Bol. Soc. Paran. Mat.
(3s.) **v. 23** 1-2 (2005): 29–40.

Spin-structures and 2-fold coverings *

Daciberg L. Gonçalves, Claude Hayat and Maria Hermínia de Paula Leite Mello

ABSTRACT: We prove that the existence of a *Spin*-structure on an oriented real vector bundle and the number of them can be obtained in terms of 2-fold coverings of the total space of the $SO(n)$-principal bundle associated to the vector bundle. Basically we use theory of covering spaces. We give a few elementary applications making clear that the *Spin*-bundle associated to a *Spin*-structure is not sufficient to classify such structure, as pointed out by [6].

Key Words: Buldles, principal bundles, orientable bundles, spin-structure, fundamental group, Stiefel-Whithney classes.

Contents

1. Introduction

Let ξ be an oriented n-real vector bundle over a CW-complex X. So ξ has structural group $SO(n)$. There is a classical definition of a *Spin*-structure of ξ which is given in section 2. This definition is given by two items which are concerning to the existence of a *Spin*-principal bundle and a 2-fold covering, where some relations hold. The main purpose of this note is to give a proof that the above definition is equivalent just to the existence of some 2-fold coverings. More precisely, the *Spin*-structures are certain 2-fold coverings of the total space of the

* This work was done during the visit of the first and third authors to the Laboratoire de Mathématiques Emile Picard, Université Toulouse III during the periods 30^{th} September to 1^{th} November 2004 and 15^{th} September to 6^{th} October 2003, respectively and the visit of the second author to the Departamento de Matemática do IME-Universidade de São Paulo during the period 21^{th} July to 7^{th} December 2004. The visits were supported by the international Cooperation Capes/Cofecub project number 364/01.
2000 Mathematics Subject Cassification: 57R15, 57ML05.
Date submission 12 -Jul-2005.

associated $SO(n)$-principal bundle, $SO(n) \overset{i}{\hookrightarrow} P_{SO(n)}(\xi) \overset{p}{\longrightarrow} X$ under the usual relation between two covering spaces. Based on this, we give an alternative definition (see Definition 2.3 of a *Spin* structure).

Using this equivalent definition we easily obtain some known results about *Spin*-structure, including ones about existence and classification of the *Spin*-structures. Also it is natural to ask what happens if we look only at the *Spin*-bundle which arises in the definition of a *Spin*-structure. Namely, if for a given *Spin*-structure we consider only the *Spin*-bundle associated to it, is this sufficient to classify the *Spin*-structure? We make several calculations which illustrate that this is not the case. The principal bundle maps are an essential part of the structure.

The group $SO(n)$ has fundamental group Z_2 for $n > 2$ and Z for $n = 2$. For $n > 1$, let $Spin(n)$ be the group which is the connected 2-fold covering of the group $SO(n)$. We denote that covering by $Z_2 \hookrightarrow Spin(n) \overset{\lambda}{\longrightarrow} SO(n)$.

For the case where $n = 1$, we define a *Spin*-structure of an oriented n-real vector bundle ξ to be simply a 2-fold covering of the basis X. Since there is always a 2-fold covering of X, for example $X \times Z_2$, there is always a *Spin*-structure. In this case, we define two *Spin*-structures to be equivalent if the correspondent 2-fold coverings are equivalent as covering spaces (see [5], Chapter V section 6). So the study of *Spin*-structures of an oriented 1-real vector bundle ξ over a space X corresponds to the classical study of 2-fold coverings of X. Also, recall that there is only one orientable 1-real vector bundle ξ over a space X which is the trivial bundle.

From now on let $n > 1$ and let us assume that the covering spaces are connected.

This note contains two sections besides this one. In section 2 we state and prove the main result which is Theorem 2.1. Then we give an alternative definition, Definition 2.3, of a *Spin*-structure, and we show few results using this new definition. In section 3 we compute the set of *Spin*-structures in several examples and we look at the set of the *Spin*-bundles obtained from the *Spin*-structures.

Similar results as the ones in this note, were obtained by M. Schulz in his Phd. thesis [8].

2. Statement of the main theorem

2.1. Definitions and Theorem. Let ξ be an oriented n-real vector bundle over a CW-complex X. So ξ has structural group $SO(n)$. Consider the associated $SO(n)$-principal bundle, $SO(n) \overset{i}{\hookrightarrow} P_{SO(n)}(\xi) \overset{p}{\longrightarrow} X$, where $P_{SO(n)}(\xi)$ is the space of all oriented orthonormal frames.

