not of type $\widetilde{\mathrm{A}}_n$.

First consider the case that M is an indecomposable module in one of the following families : $\mathcal{P}_0, \mathcal{T}_0, \cup_{\gamma \in P^1(k)} \mathcal{T}$, then there exists an indecomposable injective module I such that $Hom(M, I) \neq 0$. It follows that M and I are separated by a separating tubular family, then this non zero morphism factor trough a orthogonal tubular family, in particular factors trough five orthogonal bricks, then by Nazarova theorem it follows that $Hom_B(M, modB)$ is wild and, see in prop. 3.3 of [18], the corresponding quadratic form is strongly indefinite.

Now, consider the case that M belongs to $\mathcal{Q}_\infty$. Observe that this pre-injective component corresponds to the pre-injective component of the algebra B_∞, that is, the pre-injective component of a tame concealed algebra not of type $\widetilde{\mathrm{A}}_n$. Then, the situation is similar to the one consider in theorem 2.2 of [16] and the result follows with the same arguments.

It only remains to consider the case when M belongs to $\mathcal{T}_\infty$. The analysis is analogous to the one in the case of a tilted algebra of euclidian type considered in theorem 2.3 of [8]. For the convenience of the reader we repeat some of these arguments here. If $M_\infty = M|_{B_\infty}$ is such that $M_\infty = 0$, then $suppM$ is contained in a branch R and the vector-space category $Hom(M, B - mod)$ is the same as $Hom(M, R - mod)$. It follows from [14], that if $Hom(M, R - mod)$ is wild then $q_{R[M]}$ is strongly indefinite. Since $R[M]$ is a convex subcategory of $B[M]$, then $q_{R[M]}$ strongly indefinite implies $q_{B[M]}$ strongly indefinite. In case that $M_\infty \neq 0$ and $B_\infty[M_\infty]$ is wild, since the pre-injective component of B is A_n-free , it follows by [16], that $q_{B_\infty[M_\infty]}$ is strongly indefinite. Since $B_\infty[M_\infty]$ is a convex subcategory of $B[M]$ so $q_{B[M]}$ is strongly indefinite.

Now, consider the case when $B_\infty[M_\infty]$ is tame , but $B[M]$ is wild. Since $B_\infty[M_\infty]$ is tame, there are two possibilities: either M_∞ is a ray module or M_∞ is a module of regular length two in the tube of rank $n-2$ and B_∞ is tame concealed of type $\widetilde{D}_n$.

In case that M is a ray module over B, the same argument that in [8] shows that $B[M]$ is an algebra with acceptable projective modules. Also if $M = M_\infty$ and therefore, M is a ray module over B_∞, then again $B[M] = B[M_\infty]$ is an algebra with acceptable projective modules. It follows by [18] that $B[M]$ is wild if and only if $q_{B[M]}$ is strongly indefinite.

Suppose M is not a ray module over B, $M \neq M_\infty$ and M_∞ is a ray module. It is not difficult to show that category $Hom(M, B - mod)$ has three pieces, that is, the ray of $\mathcal{T}_e$ that starts in M_∞, $Hom(M_\infty, \mathcal{Q}_\infty)$ where $\mathcal{Q}_\infty$ is the pre-injective component of B_∞ and the subcategory given by the successors of M in the tube, that are not B_∞-modules. Since $B_\infty[M_\infty]$ is tame, $Hom(M_\infty, \mathcal{Q}_\infty)$ is given by some of the patterns given in [[19], pag 254]. Here, we are using the results given by Ringel, in [[19], pag 254], theorem 3, and so, we follow the notations and definitions given there. The fundamental case that remains to consider is when M is not an injective module, since in case that M is an injective module $B[M]$ is a coil enlargement of B and so is tame. Now, consider the case that M is an injective module and there exists a sectional path $M \to Y_0 \to \ldots \to Y_t$ with $t \geq 1$. In first place, we observe that $Hom_B(Y_i, X) = 0$ for and $Hom(\tau^{-1}M, X) = 0$ for all pre-injective X.

In particular, $Hom(Y_i, X) = Hom(\tau^{-1}M, X) = 0$ for all X such that $Hom(M_\infty, X)\ 0 \neq$ with X in the pre-injective component.

