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Spectrum of the $\mathrm{A_p}$-Laplacian Operator

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ABSTRACT: This work deals with the nonlinear boundary eigenvalue problem

$$(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,I)}) \qquad \begin{cases} -\mathcal{A}_p u = \lambda\rho(x)|u|^{p-2}u & in \quad I =]a,b[, \\ u(a) = u(b) = 0, \end{cases}$$

where $\mathcal{A}_p$ is called the $\mathbf{A}_p$-Laplacian operator and defined by $\mathcal{A}_p u = \left(\Lambda(x)|u'|^{p-2}u'\right)'$, $p > 1$, λ is a real parameter, ρ is an indefinite weight, a, b are real numbers and $\Lambda \in C^1(I) \cap C^0(\bar{I})$ and it is nonnegative on $\bar{I}$.

We prove in this paper that the spectrum of the $\mathbf{A}_p$-Laplacian operator is given by a sequence of eigenvalues. Moreover, each eigenvalue is *simple, isolated* and verifies the strict monotonicity property with respect to the weight ρ and the domain I. The $k-$th eigenfunction corresponding to the k-th eigenvalue has exactly $k-1$ zeros in (a,b). Finally, we give a simple *variational formulation of eigenvalues.*

Keywords: A_p-Laplacian spectrum, Nonlinear eigenvalue problem.

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1. Introduction

The operator defined by $\mathcal{A}_p u = \big(\Lambda(x)|u'|^{p-2}u'\big)'$ in a real interval $I=]a,b[$ is called $\mathbf{A}_p$-Laplacian. The spectrum of the $\mathbf{A}_p$ Laplacian operator is defined as the set $\sigma_p(\mathcal{A}_p,\Lambda,\rho)$ of $\lambda \equiv \lambda(\Lambda,\rho,I)$ for which there exists a nontrivial solution $u \in W_0^{1,p}(I)$ in a weak sense of problem

$$(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,I)}) \qquad \begin{cases} -\mathcal{A}_p u = \lambda\rho(x)|u|^{p-2}u & in \quad I=]a,b[, \\ u(a)=u(b)=0, \end{cases} \tag{1}$$

where $p>1$, Λ is a positive real valued function such that $\Lambda \in C^1(I)\cap C^0(\bar{I})$ and $\rho \in M^+(I)$ is the weight. $M^+(I)$ is defined by

$$M^+(I) = \{\rho \in L^\infty(I)/meas\{x\in I,\ \rho(x)>0\}\neq 0\}\,. \tag{2}$$

In the same way we define $M^-(I)$ by

$$M^-(I) = \{\rho \in L^\infty(I)/meas\{x\in I,\ \rho(x)<0\}\neq 0\}\,. \tag{3}$$

The values $\lambda(\Lambda,\rho,I)$ for which there exists a nontrivial solution of problem (1) are called eigenvalues and the corresponding solutions are the eigenfunctions. We denote by $\sigma_p^+(\mathcal{A}_p,\Lambda,\rho)$ the set of all positive eigenvalues and by $\sigma_p^-(\mathcal{A}_p,\Lambda,\rho)$ the set of all negative ones.

2. Existence of eigenvalues by minimax techniques

In this section we study the existence of a sequence of eigenvalues for problem (1). The method used is an adaptation of the Ljusternik-Shnirelmann theory. For more details about the theory see [1,10,14,16,17,20].

The Sobolev space $W_0^{1,p}(I)$ is endowed with the norm $\|u\|_{1,p,\Lambda} = \left(\int_I \Lambda(x)|u'(x)|^p dx\right)^{\frac{1}{p}}$.

The equivalence between the previous norm and the usual one is obvious. ($\Lambda \in C^1(I)\cap C^0(\bar{I})$ and $\Lambda(x)>0$ in $\bar{I}$, then $\delta \le \Lambda(x) \le M$ for all $x\in\bar{I}$).

The weight ρ is assumed to belong to $M^+(I)$. Consider $B : W_0^{1,p}(I) \longrightarrow \mathbb{R}$ defined by $B(u) = \frac{1}{p}\int_I \rho(x)|u|^p\,dx$, that is the potential of $b : W_0^{1,p}(I) \longrightarrow W_0^{-1,p'}(I)$, defined by $b(u) = \rho(x)|u|^{p-2}u$. By the compactness of the Sobolev inclusion, b is compact and uniformly continuous on bounded sets and as a consequence B is compact. Moreover, b is odd and B is even. The idea is to obtain critical points of $B(u)$ on the manifold

$$\mathcal{M} = \left\{u \in W_0^{1,p}(I)\,|\,\frac{1}{p}\int_I \Lambda(x)|u'|^p = \alpha\right\}. \tag{4}$$

We introduce the classical *genus* function due to *Krasnoselskii* [13], we prefer the definition given by *Coffman* [6]. Given X a Banach space, we consider the class $\Sigma = \{A\subset X\,|\,A\ closed,\ A=-A\}$.