Recall (see [4] p.371-372) that a *G-principal bundle* $G \hookrightarrow P \overset{p}{\to} X$, where G is a group, is given by an *atlas* (U_m, k_m). It means an open covering $\{U_m\}$ of X and homeomorphisms $k_m\colon V_m = p^{-1}(U_m) \to U_m \times G$ such that

$$\begin{array}{ccc} (U_m \cap U_n) \times G & \overset{k_m \circ k_n^{-1}}{\longrightarrow} & (U_m \cap U_n) \times G \\ (x, u) & \longmapsto & (x, \mu_G(k_{mn}(x), u)). \end{array}$$

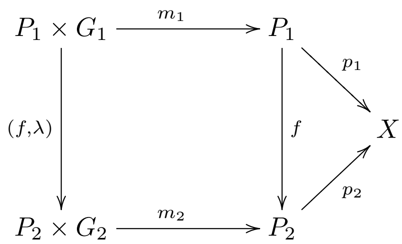
where μ_G is the product in G and k_{mn} are continuous functions $U_m \cap U_n \to G$. They verify the cocycle conditions:

$$k_{im}(x)k_{mt}(x)k_{ti}(x) = 1_G,$$

and from the definition one can define a right action of G on the total space of the bundle, which commutes with the projection and we denote it by $m : P \times G \to P$.

We recall other definitions which are going to be used.

Definition 2.1 *Given two principal bundles* $G_i \hookrightarrow P_i \xrightarrow{p_i} X$ $i = 1, 2$ *a principal bundle homomorphism is a pair* (f, λ) *where* $f : P_1 \to P_2$ *is a continuous map and* $\lambda : G_1 \to G_2$ *is a group homomorphism such that the following diagram is commutative:*



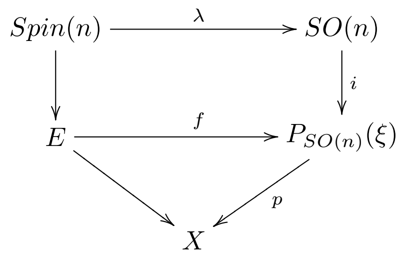
(1)

where m_i *is the action of* G_i *on* P_i.

The group $SO(n)$ has fundamental group Z_2 for $n > 2$ and Z for $n = 2$. For $n > 1$, let $Spin(n)$ be the group which is the connected 2-fold covering of the group $SO(n)$. We denote that covering by $Z_2 \hookrightarrow Spin(n) \xrightarrow{\lambda} SO(n)$ where λ is a group homomorphism. As a result of our discussion in the introduction we will consider always $n > 1$ and assume that the covering spaces are connected.

Definition 2.2 *[6,4] Let* ξ *be an oriented* n*-real vector bundle over a CW-complex* X. *Consider the associated* $SO(n)$*-principal bundle* $SO(n) \overset{i}{\hookrightarrow} P_{SO(n)}(\xi) \xrightarrow{p} X$, *we have:*

1- A Spin-structure on ξ *is a pair* (η, f) *where* η: $Spin(n) \hookrightarrow E \xrightarrow{\pi} X$ *is a* $Spin(n)$*-principal bundle and* $f : E \to P_{SO(n)}(\xi)$ *is a 2-fold covering such that the following diagram commutes:*



(2)

where $\lambda : Spin(n) \to SO(n)$ is the 2-fold covering of $SO(n)$ and (f, λ) is a map of principal bundles (see Definition 2.1).

2- The Spin-structures (η, f) and (η', f') are equivalent if there exists an isomorphism $\psi\colon \eta \to \eta'$ such that $f' \circ \psi = f$.

The condition above, $f' \circ \psi = f$, is the classical equivalence of two coverings.

$$\begin{array}{ccc} E & \xrightarrow[\approx]{\psi} & E' \\ & {}_{f}\searrow \quad \swarrow_{f'} & \\ & P_{SO(n)}(\xi) & \end{array} \tag{3}$$

Given a differentiable manifold which is oriented, we have the notion of *Spin*-manifold. For, consider the tangent bundle of the manifold which is a $SO(n)$-bundle as result of the given orientation. So we can apply the Defintion 2.2. When there is a *Spin*-structure we say that the oriented manifold admits a *Spin*-structure or it is *Spinable.*