Moreover, $Hom(Y_i, \tau^{-1}M) = 0 = Hom(\tau^{-1}M, Y_j)$ for $\forall j \geq 1$. Hence, by [[19] (3.1)] we find one of the following path-incomparable (see [9]) subcategories in $\mathcal{Q}_\infty$, with the only exception of the case $(\tilde{D}_n, n-2)$: $\mathbb{K}_1 = \{A, B, C\}$, (in cases: $(\tilde{D}_4, 1), (\tilde{D}_6, 2), (\tilde{D}_7, 2), (\tilde{D}_8, 2), (\tilde{E}_6, 2), (\tilde{E}_7, 3), (\tilde{E}_7, 4), (\tilde{E}_8, 5)$) and $\mathbb{K}_2 = \{A, B \to C\}$ in cases $(\tilde{D}_5, 2)$ and $(\tilde{E}_6, 3)$. So, in each case, adding the objects $Y_1, \tau^{-1}M$ to the categories $\mathbb{K}_1$ or $\mathbb{K}_2$ it follows that $Hom(M, B-mod)$ is wild and that $q_{B[M]}$ is strongly indefinite.

We compute the quadratic form for the case $(\tilde{D}_5, 2)$, the other cases are similar. Let $\tilde{L}$ be the B-module $\tilde{L} = 2Y_1 \oplus 2\tau^{-1}M \oplus 2A \oplus B \oplus C$ and $L = \tilde{L} \oplus 4S_e$, then $q_{B[M]}(\underline{dim}L) = \chi_{B[M]}(\underline{dim}L) + 4dim_k Ext^2(M, \tilde{L}) = \chi_{B[M]}(\underline{dim}L) = \chi_{B[M]}(\underline{dim}\tilde{L}) + 4^2 - 4(8) = 15 + 16 - 32 = -1$.

Now, consider the case $(\tilde{D}_n, n-2)$. The pattern is given by:

$$\begin{array}{ccccccc}
 & & & \overset{A}{\circ} & & & \\
 & & \nearrow & & \searrow & & \\
\circ \to \cdots \circ \to & \circ & & & & \circ & \to \cdots \\
 & & \searrow & & \nearrow & & \\
 & & & \underset{B}{\circ} & & &
\end{array}$$

If $t > 1$, considering that $\mathbb{K} = \{A, B, \tau^{-1}M, Y_1 \to Y_2\}$ is wild, again the quadratic form is strongly indefinite. On the other hand, the case $t = 1$ split in two possibilities with the same behavior that in [8] and the result holds.

It remains to look at the case that M_∞ is a module of regular length 2 in a tube of rank $n-2$ and B_∞ is tame concealed algebra of type $\tilde{D}_n$. If $M = M_\infty$ lies in a stable tube, then $Hom(M, B-mod) = Hom(M_\infty, B_\infty - mod)$ and therefore both are tame or both wild. So, suppose that M belongs to a co-inserted tube. Since M_∞ has regular length 2, there exist E_1 and E_0 ray-modules over B_∞ such that $\tau E_0 = E_1 \to M_\infty \to E_0$ is the almost split sequence ending at E_0. Let $E_0, E_1, \cdots E_{n-3}$ be the ray-modules of the tube.

Observe that if $M = M_\infty$, then $Hom(M, B-mod)$ has the same pattern that $Hom(M_\infty, B_\infty - mod)$. If M is a B_∞-module, then
$Hom_B(M, N) \neq 0$ for modules N in the same tube that M or for modules N in the pre-injective component. Hence, since $Hom(M, N) = Hom(M_\infty, N_\infty)$ it follows that the pattern is one of patterns given in [19], and then is tame. Considering the situations when the branch is co-inserted in E_j for some j , in any cases the situation is similar to the one in [8] and the result holds. $\square$

The condition that B has each directed components of tree type, for tubular algebras is in fact equivalent to the condition of strongly simple connectedness, see

[2]. Then we have the following corollary:

Corollary 3.1A *Let B be a strongly simply connected tubular algebra, let M be an indecomposable module, and consider $\Lambda = B[M]$. Then B is tame if and only if q_Λ is weakly non negative.*

4. One Point Extensions Of Tame Quasi-Tilted Algebras Of Canonical Type

In this section, we consider now the case where B is a tame quasi-tilted algebra of canonical type. We use strongly in this work the characterization of B as a semi-regular n-iterated tubular algebra, with $n = 0$ or $n = 1$ (see [21],[18] and [4]).