Definition 2.1 *(cf [1,17,20] Consider the map, γ, defined as follows*

$$\begin{array}{rccc} \gamma: & \Sigma & \longrightarrow & \mathbb{N}\cup\infty \\ & A & \longrightarrow & \gamma(A) \end{array}$$

where $\gamma(A) = \min\left\{k \in \mathbb{N} \,|\, \exists\varphi \in C(A, \mathbb{R}^k - \{0\}, \varphi(x) = -\varphi(-x)\right\}$. If the infimum does not exist, then we define $\gamma(A) = +\infty$. We call $\gamma(A)$ the genus of $A \in \Sigma$.

We will prove the existence of a sequence of critical values and critical points by using a *mini-max* argument on the class of sets defined below. For each $k \in \mathbb{N}$ consider

$$\mathcal{C}_k = \{C \subset \mathcal{M} \,|\, C \; compact, \; C = -C, \; \gamma(C) \geq k\}. \tag{5}$$

The main result on the existence of eigenvalues is the following

Theorem 2.1 *(cf [2] , [11]) Let $\mathcal{C}_k$ be defined by (5), let $\rho \in M^+(I)$ and let β_k be defined by*

$$\beta_k = \sup_{C\in\mathcal{C}_k} \min_{u\in C} B(u). \tag{6}$$

Then, $\beta_k > 0$, and there exists $u_k \in \mathcal{M}$ such that $B(u) = \beta_k$, and u_k is a solution of problem (1) for $\lambda_k = \frac{\alpha}{\beta_k}$.

Proposition 2.1 *(cf [2] , [11]) Let β_k be defined in (6). Then $\lim\limits_{k\to+\infty} \beta_k = 0$. As a consequence $\lambda_k = \alpha\beta_k^{-1} \to +\infty$ as $k \to +\infty$*

Proof *(cf [11])* Consider $\{E_j\}_{j\geq 1}$ sequence of linear subspaces in $W_0^{1,p}(I)$ such that

$$i) E_k \subset E_{k+1}; \quad ii)\overline{\mathcal{L}(\cup E_k)} = W_0^{1,p}(I); \quad iii) \dim E_k = k.$$

Define $\tilde{\beta}_k = \sup_{C\in\mathcal{C}_k} \min_{u\in C\cap E_{k-1}^c} B(u)$, where E_k^c is the linear and topological complementary of E_k. Obviously $\tilde{\beta}_k \geq \beta_k > 0$. Now, if for some positive constant, $\delta > 0$, $\tilde{\beta}_k > \delta > 0$ for all $k \in \mathbb{N}$, then for each $k \in \mathbb{N}$ there exists $C_k \in \mathcal{C}_k$ such that $\tilde{\beta}_k > \min\limits_{u\in C_k\cap E_{k-1}^c} B(u) > \delta$, and then there exists $u_k \in C_k \cap E_{k-1}^c$ such that $\tilde{\beta}_k > B(u_k) > \delta$. In this way $\{u_k\} \subset \mathcal{M}$, $B(u_k) > \delta > 0$ for all $k \in \mathbb{N}$, hence for some subsequences,

$$\begin{array}{lllllll} u_k & \rightharpoonup & v, & as & k & \to & \infty, \quad weakly\ in\ W_0^{1,p}(I) \\ u_k & \to & v, & as & k & \to & \infty, \quad in\ L^p(I). \end{array}$$

As a consequence $B(v) > \delta$ and this is a contradiction because $u_k \in E_{k-1}^c$ implies $v = 0$.

Remark 2.1 *It is obvious to show the following Corollary*

Corollary 2.1A *Let $\mathcal{C}_k$ be defined by (5), let $\rho \in M^-(I)$ and let β_{-k} be defined by*

$$\beta_{-k} = \inf_{C\in\mathcal{C}_k} \max_{u\in C} B(u). \tag{7}$$

Then, $\beta_{-k} < 0$, and there exists $u_k \in \mathcal{M}$ such that $B(u) = \beta_{-k}$, and u_k is a solution of problem (1) *for $\lambda_{-k} = \frac{\alpha}{\beta_{-k}}$. Moreover, $\lim_{k\to+\infty} \beta_{-k} = 0$. As a consequence $\lambda_{-k} = \alpha\beta_{-k}^{-1} \to -\infty$ as $k \to +\infty$*

In [7] a similar result is given in the linear case i.e $p = 2$ and $\Lambda \equiv 1$. So, in the next we are interesting by the positive eigenvalues, we will give a remark about the negative eigenvalues at the end of the paper.