We recall some constructions of *connected* covering. If $p : M \to N$ is a 2-fold covering of a space N, then $p_\sharp(\pi_1 M)$ is the kernel of an epimorphism $\varphi\colon \pi_1 N \to Z_2$. Conversely, let $\varphi\colon \pi_1 N \to Z_2$ be an epimorphism. If $\widetilde{N}$ denotes the universal cover of N, then the projection $p\colon M = \widetilde{N}/\ker\varphi \to N$ is a 2-fold covering such that $p_\sharp(\pi_1 M) = \ker\varphi$ and the following diagram is commutative (we identify $\ker\varphi$ and Z_2 with subsets of the corresponding sets):

$$\begin{array}{ccccc} & & \pi_1 N & & \mathbb{Z}_2 \\ & & \downarrow & \swarrow & \\ \ker\varphi & \hookrightarrow & \widetilde{N} & \longrightarrow & \widetilde{N}/\ker\varphi \\ & & \downarrow & \swarrow_{p} & \\ & & N & & \end{array}$$

For example, $\pi_1 SO(2) = Z$ and $\pi_1 SO(n) = Z_2$, $n \geq 3$, admit only one epimorphism to Z_2 and, hence, there is a unique (up to covering-equivalence) connected 2-fold covering of $SO(2)$ and $SO(n)$ $n \geq 3$, resp..

Now we state the main theorem, which gives an alternative definition of the existence of a *Spin*-structure.

Let $f\colon\ E = \tilde{P}/\ker\varphi \to P = P_{SO(n)}(\xi)$ be a 2-fold covering and consider the homotopy exact sequence:

$$1 \to \pi_1 E \xrightarrow{f_\sharp} \pi_1 P \xrightarrow{\varphi} Z_2 \to 0$$

Theorem 2.1 *An oriented n-real vector bundle ξ admits a Spin-structure if and only if there exists a 2-fold covering $f\colon\ E = \tilde{P}/\ker\varphi \to P = P_{SO(n)}(\xi)$ such that*

$\varphi \circ i_\sharp : \pi_1 SO(n) \to Z_2$ is an epimorphism, where $SO(n) \to P_{S0(n)}(\xi) \to X$ is the associated $SO(n)$-principal bundle of the oriented vector bundle ξ. Further, the set of equivalence classes of 2-fold coverings (as defined by means of diagram 3) as above is in one-to-one correspondence with the set of equivalent classes of Spin-structures (as in Definition 2.2) of the oriented bundle.

Based on the Theorem above we can give the following alternative definition of a *Spin*-structure on ξ:

Definition 2.3 *Let ξ be an oriented n-real vector bundle over a CW-complex X. Consider the associated $SO(n)$-principal bundle, $SO(n) \overset{i}{\hookrightarrow} P_{SO(n)}(\xi) \xrightarrow{p} X$.*

A Spin-structure on ξ is an epimorphism $\varphi\colon \pi_1 P_{SO(n)}(\xi) \to Z_2$ such that $\varphi \circ i_\sharp \colon \pi_1 SO(n) \to Z_2$ is an epimorphism.

Now we derive some results using this definition.

Remark 2.1 *Given an orientable bundle, one can choose an orientation. If the base X is connected then there are two possible orientations. In any case if ξ is an orientable bundle and ξ_1 is an oriented bundle obtained from ξ by giving an orientation, we can ask for the number of Spin-structures (possibly zero) of this oriented bundle ξ_1. It is not difficult to see that the number of Spin-structures for ξ_1 is independent of the choice of the orientation of the bundle ξ. In particular there is a Spin-structure of the bundle with respect to one orientation if and only if there is a Spin-structure with another orientation.*

2.2. CLASSICAL RESULTS. As before, let $P = P_{SO(n)}(\xi)$ and define

$$A = \{\varphi\colon \pi_1 P \to Z_2 \mid \varphi \circ i_\sharp \quad \text{is an epimorphism}\}.$$

Corollary 2.1A *The cardinality of the set $S(\xi)$ of the Spin−structures on ξ equals the cardinality of A. The set A is either empty or*

$$\#A = \#Hom(\pi_1 X, Z_2) = \#H^1(X; Z_2).$$

Proof: The exact sequence on homotopy of the fibration $SO(n) \hookrightarrow P \to X$

$$\pi_1 SO(n) \xrightarrow{i_\sharp} \pi_1 P \to \pi_1 X \to 0$$

gives the following exact sequence

$$0 \to Hom(\pi_1 X, Z_2) \to Hom(\pi_1 P, Z_2) \xrightarrow{\tilde{\imath}} Z_2 \tag{4}$$

because $Hom(\pi_1 SO(n), Z_2) = Z_2$ for $n > 1$. When A is not empty then $\tilde{\imath}$ is an epimorphism. Hence, $Hom(\pi_1 P, Z_2)$ is decomposed into the two cosets modulo $ker\tilde{\imath} = Hom(\pi_1 X, Z_2)$. The non-trivial coset is A. □