Theorem 4.1 *Suppose that B is a strongly simply connected tame quasi-tilted algebra of canonical type, and M is an indecomposable B-module. Then $B[M]$ is tame if and only if the corresponding Tits form is weakly non negative.*

Proof: If B is a tilted algebra, then B is a tilted algebra of euclidian type and the result follows from [8]. Then we consider the case when B is not tilted. Since B is non tilted, it follows (by [21]) that $B = {}_{i=1}^{t}[E_i, R_i]C[E'_j, R'_j]_{j=1}^{s}$, where C is a tame concealed algebra, E_i and E'_i are ray modules in tubes of the separating tubular family of C, and R_i , R'_i branches. Also the E_i and E'_i do not lie in the same tube. We call $B^+ = [E_i, R_i]C$ and $B^- = C[E'_i, R'_i]$

The proof is done considering the Auslander-Reiten quiver of B and the possibilities for M an indecomposable B-module. In the case that M is in the pre-injective component of B, that is the pre-injective component of B^-, as B^- is a tilted of euclidian type or a tubular algebra, the result follows from [8] or theorem 3.1.

We consider two possibilities:
i)- B is domestic, and so B^+ and B^- are tilted of euclidian type or
ii)- B is non domestic and then B^+ or B^- are of tubular type.

If M is in the pre-projective component of B, we know that $suppM \subset B^+$ and so $B[M] = (B^+[E'_j, R'_j]_{j=1}^{s})[M] = (B^+[M])[E'_j, R'_j]_{j=1}^{s}$. It follows that $B^+[M]$ is a full convex subcategory of $A[M]$. Now we consider the two different cases:

If B^+ is of tubular type, then $B^+[M]$ is wild, and $q_{B^+[M]}$ is strongly indefinite and the same is true for $B[M]$, see theorem 3.1.

If B^+ is tilted of euclidian type, then B^+ has a complete slice in the pre-projective component. If the vector-space category $Hom(M, B^+ - mod)$ is finite we will have that $B^+[M]$ and $B[M]$ are tame.

If B^+ is tilted and the category $Hom(M, B^+ - mod)$ is not finite, the algebra $B^+[M]$ has a component in the Auslander-Reiten quiver that contains all the projective modules and is a π-component (see [10]). Using the same argument that in [8], proposition 3.1, this component does not contain injective modules, so $B^+[M]$ is again a tilted algebra.

Now, consider that M is a module in a tube. Assume that B^+ and B^- are tilted of euclidian type. Then the Auslander-Reiten quiver of B has a semi-regular tubular family, where some tubes contain projective modules (with support in B^-) and some tubes contain injective modules (with support in B^+). Observe that,by [21] each component Γ of Γ_B is contained in B^+-mod or in B^--mod. If M belongs to a tube that contains projective modules, or a stable tube, then, the vector-space categories $Hom(M, B - mod)$ and $Hom(M, B^- - mod)$ are isomorphic. By other hand, $B^-[M]$ is a convex subcategory of $B[M]$ by 3.1, and since B^- has the pre-injective component of tree type, the result follows from [8].

Now, consider the case where M belongs to a tube with injective modules (so, $suppM \subset B^+$). Let $M_0 = M|_C$, by 3.1 $C[M_0]$ is a convex subcategory in $B^+[M]$, also, as $B = B^+[E'_j, R'_j]$ then, $B[M] = (B^+[M])[E'_j, R'_j]$ and $B^+[M]$ is a convex subcategory of $B[M]$. So, if $C[M_0]$ is wild and C is of tree type then also $B^+[M]$ is wild, and $q_{B[M]}$ is strongly indefinite.