By Theorem 1, the values given by

$$\frac{1}{\lambda_k(\Lambda,\rho,I)} = \sup_{C\in\mathcal{C}_k} \min_{u\in C} \frac{\displaystyle\int_I \rho(x)|u|^p\,dx}{\displaystyle\int_I \Lambda(x)|u'|^p\,dx} \tag{8}$$

are positive eigenvalues of problem (1). Put $S_\Lambda = \left\{ u \in W_0^{1,p}(I) \mid \int_I \Lambda(x)|u'|^p\,dx = 1 \right\}$ then, S_Λ is the unit sphere of $(W_0^{1,p}(I), \|u\|_{1,p,\Lambda})$. Next, define $\mathcal{B}_k = \{C \in \mathcal{C}_k \mid C \subset S_\Lambda\}$. then (8) can be rewritten as

$$\frac{1}{\lambda_k(\Lambda,\rho,I)} = \sup_{C\in\mathcal{B}_k} \min_{u\in C} \int_I \rho(x)|u|^p\,dx \tag{9}$$

Remark 2.2 *The main open problem is to show that the sequence given by relation* (9) *contains all the eigenvalues of problem (1). This is true in the next situations $\Lambda(x) \equiv 1$ and $\rho(x) \equiv 1$ cf [8,10,12,15]. $\Lambda(x) \equiv 1$ and $\rho(x) \geq 0$ and continuous cf [9].*

The problem (1) is still open for $\Lambda(x) \not\equiv 1$ and/or $\rho \in L^\infty(I)$. The authors treated, in first time, the situation $\Lambda(x) \equiv 1$ and $\rho \in L^\infty(I)$, after they remarked that problem (1) can also be treated by the same reasonnement with some little modifications. So, in this paper we will resolve problem (1) and, in particular, recover the nonlinear Sturm-Liouville eigenvalue studied in [4,5,7,8,9,10,12,15].

3. A_p-Laplacian spectrum

3.1. Main results. The following theorem contains the main results of the paper

Theorem 3.1 *For all $p > 1$, $\rho \in M^+(I)$ and a positive function $\Lambda \in C^1(I)\cap C^0(\bar{I})$, the problem* (1) *has a nontrivial solution if and only if λ belongs to an increasing sequence $(\lambda_k)_{k\geq 1}$. Moreover,*

1. *Each eigenvalue $\lambda_k(\Lambda,\rho,I)$ is simple and any corresponding eigenfunction takes the form $\alpha v_k(x)$ with $\alpha \in \mathbb{R}$; namely the multiplicity of each eigenvalue is 1. Moreover $v_k(x)$ has exactly k-1 simple zeros.*
2. *Each $\lambda_k(\Lambda,\rho,I)$ verifies the strict monotonicity property with respect to the weight ρ and the domain I.*

3. $\sigma_p^+(\mathcal{A}_p, \Lambda, \rho) = \{\lambda_k(\Lambda, \rho, I),\ k = 1,\, 2,\, \cdots\}$. *The eigenvalues are ordered as*

$$0 < \lambda_1(\Lambda, \rho, I) < \lambda_2(\Lambda, \rho, I) < \lambda_3(\Lambda, \rho, I) < \cdots < \lambda_k(\Lambda, \rho, I) \to +\infty$$

as $k \to +\infty$.

As we will remark in the proof of Theorem 2, we have a simple variational formulation for the eigenvalues.

Corollary 3.1A *The eigenvalues of problem* (1) *are given by the simple variational formulation,*

$$\frac{1}{\lambda_k(\Lambda, \rho, I)} = \sup_{E \in \mathcal{E}_k} \min_{E \cap S_\Lambda} \int_a^b \rho(x)|v|^p \, dx \tag{10}$$

$\mathcal{E}_k = \Big\{E/E$ *is a* k *dimensional subspace of* $\Big(W_0^{1,p}(I), \|.\|_{1,p,\Lambda}\Big)\Big\}$ *and* S_Λ *is the unit sphere of* $\Big(W_0^{1,p}(I), \|.\|_{1,p,\Lambda}\Big)$

3.2. TECHNICAL LEMMAS. Before starting the proof of Theorem 2, we give some lemmas whose will be frequently used later.

Lemma 3.1 $\lambda_1(\Lambda, \rho, I)$ *is the unique eigenvalue which has an eigenfunction with constant sign. Simple; that is, if* u *and* v *are two eigenfunctions corresponding to the eigenvalue* $\lambda_1(\Lambda, \rho, I)$*, then* $v = \alpha u$ *for some* α*. Isolated; that is,* $\lambda_1(\Lambda, \rho, I)$ *is the unique eigenvalue in* $[0, a]$ *for some* $a > \lambda_1(\Lambda, \rho, I)$*. Finally,* $\lambda_1(\Lambda, \rho, I)$ *is given by the variational formulation*

$$\frac{1}{\lambda_1(\Lambda, \rho, I)} = \sup_{v \in S_\Lambda} \int_I \rho(x)|v|^p \, dx = \int_I \rho(x)|\phi_1|^p \, dx$$

where $\phi_1 \in S_\Lambda$ *is an eigenfunction corresponding to* $\lambda_1(\Lambda, \rho, I)$

Proof: The proof is an adaptation of the ones given in [2,11]. Let ϕ_1 be a positive eigenfunction corresponding to $\lambda_1(\Lambda, \rho, I)$, by the Maximum Principle (*cf* [19]) $\phi_1'(a) > 0$ and $\phi_1'(b) < 0$.