In fact, the exact sequence (4) is part of a longer sequence. This longer sequence is obtained as follows: consider the Serre spectral sequence of the fibration $SO(n) \overset{i}{\hookrightarrow} P_{SO(n)}(\xi) \overset{p}{\longrightarrow} X$. It gives the so called Serre exact sequence [1, th5.12]:

$$0 \to H^1(X; Z_2) \to H^1(P; Z_2) \to H^1(SO(n); Z_2) \overset{w}{\longrightarrow} H^2(X; Z_2) \tag{5}$$

where the image of the generator of $H^1(SO(n); Z_2) = Z_2$ by the morphism w is the second Stiefel-Whitney class of ξ (see [4]).

This exact sequence can be rewritten under an equivalent form:

$$0 \to Hom(\pi_1 X, Z_2) \overset{p^*}{\longrightarrow} Hom(\pi_1 P, Z_2) \overset{i^*}{\longrightarrow} Z_2 \overset{w}{\longrightarrow} H^2(X; Z_2). \tag{6}$$

Using this sequence together with the previous Corollary we obtain the following well known result:

Corollary 2.1B *Let ξ be an oriented n-vector bundle over a CW complex X. Then ξ admits a Spin-structure if and only if the second Stiefel-Whitney class of ξ is zero.*

Corollary 2.1C *The projective space RP^5 of dimension 5 does not admit a Spin-structure.*

Proof: The projective space RP^5 is known to be orientable. The second Stiefel-Whitney class of its tangent bundle by [7] p. 46 is nontrivial. So by the previous Corollary the result follows. □

2.3. PROOF OF THEOREM 2.1. One direction of the statement is clear. For, given a *Spin*-structure (η, f) take the double covering $f : E \to P$. From the diagram 2 it follows that this covering has the property that $\varphi \circ i_\# : \pi_1(SO(n)) \to Z_2$ is an epimorphism.

For the converse, consider $\varphi \in A$ and $f : E = \tilde{P}/ker\varphi \to P$. We will first show that $p \circ f : E \to X$ is a locally trivial bundle with fiber *Spin*(n) and then that it is in fact a principal bundle.

By hypothesis $SO(n) \hookrightarrow P \overset{p}{\longrightarrow} X$ is a $SO(n)$-principal bundle. Let us denote its atlas by (U_m, k_m) where $\{U_m\}$ is an open covering of X and $k_m : p^{-1}(U_m) \to U_m \times SO(n)$ a trivialization of the principal bundle. The injection $j\colon V_m = p^{-1}(U_m) \hookrightarrow P$ denotes the injection as a subset of P. The restriction $f' = f\mid_{f^{-1}(V_m)}$ is a 2-fold covering of V_m. By [5] Proposition 11.1 p. 177, we have the equality:

$$(*) \quad f'_\sharp \pi_1 f^{-1}(V_m) = j_\#^{-1} f_\sharp \pi_1 E,$$

from which it is easy to prove that $\ker \varphi \circ j_\sharp = f'_\sharp \pi_1 f^{-1}(V_m)$. As the open sets U_m in the atlas (U_m, k_m) can be taken contractible, the homeomorphisms k_m induce isomorphisms in the fundamental group of $\pi_1(V_m)$ and $\pi_1(SO(n))$. Then, the hypothesis $\varphi \circ i_\sharp$ being surjective implies that $\varphi \circ j_\sharp$ is also surjective, so $f^{-1}(V_m)$ is connected.

Now, if $y \in V_m$, we have $p \circ f'(f^{-1}(y)) = p(y)$, hence $f^{-1} \circ k_m^{-1}(U_m \times SO(n))$ is homeomorphic to $U_m \times H$ for some H, which is a non-trivial 2-fold covering of $U_m \times SO(n)$ inducing the identity on U_m. So H is a non-trivial 2-fold covering of $SO(n)$, which is unique. This proves that $H = Spin(n)$ and that there exists a homeomorphism $h_m\colon (p \circ f)^{-1}(U_m) \to U_m \times Spin(n)$ verifying

$$(**) \quad (id_{U_m} \times \lambda) \circ h_m = k_m \circ f'.$$

See the diagram below:

$$\begin{array}{ccc} f^{-1}(V_m) = \quad f^{-1}(p^{-1}(U_m)) & \xrightarrow[\approx]{h_m} & U_m \times \mathrm{Spin}(n) \\ \Big\downarrow {\scriptstyle f'=f_{|V_m}} & & \Big\downarrow {\scriptstyle \mathbb{I}\times\lambda} \\ p^{-1}(U_m) = V_m & \xrightarrow[k_m]{\approx} & U_m \times SO(n) \end{array}$$

This means that $p \circ f : E \to X$ is a locally trivial bundle with fiber $Spin(n)$. It remains to show that it is a principal bundle.