Then, assume that $C[M_0]$ is tame. If $M = M_0$, and M is a ray-module, then $C[M_0]$ is domestic or tubular and $B^+[M] = ([E_i, R_i]C)[M]$ is an iterated tubular algebra. Moreover $B[M] = B^+[M][E'_j, R'_j]$ is also iterated tubular, and so is tame. If $M \neq M_0$, and M is a ray-module, by similar arguments to the one in 3.1, $B[M]$ is an algebra with acceptable projective modules and so, by theorem 3.4 of [18], the representation type is determined by the quadratic form. And by the other hand, if the support of M is contained in the branch, the result follows from [14]. So, assume that M is not a ray module, that $M_0 \neq 0$ and that $C[M_0]$ is tame , so M_0 is a ray-module or is a module of level two in a tube of rank $n-2$, and C is a $\tilde{\mathrm{D}}_n$ concealed algebra.

The case that M_0 is a ray module is solved with a similar argument that in 3.1.

Now, consider that C is a concealed algebra of $\tilde{D}_n$-type, and that $C[M_0]$ is a 2-tubular algebra. Also, $M \neq M_0$. Observe that the pre-injective C-modules can be immersed in the pre-injective component of B^-, that is, the pre-injective component of B (see [11]). It follows that there exist a faithful functor $F : C - mod \to B - mod$ such that $F(X)$ is pre-injective if X is pre-injective. Moreover, if X is a pre-injective C-module such that dim $\mathrm{Hom}_C(M_0, X) = 2$ then dim $\mathrm{Hom}_B(M, F(X)) = 2$. So, the vector-space category $Hom(M, B - mod)$ contains the vector-space category given by the pattern $\tilde{D}^{n-2}_{n-2}$ as in [19], pag 253.

Look at the vector-space category $Hom(M, B - mod)$ whose objects are $Hom(M, X)$ for X in the tube. Let $E_0, E_1, \cdots E_{n-3}$ be the ray-modules over C of the tube where M_0 lies. Assume that M_0 is the middle term of the almost split sequence $0 \to E_1 \to M_0 \to E_0 \to 0$. Again, consider the possibilities that the branch is co-inserted in some of the ray modules E_j these cases are analogous to the case of B tilted of euclidian type considered in [8], pag. 8.

Now, consider the non domestic case. There are two possibilities:

i) If B^+ is tubular and B^- is domestic, since B is tame, the Auslander-Reiten quiver of B^+ is given by $\mathcal{P}_0, \mathcal{T}_0, \cup_{\gamma \in P^1(k)} \mathcal{T}, \mathcal{T}_\infty, \mathcal{Q}_\infty$ and the Auslander-Reiten quiver of B is given by $\mathcal{P}_0, \mathcal{T}_0, \cup_{\gamma \in P^1(k)} \mathcal{T}_\gamma, \mathcal{T}^-_\infty, \mathcal{Q}^-_\infty$ where $\mathcal{Q}^-_\infty$ is the preinjective component of A and of B^- and the new projective modules are inserted in stable tubes belonging to $\mathcal{T}_\infty$. We denote $\mathcal{T}^-_\infty$ the new family of tubes. In this case, considering all the possibilities for M, shortly saying, if $suppM \subset B^+$ or $suppM \subset B^-$ the result follows from the previous cases.

ii) If B^- is tubular and B^+ is domestic, as B is tame, the Auslander-Reiten quiver of B is given by: $\mathcal{P}_0^+, \mathcal{T}_0^+, \cup_{\gamma \in P^1(k)} \mathcal{T}_\gamma, \mathcal{T}_\infty, \mathcal{Q}_\infty$ and the new injective modules, having support in B^+ are inserted in the $\mathcal{T}_0$. Let M be an indecomposable module in a tube in $\mathcal{T}_0^+$, that is a tube containing injective modules, and so $suppM \subset B^+$. The vector-space category $Hom(M, -)$ is finite for those modules with the support contained in the branch, then the result follows from [14]. If $Hom(M, -)$ is infinite there exists an injective module I, outside of the tube, such that $Hom(M, I) \neq 0$, but again this morphism factors through infinite families of tubes and the result follows as in theorem 3.1,or [18].