Lemma 3.2 *The restriction of a solution* $(u, \lambda(\Lambda, \rho, I))$ *of problem* (1)*, on a nodal interval* J*, is an eigenfunction of problem* $(\mathcal{V}.\mathcal{P}_{(\Lambda_{/J}, \rho_{/J}, J)})$*, and we have*

$$\lambda(\Lambda, \rho, I) = \lambda_1(\Lambda_{/J}, \rho_{/J}, J). \tag{11}$$

Proof: A nodal domain is a component of $I \backslash Z(u)$, where u is a solution of problem (1) and $Z(u) = \{x \in I \,|\, u(x) = 0\}$.
Let $v \in W_0^{1,p}(J)$ and let $\tilde{v}$ be its extension by zero on I. It is obvious that $\tilde{v} \in W_0^{1,p}(I)$. Multiply $(\mathcal{V}.\mathcal{P}_{(\Lambda, \rho, I)})$ by $\tilde{v}$ we get

$$\int_J \Lambda(x)|u'|^{p-2}u'v' \, dx = \lambda(\Lambda, \rho, I) \int_J \rho(x)|u|^{p-2}uv \, dx$$

for all $v \in W_0^{1,p}(J)$. Hence the restriction of u on J is a solution of problem $(\mathcal{V}.\mathcal{P}_{(\Lambda_{/J}, \rho_{/J}, J)})$ with constant sign. We then have (lemma 1) $\lambda(\Lambda, \rho, J) = \lambda_1(\Lambda_{/J}, \rho_{/J}, J)$. ∎

Lemma 3.3 *Each solution $(u, \lambda(\Lambda, \rho, I))$ of problem (1) has a finite number of simple zeros.*

Proof: This Lemma plays an essential role in our work. We start by showing that u has a finite number of nodal domains. Assume that there exists a sequence I_k, $k \geq 1$, of nodal intervals, $I_i \cap I_j = \emptyset$ for $i \neq j$. We deduce by lemma 2 that

$$\lambda(\Lambda, \rho, I) = \lambda_1(\Lambda_{/I_k}, \rho_{/I_k}, I_k) \geq \frac{\delta\lambda_1(1,1,I_k)}{C} = \frac{\delta\lambda_1(1,1,]0,1[)}{C(meas(I_k))^p}, \tag{12}$$

where $C = \|\rho\|_\infty$ and $\delta = \min_{x\in\bar{I}} \Lambda(x)$. From equation (12) we deduce $meas(I_k) \geq (\frac{\delta\lambda_1}{\lambda C})^{\frac{1}{p}}$, for all k, $\lambda_1 = \lambda_1(1,1,]0,1[)$ and $\lambda = \lambda(\Lambda, \rho, I)$, so

$$meas(I) = \sum_{k\geq 1} meas(I_k) = +\infty.$$

This yield a contradiction.

Let $\{I_1, I_2, \cdots I_k\}$ be the nodal domains of u. Put $I_i =]a_i, b_i[$, where $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots a_k < b_k \leq b$. It is clear that the restriction of u on $]a, b_1[$ is a nontrivial eigenfunction with constant sign corresponding to $\lambda(\Lambda, \rho, I)$. The maximum principle (*cf* [19]) yields either $u(t) > 0$ or $u(t) < 0$ for all $t \in]a, b_1[$, so $a = a_1$, and, $u'(a_1) \neq 0$ and $u'(b_1) \neq 0$. By a similar argument we prove that $b_1 = a_2$, $b_2 = a_3, \cdots b_k = b$, so u has a finite number of simple zeros. ■

Lemma 3.4 *(cf [18]) Let u be a solution of problem (1) and $u \in W_0^{1,p}(I) \cap L^\infty(I)$ then $u \in C^{1,\alpha}(I) \cap C^1(\bar{I})$ for some $\alpha \in (0,1)$.*

3.3. Proof of Theorem 2. For $n = 1$, as proved in lemma 1 $\lambda_1(\Lambda, \rho, I)$ is simple, isolated and any corresponding eigenfunction has constant sign. Hence it has no zero in (a, b). For the strict monotonicity property (SMP in brief) we state the proposition

Proposition 3.1 *$\lambda_1(\Lambda, \rho, I)$ verifies the strict monotonicity property with respect to weight ρ and the domain I, i.e., if $\rho_1, \rho_2 \in M^+(I)$, $\rho_1(x) \leq \rho_2(x)$ and $\rho_1(x) < \rho_2(x)$ in some subset of I of nonzero measure then,*

$$\lambda_1(\Lambda, \rho_2, I) < \lambda_1(\Lambda, \rho_1, I) \tag{13}$$

and, if J is a strict sub interval of I such that $\rho_{/J} \in M^+(J)$ then,

$$\lambda_1(\Lambda, \rho, I) < \lambda_1(\Lambda_{/J}, \rho_{/J}, J). \tag{14}$$