By hypothesis $k_m \circ k_n^{-1}(x, u) = (x, \mu_{SO(n)}(k_{mn}(x), u))$ where $\mu_{SO(n)}$ is the product in $SO(n)$. By construction, there was defined continuous maps

$$h_{mn} : U_m \cap U_n \to Spin(n)$$

such that $h_m \circ h_n^{-1}(x, v) = (x, h_{mn}(x)(v))$. The relation $(**)$ gives the following diagram

$$\begin{array}{ccc} (U_m \cap U_n) \times \mathrm{Spin}(n) & \xrightarrow{h_m \circ h_n^{-1}} & (U_m \cap U_n) \times \mathrm{Spin}(n) \\ \Big\downarrow {\scriptstyle id\times\lambda} & & \Big\downarrow {\scriptstyle id\times\lambda} \\ (U_m \cap U_n) \times SO(n) & \xrightarrow{k_m \circ k_n^{-1}} & (U_m \cap U_n) \times SO(n) \end{array} \tag{7}$$

which means that

$$\lambda(h_{mn}(x)(v)) = \mu_{SO(n)}(k_{mn}(x), \lambda(v)), \quad v \in Spin(n).$$

The homomorphism λ is surjective. There exists 2 preimages in $Spin(n)$ of $k_{mn}(x) \in SO(n)$, denoted by $v_{mnx_i}, i = 1, 2$, such that:

$$\lambda(h_{mn}(x)(v)) = \lambda(\mu_{Spin(n)}(v_{mnx_i}, v)).$$

In particular for $v = e$ the neutral element of $Spin(n)$

$$\lambda(h_{mn}(x)(e)) = \lambda(\mu_{Spin(n)}(v_{mnx_i}, e)).$$

One v_{mnx_i} is equal to $h_{mn}(x)(e)$, then

$$\lambda(h_{mn}(x)(v)) = \mu_{SO(n)}(\lambda(h_{mn}(x)(e)), \lambda(v)) = \lambda(\mu_{Spin(n)}(h_{mn}(x)(e), v)).$$

So $h_{mn}(x)(v)) = \mu_{Spin(n)}(h_{mn}(x)(e), v)$ or $t_{mnxv}(\mu_{Spin(n)}(h_{mn}(x)(e), v))$ where t_{mnxv} is the action of Z_2. Because $t_{mnxe} = 1$ and it is continuous in v, it is constant and $t_{mnxv} = 1$. This proves that the action of $h_{mn}(x)$ on $Spin(n)$ is by translation, hence $Spin(n) \hookrightarrow E \xrightarrow{p \circ f} X$ is a $Spin(n)$-principal bundle. □

3. What about the *Spin*−principal bundle which is given in a *Spin*−structure as defined in Definition 2.2?

Recall that in the Definition 2.2 a *Spin*-structure is a pair (η, f) where η is a *Spin*-principal bundle. In [6], Milnor pointed out that there may exist only one $Spin(n)$-principal bundle over X, up to bundle equivalence, but different *Spin*-structures on ξ, where ξ is an oriented bundle over X.

A slightly more general situation can be described as follows. We can construct a map which associates to each *Spin*-structure (η, f) the *Spin*-principal bundle η. It is natural to ask if the *Spin*-structure can be distinguished by its *Spin*-principal bundle. In this section we compute the set of *Spin*-structures as well the set of all *Spin*-principal bundles obtained from the *Spin*-structures. In some cases the examples show that the answer of the question above is "yes" and in the other cases the answer is "no". The examples where the answer is "no" illustrate precisely the situation pointed out by Milnor [6].

Our first example is an orientable bundle of dimension 2 over S^1.