iii) Finally, consider that B^+ and B^- are tubular, in this case the Auslander-Reiten quiver of B is $\mathcal{P}_0^+, \mathcal{T}_0^+, \cup_{\gamma \in P^1(k)}$
$\mathcal{T}_\gamma^+, \mathcal{T}_\infty^+ = \mathcal{T}_0^-, \cup_{\gamma \in P^1(k)} \mathcal{T}_\gamma^-$, $\mathcal{Q}_\infty^-$. Since $suppM \subset B^+$ or $suppM \subset B^-$ all cases were already considered. $\square$

As in the case of tubular algebras, if B is a tame quasi-tilted algebra that is not tilted, then B is strongly simply connected if and only if B^+ and B^- are strongly simply connected, if and only if each directed component of B^+ and B^- is of tree type, see [4].

Remark 4.1 *The statement of 4.1 remains true if we replace the hypotheses of strongly simply connectedness by the following one, B^+, B^-, and C have the pre-injective components of tree type.*

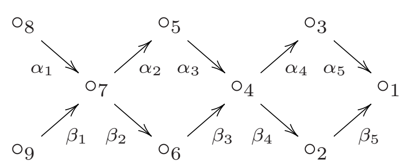
5. One Point Extensions Of Semi-Regular Iterated Tubular Algebras

In this section we consider one-point extensions of semi-regular iterated tubular algebras by indecomposable modules. We recall the structure of the Auslander-Reiten quiver of a semi-regular iterated tubular algebra B, see [18], in order to consider the vector-space categories that arise in the one point extension. For this purpose, consider the construction of B by steps. Assuming that B is a n-iterated tubular algebra (with $n \geq 2$) we consider: $A_1 = [E_i^1, R_i^1]C_1$ that is tilted or tubular, and $B_1 = A_1[E_i^2, R_i^2] = [E_i^1, R_i^1]C_1[E_i^2, R_i^2]$ that is quasi-tilted, because B is semi-regular n-iterated tubular. If we are going to extend one more step, we need that $C_1[E_i^2, R_i^2]$ be a tubular algebra, that is, $C_1[E_i^2, R_i^2] = [E_i^3, R_i^3]C_2$ and $B_2 = [E_i^3, R_i^3]C_2[E_i^4, R_i^4]$ that is also quasi-tilted, and if B is n-iterated, we have the n-quasi-tilted algebras B_1 , B_2 , ... , $B_n = [E_i^{n-1}, R_i^{n-1}]C_{n-1}[E_i^n, R_i^n]$.

Observe that all B_i^+ and B_i^- are tubular, except, maybe, the first one B_1^+ and the last one B_n^-.

We recall an example from [18] which will be useful for understand the general situation.

Let A given by the quiver



$\alpha_i\alpha_{i+1}\alpha_{i+2} = \lambda_i\alpha_i\beta_{i+1}\beta_{i+2}$, $\beta_i\alpha_{i+1}\alpha_{i+2} = \mu_i\beta_i\beta_{i+1}\beta_{i+2}$, $\mathrm{rad}^4 A = 0$, with all λ_i, μ_j different elements of the field k, and $\lambda_i \neq \mu_j$
we have four tame concealed algebras $C_i s$

$$C_0 = \begin{tikzcd} & \circ_3 \arrow[dr] & \\ \circ_4 \arrow[ur] \arrow[dr] & & \circ_1 \\ & \circ_2 \arrow[ur] & \end{tikzcd} \quad C_2 = \begin{tikzcd} & \circ_5 \arrow[dr] & \\ \circ_7 \arrow[ur] \arrow[dr] & & \circ_4 \\ & \circ_6 \arrow[ur] & \end{tikzcd}$$

$$C_1 = \begin{tikzcd} \circ_5 \arrow[dr] & & \circ_3 \\ & \circ_4 \arrow[ur] \arrow[dr] & \\ \circ_6 \arrow[ur] & & \circ_2 \end{tikzcd} \quad C_3 = \begin{tikzcd} \circ_8 \arrow[dr] & & \circ_5 \\ & \circ_7 \arrow[ur] \arrow[dr] & \\ \circ_9 \arrow[ur] & & \circ_6 \end{tikzcd}$$

and two tame quasi-tilted algebras B_0 and B_1.