Proof: Let $\rho_1, \rho_2 \in M^+(I)$ as in proposition 1, recall that the principal eigenfunction $\phi_1 \in S_\Lambda$ corresponding to $\lambda_1(\Lambda, \rho, I)$ has no zero in I i.e $\phi_1(t) \neq 0$ for all $t \in I$. By (9) we get

$$\begin{aligned} \tfrac{1}{\lambda_1(\Lambda,\rho_1,I)} = \int_I \rho_1(x)|\phi_1|^p\,dx &< \int_I \rho_2(x)|\phi_1|^p\,dx \\ &\leq \sup_{v\in S_\Lambda} \int_I \rho_2(x)|v|^p\,dx \\ &= \tfrac{1}{\lambda_1(\Lambda,\rho_2,I)}. \end{aligned} \tag{15}$$

Then inequality (13) is proved, to prove inequality (14), let J be a strict sub interval of I and $\rho_{/J} \in M^+(J)$. Let $u_1 \in S_\Lambda(J)$, the unit sphere of $W_0^{1,p}(J)$, be the positive eigenfunction of $(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,J)})$ corresponding to $\lambda_1(\Lambda, \rho_{/J}, J)$, and denote by $\tilde{u}_1$ the extension by zero on I. Then,

$$\begin{aligned} \tfrac{1}{\lambda_1(\Lambda,\rho_{/J},J)} = \int_J \rho(x)|u_1|^p\,dx &= \int_I \rho(x)|\tilde{u_1}|^p\,dx \\ &< \sup_{v\in S_\Lambda} \int_I \rho(x)|v|^p\,dx \\ &= \tfrac{1}{\lambda_1(\Lambda,\rho,I)}. \end{aligned} \tag{16}$$

The last strict inequality holds from the fact that $\tilde{u}_1$ vanishes in I/J so can not be an eigenfunction of problem (1).

For $n = 2$ we start by proving that $\lambda_2(\Lambda, \rho, I)$ has a unique zero in (a, b).

Proposition 3.2 *There exists a unique real $c_{2,1} \in I$ for which we have $Z(u) = \{c_{2,1}\}$ for any eigenfunction u corresponding to $\lambda_2(\Lambda, \rho, I)$. For this reason, we will say $c_{2,1}$ is the zero of $\lambda_2(\Lambda, \rho, I)$.*

Proof: Let u be an eigenfunction corresponding to $\lambda_2(\Lambda, \rho, I)$. u changes sign in I (lemma 1). Consider $I_1 =]a, c_1[$ and $I_2 =]c_2, b[$ two nodal domains of u, by lemma 2, $\lambda_1(\Lambda, \rho_{/I_1}, I_1) = \lambda_2(\Lambda, \rho, I) = \lambda_1(\Lambda, \rho_{/I_2}, I_2)$. Assume that $c_1 < c_2$, choose $d \in]c_1, c_2[$ and put $J_1 =]a, d[$, $J_2 =]d, b[$, hence $J_1 \cap J_2 = \emptyset$, and for $i = 1, 2$, $I_i \subset J_i$ strictly, and $\rho_{/J_i} \in M^+(J_i)$, making use of proposition 2, by (14), we get

$$\lambda_1(\Lambda, \rho_{/J_1}, J_1) < \lambda_1(\Lambda, \rho_{/I_1}, I_1) = \lambda_2(\Lambda, \rho, I) \tag{17}$$

and

$$\lambda_1(\Lambda, \rho_{/J_2}, J_2) < \lambda_1(\Lambda, \rho_{/I_2}, I_2) = \lambda_2(\Lambda, \rho, I). \tag{18}$$

Let $\phi_i \in S_\Lambda$ be an eigenfunction corresponding to $\lambda_1(\Lambda, \rho_{/J_i}, J_i)$, by lemma 1 we have for $i = 1, 2$

$$\frac{1}{\lambda_1(\Lambda, \rho, J_i)} = \int_{J_i} \rho(x)|\phi_i|^p\,dx.$$

Put $\tilde{\phi}_i$ the extension by zero of ϕ_i on I and consider the two dimensional subspace $F = \langle\tilde{\phi}_1, \tilde{\phi}_2\rangle$. Let $K_2 = F \cap S_\Lambda \subset W_0^{1,p}(I)$, obviously $\gamma(K_2) = 2$, remark that for $v = \alpha\tilde{\phi}_1 + \beta\tilde{\phi}_2$, $\|v\|_{1,p,\Lambda} = 1 \Longleftrightarrow |\alpha|^p + |\beta|^p = 1$, hence by lemma 1 and (9), (17), (18) we obtain,