3.1. SPIN-STRUCTURES OVER S^1. Let $\xi : \mathbb{R}^2 \hookrightarrow \mathbb{R} \times TS^1 \longrightarrow S^1$ be the 2-vector bundle over S^1, where $TS^1 = R \times S^1$ is the tangent space of S^1. The principal $SO(2)$−fibre bundle associated to ξ is $SO(2) \hookrightarrow P_{SO(2)}(\xi) \longrightarrow S^1$. In fact we have $SO(2) = S^1$ and $P_{SO(2)}(\xi) = S^1 \times S^1$.

a) The index 2 subgroups of $\pi_1 P_{SO(2)}(\xi)$

The subgroups of index 2 of $\pi_1 P_{SO(2)}(\xi) = Z \times Z$ are the kernels of surjective homomorphisms of $Z \times Z$ to Z_2. There are three surjective homomorphisms:

$$\begin{aligned} \varphi_1 &: Z \times Z \to Z_2, \quad (1,0) \mapsto 1, (0,1) \mapsto 0; \\ \varphi_2 &: Z \times Z \to Z_2, \quad (1,0) \mapsto 0, (0,1) \mapsto 1; \\ \varphi_3 &: Z \times Z \to Z_2, \quad (1,0) \mapsto 1, (0,1) \mapsto 1. \end{aligned}$$

Then

$$\begin{aligned} H_1 &:= \ker \varphi_1 &&= 2Z \times Z, \\ H_2 &:= \ker \varphi_2 &&= Z \times 2Z, \\ H_3 &:= \ker \varphi_3 &&= \{(a, a+2k) \mid a \in Z, k \in Z\} \quad \cong \Delta \oplus (\{0\} \times 2Z) \\ & &&\text{with } \Delta = \{(k,k) \mid k \in Z\}. \end{aligned}$$

b) Description of the 2-fold coverings of $P_{SO(2)}(\xi)$

The universal cover of $S^1 \times S^1$ is:

$$\mathbb{Z} \times \mathbb{Z} \overset{(\times 2\pi, \times 2\pi)}{\hookrightarrow} \mathbb{R} \times \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} = S^1 \times S^1.$$

The operation of H_i on $\mathbb{R}\times\mathbb{R}$ is the restriction of the operation of $\mathbb{Z}\times\mathbb{Z}$ on $\mathbb{R}\times\mathbb{R}$. We denote by $E_i = (\mathbb{R}\times\mathbb{R})/H_i$ the 2-fold covering of $S^1\times S^1$ with fundamental group H_i. The projection $f_i\colon E_i \to S^1\times S^1$ is defined by the diagram

$$\begin{array}{ccc}
Z\times Z & & \\
\big\downarrow {\scriptstyle (\times 2\pi,\times 2\pi)} & & \\
\mathbb{R}\times R & \longrightarrow & (R\times R)/H_i = E_i \\
\big\downarrow & \swarrow {\scriptstyle f_i} & \\
R/2\pi\mathbb{Z}\times R/2\pi\mathbb{Z} & &
\end{array}$$

Now we have to select the double coverings which provide the Spin-structures.

c) Eliminate one of the coverings of $S^1\times S^1$

i) An element of E_1 is a coset

$$(\vartheta,\mu)+H_1 = \{(\vartheta+4k_1\pi, \mu+2k_2\pi) \mid (\vartheta,\mu)\in R\times R, k_1,k_2\in Z\}.$$

Remark that (ϑ,μ) and $(\vartheta+2\pi,\mu)$ are not in the same class mod H_1. As usual, it is possible to define

$$f_1((\vartheta,\mu)+H_1) = (\vartheta+2\pi Z, \mu+2\pi Z);$$

now

$$f_1^{-1}(0,0) = \{H_1, (2\pi,0)+H_1\} \cong Z_2.$$

ii) An element of E_2 is a coset

$$(\vartheta,\mu)+H_2 = \{(\vartheta+2k_1\pi, \mu+4k_2\pi) \mid (\vartheta,\mu)\in R\times R, k_1,k_2\in Z\}.$$

As usual, it is possible to define

$$f_2((\vartheta,\mu)+H_2) = (\vartheta+2\pi Z, \mu+2\pi Z);$$

then

$$f_2^{-1}(0,0) = \{H_2, (0,2\pi)+H_2\} \cong Z_2.$$

iii) The operation of H_3 on $R\times R$ is the restriction of the operation of $Z\times Z$ on $R\times R$.

$$\begin{array}{ccc}
H_3\times R\times R & \longrightarrow & R\times R \\
((k_1, k_1+2k_2),(\vartheta,\mu)) & \mapsto & (\vartheta+2k_1\pi, \mu+2k_1\pi+4k_2\pi).
\end{array}$$