$$B_1 = \begin{tikzcd} & \circ_5 \arrow[dr, "\alpha_3"'] & & \circ_3 \arrow[dr, "\alpha_5"'] & \\ \circ_7 \arrow[ur, "\alpha_2"'] \arrow[dr, "\beta_2"'] & & \circ_4 \arrow[ur, "\alpha_4"'] \arrow[dr, "\beta_4"'] & & \circ_1 \\ & \circ_6 \arrow[ur, "\beta_3"'] & & \circ_2 \arrow[ur, "\beta_5"'] & \end{tikzcd}$$

$$B_2 = \begin{tikzcd} \circ_8 \arrow[dr, "\alpha_1"'] & & \circ_5 \arrow[dr, "\alpha_3"'] & & \circ_3 \\ & \circ_7 \arrow[ur, "\alpha_2"'] \arrow[dr, "\beta_2"'] & & \circ_4 \arrow[ur, "\alpha_4"'] \arrow[dr, "\beta_4"'] & \\ \circ_9 \arrow[ur, "\beta_1"'] & & \circ_6 \arrow[ur, "\beta_3"'] & & \circ_2 \end{tikzcd}$$

with the induced relations. The Auslander-Reiten quiver of A is given by the pre-projective component of C_0, a semi-regular tubular family $\mathcal{T}_0$ of B_1-modules, separating tubular families $\mathcal{T}_\gamma$ with $\gamma \in \mathbb{P}^1$ of B_1-modules, a semi-regular tubular family $\mathcal{T}_1$ containing semi-regular tubes, each of them formed of B_1-modules or B_2-modules, separating tubular families $\mathcal{T}_\delta$ with $\delta \in \mathbb{P}^1$ of B_2-modules and a semi-regular tubular family $\mathcal{T}_2$ of B_2-modules, and the pre-injective component of C_3.

The following lemma state the general situation, and allows us to apply the results obtained in section 2 to the situation of semi-regular iterated tubular algebras.

Lemma 5.1 *Let B be a semi-regular n-iterated tubular algebra and let M be an indecomposable A-module. Then there exists B_i a tame quasi-tilted algebra such that*
a) B_i is a full convex subcategory of B.
b) supp$M \subset B_i$.

Proof:
a) It follows by construction of iterated tubular algebras.
b) We call $\mathcal{T}_0$ the first tubular family containing projective modules (may have also injective modules), $\mathcal{T}_i$ the following families containing projective and injective modules, $i = 1, \cdots, t$ and $\mathcal{T}_\infty$ the last family containing injective modules (may have also projective modules). If B_1^+ is tilted, then $B_1^+ = [E_i^1, R_i^1]C_1$, then a projective module P_j belongs to $\mathcal{T}_0$ if $j \in B_1^- \setminus C_1$. In this case an injective module I_k belongs to $\mathcal{T}_0$ if $k \in B_1^+ \setminus C_1$. If B_1^+ is tubular, then we have two tame concealed algebras C_0 and C_1, then a projective module $P_j \in \mathcal{T}_0$ if $j \in B_1^+ \setminus C_0$.

Observe that by construction an indecomposable projective B-module P_j belongs to an inserted tube of a semi-regular tubular family $\mathcal{T}_i$ with $i \neq 0, \infty$ if and only if $j \in C_{i+1} \setminus C_i$. Also, an indecomposable injective B-module I_k belongs to a co-inserted tube of the semi-regular tubular family $\mathcal{T}_i$ if and only if $i \neq 0, \infty$ for $k \in C_{i-1} \setminus C_i$.

If B_n^- is tilted then a projective module P_j belongs to $\mathcal{T}_\infty$ if $j \in B_n^- \setminus C_n$. In this case an injective module $I_k$belongs to $\mathcal{T}_\infty$ if $k \in B_n^+ \setminus C_n$. If B_n^- is tubular $I_k \in \mathcal{T}_\infty$ if $k \in B_n^- \setminus C_{n+1}$

Let M be a B-module, then if M belongs to the pre-projective component or to the tubular family $\mathcal{T}_0$ or to one of the tubular families between this family and $\mathcal{T}_1$ inclusive, then M is a B_1-module, the module over the quasi-tilted algebra B_1. If M belongs to a tubular family between the tubular families $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$, $i = 1, \cdots, t-1$ then the support of M is in B_i, since $Hom_B(P_j, M) = 0$ for all $j \in C_r$ with $r > i+1$ and $Hom_B(M, I_k) = 0$ for all $k \in C_l$, $l < i-1$.