$$\begin{aligned} \tfrac{1}{\lambda_2(\Lambda,\rho,I)} &\geq \min_{v\in K_2} \int_I \rho(x)|v|^p\,dx \\ &= \min_{v=\alpha\tilde{\phi}_1+\beta\tilde{\phi}_2\in K_2} \left(|\alpha|^p \int_{J_1} \rho(x)|\phi_1|^p\,dx + |\beta|^p \int_{J_2} \rho(x)|\phi_2|^p\,dx\right) \\ &= |\alpha_0|^p \int_{J_1} \rho(x)|\phi_1|^p\,dx + |\beta_0|^p \int_{J_2} \rho(x)|\phi_2|^p\,dx \\ &= \tfrac{|\alpha_0|^p}{\lambda_1(\Lambda,\rho_{/J_1},J_1)} + \tfrac{|\beta_0|^p}{\lambda_1(\Lambda,\rho_{/J_2},J_2)} > \tfrac{|\alpha_0|^p+|\beta_0|^p}{\lambda_2(m,I)} = \tfrac{1}{\lambda_2(\Lambda,\rho,I)}, \end{aligned}$$

contradiction, hence $c_1 = c_2$. On the other hand, let v be another eigenfunction corresponding to $\lambda_2(\Lambda, \rho, I)$, denote d its unique zero in (a, b). Assume, for example, that $c < d$, by lemma 2 and relation (18), we get

$$\lambda_2(\Lambda, \rho, I) = \lambda_1(\Lambda, \rho_{/]a,d[},]a, d[) < \lambda_1(\Lambda, \rho_{/]a,c[},]a, c[) = \lambda_2(\Lambda, \rho, I).$$

Contradiction so $c = d$. So, we have proved that each eigenfunction corresponding to $\lambda_2(\Lambda, \rho, I)$ vanishes at a unique point in (a, b), and the (simple) zero is the same of all eigenfunctions, which completes the proof of proposition 3. ■

Lemma 3.5 *$\lambda_2(\Lambda, \rho, I)$ is simple, hence $\lambda_2(\Lambda, \rho, I) < \lambda_3(\Lambda, \rho, I)$.*

Proof: Let u and v be two eigenfunctions corresponding to $\lambda_2(\Lambda, \rho, I)$, by lemma 2 the restrictions of u and v on $]a, c_{2,1}[$ and $]c_{2,1}, b[$ are eigenfunctions corresponding to $\lambda_1(\Lambda, \rho_{/]a,c_{2,1}[},]a, c_{2,1}[)$ and $\lambda_1(\Lambda, \rho_{/]c_{2,1},b[},]c_{2,1}, b[)$, respectively. Making use of the simplicity of the first eigenvalue, we get $u = \alpha v$ in $]a, c_{2,1}[$ and $u = \beta v$ in $]c_{2,1}, b[$, but both of u and v are eigenfunctions, then by lemma 3, there are in $C^1(I)$, the maximum principle (*cf* [19]) tell us that $u'(c_{2,1}) \neq 0$, so $\alpha = \beta$ i.e $\lambda_2(\Lambda, \rho, I)$ is simple. Finally, by the simplicity of $\lambda_2(\Lambda, \rho, I)$ and the theorem of multiplicity (*cf* [17]) we conclude that $\lambda_2(\Lambda, \rho, I) < \lambda_3(\Lambda, \rho, I)$. ■

Proposition 3.3 *$\lambda_2(\Lambda, \rho, I)$ verifies the SMP with respect to the weight ρ and the domain I.*

Proof: Let ρ_1 and $\rho_2 \in M^+(I)$ such that, $\rho_1(x) \leq \rho_2(x)$ a.e in I and $\rho_1(x) < \rho_2(x)$ in some subset of nonzero measure, $c_{2,1}$ and $c'_{2,1}$ are the zeros of $\lambda_2(\Lambda, \rho_1, I)$ and $\lambda_2(\Lambda, \rho_2, I)$ respectively. We have to treat three situations

1. $c_{2,1} = c'_{2,1} = c$, then $meas\,(\{x \in I/\, \rho_1(x) < \rho_2(x)\} \cap]a, c[) \neq 0$, or $meas(\{x \in I/\, \rho_1(x) < \rho_2(x)\} \cap]c, b[) \neq 0$, by lemma 2 and (13) we obtain

$$\begin{array}{rcl}\lambda_2(\Lambda, \rho_2, I) & = & \lambda_1(\Lambda, {\rho_2}_{/]a,c[},]a, c[) \\ & < & \lambda_1(\Lambda, {\rho_1}_{/]a,c[},]a, c[) \\ & = & \lambda_2(\Lambda, \rho_1, I),\end{array}$$

or

$$\begin{array}{rcl}\lambda_2(\Lambda, \rho_2, I) & = & \lambda_1(\Lambda, {\rho_2}_{/]c,b[},]c, b[) \\ & < & \lambda_1(\Lambda, {\rho_1}_{/]c,b[},]c, b[) \\ & = & \lambda_2(\Lambda, \rho_1, I).\end{array}$$

2. $c_{2,1} < c'_{2,1}$, by lemma 2 and (14), we get

$$\begin{array}{rcl}\lambda_2(\Lambda, \rho_2, I) & = & \lambda_1(\Lambda, {\rho_2}_{/]a,c'_{2,1}[},]a, c'_{2,1}[) \\ & \leq & \lambda_1(\Lambda, {\rho_1}_{/]a,c'_{2,1}[},]a, c'_{2,1}[) \\ & < & \lambda_1(\Lambda, {\rho_1}_{/]a,c_{2,1}[},]a, c_{2,1}[) \\ & = & \lambda_2(\Lambda, \rho_1, I).\end{array}$$