Then

$$f_3((\vartheta,\mu)+H_3) = (\vartheta+2\pi Z, \mu+2\pi Z)$$

is well defined. We remark that (ϑ,μ) and $(\vartheta,\mu+2\pi)$ are not in the same class mod H_3 but $(\vartheta,\mu+2\pi)$ and $(\vartheta+2\pi,\mu)$ are in the same class mod H_3; hence

$$f_3^{-1}(0,0) = \{H_3, (2\pi, 0) + H_3\} \cong Z_2.$$

Defining $i_\sharp \colon Z \to Z \times Z$ by $a \mapsto (a, 0)$, the map $\varphi_2 \circ i_\sharp$ is not surjective and the maps $\varphi_1 \circ i_\sharp$ and $\varphi_3 \circ i_\sharp$ are surjective.

$$\begin{array}{ccccccccc} & & & & \ker \varphi_\ell & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i_\sharp} & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \varphi_\ell} & & & & \\ & & & & \mathbb{Z}_2 & & & & \end{array}$$

In the sense of Definition 2.2, the two coverings E_1 and E_3 define different *Spin*-structures on ξ.

It is worth to mention that the Z_2-coverings f_1 and f_3 are equivalent to the bounding *Spin*-structure on S^1 and to the Lie group *Spin*-structure on S^1, respectively, as defined by Kirby in [3] pg. 35 and 36.

We found out that only the homomorphisms $\varphi_1 \circ i_\sharp$ and $\varphi_3 \circ i_\sharp$ are surjective. From Theorem 2.1, this property implies that (f_i, λ) is a principal bundle map. The $Spin(2)$-principal bundles associated to the *Spin*-structures E_1, E_3 are orientable S^1-bundles, so they are classified by homotopy classes of maps $S^1 \to BSpin(2) = BS^1 = CP^\infty$. Since CP^∞ is simply connected, there is only one homotopy class of maps $S^1 \to BS^1 = CP^\infty$. This homotopy class represents the trivial $Spin(2)$-principal bundle. So we conclude that the *Spin*-bundles associated to the two different *Spin*-structures are isomorphic. This gives an example of the phenomenon pointed out by Milnor in [6].

Remark 3.1 *Since $Spin(n)$-principal bundles over a space X are classified by the set of homotopy classes of maps $[X, BSpin(n)]$, the above example shows that in general one can not expect that the set of Spin-structures can be identified with the set of homotopy classes of maps $[X, BSpin(n)]$. Furthermore, the example above has the property that any map $X \to BSpin(n)$ is a homotopy lifting of a classifying map $\varphi_\xi \colon X \to BSO(n)$ of the given bundle ξ, through the map $BSpin(n) \to BSO(n)$. Hence, this shows that even if you consider the set of maps $X \to BSpin(n)$ which are homotopy liftings of a classifying map $\varphi_\xi \colon X \to BSO(n)$ of the given bundle ξ, through the map $BSpin(n) \to BSO(n)$, the set of homotopy classes of such maps will not classify the Spin-structures.*

Remark 3.2 *Although the Spin-bundles do not classify the Spin-structures, as shown by the example above, following [3] p. 34 we have the following alternative description of the Spin-structures in terms of homotopy classes of maps. Given a $SO(n)$-principal bundle let $f : X \to BSO(n)$ a map which classsify the bundle. Consider the set L of all maps $f' : X \to BSpin(n)$ which are liftings of f with respect to the map $B\lambda : BSpin(n) \to BSO(n)$. There is a one-to-one correspondence* *between the set of Spin-structures of ξ and the set of homotopy classes of maps of L. So in the example above the set L contains exactly two homotopy classes of maps. (see [3]).*

3.2. THE TRIVIAL BUNDLE OVER PROJECTIVE SPACES. This family of examples includes the example provided by Milnor in [6]:

Let X be the projective space $\mathbb{R}P^m$ of dimension m. By $\xi\colon\ \mathbb{R}^n \hookrightarrow \mathbb{R}P^m \times \mathbb{R}^n \to \mathbb{R}P^m$ we denote the trivial n- real vector bundle over $\mathbb{R}P^m$ with a fixed orientation on $\mathbb{R}^n$. This vector bundle is orientable, although the total space as well as the base are non-orientable manifolds if m is even.