Finally, if M belongs to $\mathcal{T}_\infty$ or to the pre-injective component then M is a B_n-module. Then the result follows. $\square$

Remark 5.1 *Observe that by [4] a semi-regular iterated tubular algebra is strongly simply connected if and only if each directed component of the algebras B_i and C_i is of tree type. The following result still holds if each algebra B_i and C_i has its pre-injective component of tree type.*

Theorem 5.1 *Let B be a strongly simply connected semi-regular n-iterated tubular algebra and M be an indecomposable B-module. Then $B[M]$ is tame if and only if the corresponding Tits form is weakly non negative.*

Proof : If B is 0-iterated or 1-iterated, the result follows from 4.1. So, we can assume that B is n-iterated, with $n \geq 2$. Take M an indecomposable B-module, by 5.1 there exists a quasi-tilted B_i such that $suppM \in B_i$. Consider the case that $B_i = B_n$ is the last quasi-tilted algebra, then since the morphisms in the Auslander-Reiten quiver of B go from left to right, it follows that the vector-space category $Hom_B(M, modB) = Hom_{B_n}(M, modB_n)$ with B_n quasi-tilted, as $B_n[M]$ is a full convex subcategory of $B[M]$, then it falls in the situation already considered in 4.1. Suppose now that $suppM \subset B_1$, and assume that M belongs to the pre-projective component of B, that is, the pre-projective component of B_1^+. In case that B_1^+ is a tubular algebra it follows from 3.1 that $B_1^+[M]$ and $B[M]$ are wild algebras and the respective Tits quadratic forms are strongly indefinite. In case that B_1^+ is tilted, observe that the pre-projective component of $B[M]$ is the same that the pre-projective component of $B_1^+[M]$. Since B_1 is quasi-tilted, the result follows from 4.1 and the fact that $B_1^+[M]$ is a full convex subcategory in $B[M]$.

So, suppose now that M is in a tube. Then M is a B_i-module, if $i \neq n$, then B_i^- is tubular, by 3.1 we only need to consider that M is in a tube that contains injective modules. Let $\mathcal{T}_1, \cdots, \mathcal{T}_\infty$ be the tubular families containing injective modules. Consider first that M belongs to a tubular family $\mathcal{T}_k$ with $k \neq \infty$. In case that the support of M is contained in the branch, it follows that $Hom_B(M, -)$ is finite and the result follows by [14]. Consider M in a tube with support in B_i^+ not in a branch, then there exists an injective in another tubular family $\mathcal{T}_j$, $j \in B_i$ such that $Hom_B(M, I_j) \neq 0$. Since B_i^- is tubular this morphism factors through a orthogonal tubular family. It follows from the argument of the five orthogonal modules in [18] that $Hom_{B_i}(M, modB_i)$ is wild, and since $Hom_{B_i}(M, modB_i) \subset Hom_B(M, modB)$ we get that $B[M]$ is wild. By other hand, since $Hom_B(M, I_j) \neq 0$ this morphism factors through a B_i-module X such that $q_{B_i}(\underline{dim}X) = 0$. By other way, B_i is a full convex subcategory then $q_B(\underline{dim}X) = q_{B_i}(\underline{dim}X) = 0$. Note also that $B_i[M]$ is a full convex subcategory of $\Lambda = B[M]$, then it follows that $q_\Lambda(2\underline{dim}X + e_s) = q_{B_i[M]}(2\underline{dim}X + e_s) = 1 - 2Hom_{B_i}(M, X) < 0$.

Now, consider the case that M belongs to $\mathcal{T}_\infty$ the last tubular family containing injective modules, it follows that M is a B_n-module, with B_n a quasi-tilted algebra and the result follows again from 4.1.$\square$

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