3. $c'_{2,1} < c_{2,1}$, also by lemma 2 and (14) we obtain

$$\begin{array}{rcl}\lambda_2(\Lambda,\rho_2,I) &=& \lambda_1(\Lambda,\rho_{2_{/]c'_{2,1},b[}},]c'_{2,1},b[)\\ &\leq& \lambda_1(\Lambda,\rho_{1_{/]c'_{2,1},b[}},]c'_{2,1},b[)\\ &<& \lambda_1(\Lambda,\rho_{1_{/]c_{2,1},b[}},]c_{2,1},b[)\\ &=& \lambda_2(\Lambda,\rho_1,I).\end{array}$$

For the SMP with respect to the domain, put $J=]c,d[$ a strict sub interval of I with $\rho_{/J}\in M^+(J)$, and denote $c'_{2,1}$ the zero of $\lambda_2(\Lambda,\rho_{/J},J)$, as in the SMP with respect to the weight, three situations are presented

1. $c_{2,1}=c'_{2,1}=l$, then $]c,l[$ is a strict sub interval of $]a,l[$ or $]l,d[$ is a strict sub interval of $]l,b[$, by lemma 2 and (14), we get

$$\begin{array}{rcl}\lambda_2(\Lambda,\rho,I) &=& \lambda_1(\Lambda,\rho_{/]a,l[},]a,l[)\\ &<& \lambda_1(\Lambda,\rho_{/]c,l[},]c,l[)\\ &=& \lambda_2(\Lambda,\rho_{/J},J)\end{array}$$

or

$$\begin{array}{rcl}\lambda_2(\Lambda,\rho,I) &=& \lambda_1(\Lambda,\rho_{/]l,b[},]l,b[)\\ &<& \lambda_1(\Lambda,\rho_{/]l,d[},]l,d[)\\ &=& \lambda_2(\Lambda,\rho_{/J},J)\end{array}$$

2. $c_{2,1}<c'_{2,1}$, again by lemma 2 and (14) we get

$$\begin{array}{rcl}\lambda_2(\Lambda,\rho,I) &=& \lambda_1(\Lambda,\rho_{/]c_{2,1},b[},]c_{2,1},b[)\\ &<& \lambda_1(\Lambda,\rho_{/]c'_{2,1},d[},]c'_{2,1},b[)\\ &=& \lambda_2(\Lambda,\rho_{/J},J)\end{array}$$

3. $c'_{2,1}<c_{2,1}$, for the same reason as in the last case, we get

$$\begin{array}{rcl}\lambda_2(\Lambda,\rho,I) &=& \lambda_1(\Lambda,\rho_{/]a,c_{2,1}[},]a,c_{2,1}[)\\ &<& \lambda_1(\Lambda,\rho_{/]c,c'_{2,1}[},]c,c_{2,1}[)\\ &=& \lambda_2(\Lambda,\rho_{/J},J),\end{array}$$

The proof is complete. ■

Lemma 3.6 *For each eigenfunction u corresponding to* $\lambda(\Lambda,\rho,I)$ *such that* $Z(u)=\{c\}$ *for some real number c, then*

$$\lambda(\Lambda,\rho,I)=\lambda_2(\Lambda,\rho,I).$$

Proof: We will prove that $c=c_{2,1}$. Suppose, for example, $c<c_{2,1}$. By lemma 2 and (14) we get,

$$\begin{array}{rcl}\lambda(\Lambda,\rho,I) &=& \lambda_1(\Lambda,\rho_{/]c,b[},]c,b[)\\ &<& \lambda_1(\Lambda,\rho_{/]c_{2,1},b[},]c_{2,1},b[)\\ &=& \lambda_2(\Lambda,\rho,I)\end{array}$$

on the other hand,

$$\begin{aligned}\lambda_2(\Lambda,\rho,I) &= \lambda_1(\Lambda,\rho_{/]a,c_{2,1}[},]a,c_{2,1}[)\\ &< \lambda_1(\Lambda,\rho_{/]a,c[},]a,c[)\\ &= \lambda(\Lambda,\rho,I),\end{aligned}$$

contradiction. Hence, $c = c_{2,1}$ and $\lambda(\Lambda,\rho,I) = \lambda_2(\Lambda,\rho,I)$. The proof is complete.■

For $n > 2$, we use a recurrence argument. Suppose that for all k, $1 \le k \le n$, the following hypothesis holds

H.R.1 For any eigenfunction u corresponding to $\lambda_k(\Lambda,\rho,I)$, there exists a unique $c_{k,i}$, $1 \le i \le k-1$, such that $Z(u) = \{c_{k,i},\ 1 \le i \le k-1\}$.

H.R.2 $\lambda_k(\Lambda,\rho,I)$ is simple.