The $SO(n)$-principal bundle associated to $\xi(n)$ is

$$SO(n) \hookrightarrow P_{SO(n)}(\xi) = \mathbb{R}P^m \times SO(n) \to \mathbb{R}P^m. \tag{8}$$

Now we consider two cases.

Case I- Let $n = 2$. In this case let us consider 2-fold coverings E_1, E_2 of $P_{SO(2)}(\xi)$, where now $P_{SO(2)}(\xi) = \mathbb{R}P^m \times S^1$ since $SO(2) = S^1$. The first covering is $E_1 = \mathbb{R}P^m \times \text{Spin}(2)$ which corresponds to the subgroup $\pi_1(\mathbb{R}P^m) \times 2\mathbb{Z} = \mathbb{Z}_2 + 2\mathbb{Z}$. For the second covering consider the homomorphism $\varphi_2 : \pi_1(\mathbb{R}P^m) \times \mathbb{Z} \to \mathbb{Z}_2$ such that $\varphi_2(a) = 1 = \varphi_2(h)$ where a is the generator of $\pi_1(\mathbb{R}P^m)$ and h is a generator of $\mathbb{Z}$. It is not difficult to see that $\ker\varphi_2$ is isomorphic to $\mathbb{Z}$. It is generated by the element $a + h$. So it follows that the total space of the two *Spin*-bundles do not have the same homotopy type so they can not be isomorphic as *Spin*-principal bundles.

Case II Let $m \leq 3$ and $n \geq 3$.

Because $n \geq 3$ we have that $Spin(n)$ is 2-connected, i.e. $\pi_1(Spin(n)) = \pi_2(Spin(n)) = 1$. So the classifying space for the *Spin*-principal bundles, denoted by $BSpin(n)$, is 3-connected. Hence, up to bundle equivalence, there is *only one $Spin(n)$-principal bundle over $\mathbb{R}P^m$ $(m \leq 3)$* (the trivial principal bundles). Since the trivial bundle ξ admits a *Spin*-structure, the number of *Spin*-structures on $\xi(n)$, which is the cardinality of $H^1(\mathbb{R}P^m; Z_2)$ by the Corollary 2.1A, is 2. This example also gives support to the remarks 3.1 and 3.2.

Remark 3.3 *The reader may ask if there is an example of a nontrivial $SO(n)$-principal bundle ξ which admits two different Spin-structures having ismorphic Spin-principal bundles. The answer is yes. An example is the tangent bundles of an orientable compact surface of genus greater than 1. See [2]*

Remark 3.4 *In the begining of this Section we have considered the map which associates to each Spin-structure (η, f) the Spin-principal bundle η. Another interesting related question is to study the image of the map above.*

In [4] pg. 83, 84 one can find more about the study of the set of all Spin-principal bundles which comes from the set of Spin-structures.

References

1. Cartan, H., Eilenberg, S., *Homological Algebra*, Princeton University Press, (1973).

2. Gonçalves, D. L., Hayat, C., Mello, M. H. de P. L., Zieschang, H., *Spin-structures of surfaces.* In preparation.

3. Kirby, R.C., *The Topology of 4-manifolds*, LN in Math. vol 1374, (1989).

4. Lawson, H. B., Michelson, Marie-Louise, *Spin Geometry*, Princeton Mathematical series, **38**, Princeton University Press, Princeton, N.J. (1989).

5. Massey, W.S., *Algebraic Topology: An Introduction* Harcourt, Brace & World (1967).

6. Milnor, J., *Spin structure of manifolds*, Enseignement des math. (2) 9, 198-203, (1963).

7. Milnor, J., Stasheff James D., *Characteristic Classes*, Annals of Mathematics Studies, **76**, Princeton University Press, Princeton, N.J., (1974).

8. Schulz, M., *Spin-Strukturen und Anwendungen in der Knotentheorie*, Phd Thesis, Bochum-Germany, (1987).

Daciberg L. Gonçalves
Departamento de Matemática,
IME-USP, Caixa Postal 66281 - Agência Cidade de São Paulo
05311-970 - São Paulo - SP
Brasil
e-mail: dlgoncal@ime.usp.br

Claude Hayat
Département de Mathématiques,
Laboratoire Emile Picard, UMR 5580 Université Toulouse III
118 Route de Narbonne, 31062 Toulouse Cédex
France
e-mail: hayat@picard.ups-tlse.fr

Maria Hermínia de Paula Leite Mello
Departamento de Análise Matemática,
Universidade Estadual do Rio de Janeiro- Rio de Janeiro
Brasil
e-mail mhplmello@ime.uerj.br