H.R.3 $\lambda_1(\Lambda,\rho,I) < \lambda_2(\Lambda,\rho,I) < \cdots < \lambda_{n+1}(\Lambda,\rho,I)$.

H.R.4 If $(u,\lambda(\Lambda,\rho,I))$ is a solution of $(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,I)})$ such that $Z(u) = \{c_i,\ 1 \le i \le k-1\}$ (i.e with $k-1$ simple zeros), then $\lambda(\Lambda,\rho,I) = \lambda_k(\Lambda,\rho,I)$.

H.R.5 $\lambda_k(\Lambda,\rho,I)$ verifies the Strict Monotonicity Property (SMP) with respect to the weight ρ and the domain I.

and prove them for $n+1$.

Proposition 3.4 *There exists a unique family $\{c_{n+1,i},\ 1 \le i \le n\}$ such that*

$$Z(u) = \{c_{n+1,i},\ 1 \le i \le n\}$$

for any eigenfunction u corresponding to $\lambda_{n+1}(\Lambda,\rho,I)$.

Proof: Let u be an eigenfunction corresponding to $\lambda_{n+1}(\Lambda,\rho,I)$, by H.R.3 and H.R.4, u has at least n zeros. According to lemma 3, we can consider the $n+1$ nodal domains of u, $I_1 =]a,c_1[$, $I_2 =]c_1,c_2[$, ... , $I_n =]c_{n-1},c_n[$, $I_{n+1} =]c,b[$. We will prove that $c = c_n$. Remark that the restrictions of u on $]a,c_i[$, $1 \le i \le n$, are eigenfunctions with $i-1$ zeros, by H.R.4 $\lambda_{n+1}(\Lambda,\rho,I) = \lambda_i(\Lambda,\rho_{/]a,c_i[},]a,c_i[)$. Assume that $c_n < c$, choose d in $]c_n,c[$ and put, $J_1 =]a,d[$, $J_2 =]d,b[$, remark that $J_1 \cap J_2 = \emptyset$, $]a,c_n[$ is a strict sub interval of $J_1 \subset I$, and $]c,b[$ is a strict sub interval of $J_2 \subset I$, it is clear that $\rho_{/J_i} \in M^+(J_i)$ for $i = 1,2$, by H.R.4 and H.R.5 we have

$$\begin{aligned}\lambda_n(\Lambda,\rho_{/J_1},J_1) &< \lambda_n(\Lambda,\rho_{/]a,c_n[},]a,c_n[)\\ &= \lambda_{n+1}(\Lambda,\rho,I),\end{aligned}$$

and

$$\lambda_1(\Lambda,\rho_{/J_2},J_2) < \lambda_1(\Lambda,\rho_{/]c,b[},]c,b[) = \lambda_{n+1}(\Lambda,\rho,I).$$

Denote by $\big(\phi_{n+1},\lambda_1(\Lambda,\rho_{/J_2},J_2)\big)$ a solution of $(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,J_2)})$, $(v,\lambda_n(\Lambda,\rho_{/J_1},J_1))$ a solution of $(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,J_1)})$, ϕ_i,,$1 \le i \le n$, the restrictions of v on I_i and $\tilde{\phi}_i$, their extensions, by zero, on I. Let $F_{n+1} = \langle \tilde{\phi}_1, \tilde{\phi}_2, \cdots, \tilde{\phi}_{n+1} \rangle$ and $K_{n+1} = F_{n+1} \cap S_\lambda$, then $\gamma(K_{n+1}) = n+1$. We obtain by (9) and the same proof as in proposition 3

$$\frac{1}{\lambda_{n+1}(\Lambda, \rho, I)} \geq \min_{K_{n+1}} \int_I \rho(x)|v|^p \, dx > \frac{1}{\lambda_{n+1}(\Lambda, \rho, I)}$$

contradiction, so $c = c_n$. On the other hand, let v be an eigenfunction corresponding to $\lambda_{n+1}(\Lambda, \rho, I)$. Denote by d_1, d_2, $\cdots, d_n$ the zeros of v. If $d_1 \neq c_1$, then

$$\begin{array}{rcl} \lambda_{n+1}(\Lambda, \rho, I) & = & \lambda_1(\Lambda, \rho_{/]a,d_1[},]a, d_1[) \\ & \neq & \lambda_1(\Lambda, \rho_{/]a,c_1[},]a, c_1[) \\ & = & \lambda_{n+1}(\Lambda, \rho, I), \end{array}$$

so $d_1 = c_1$, by the same argument we conclude that $d_i = c_i$ for $1 \leq i \leq n$. ∎

Lemma 3.7 *$\lambda_{n+1}(\Lambda, \rho, I)$ is simple, hence.*

$$\lambda_{n+1}(\Lambda, \rho, I) < \lambda_{n+2}(\Lambda, \rho, I).$$

Proof Let u and v be two eigenfunctions corresponding to $\lambda_{n+1}(\Lambda, \rho, I)$. The restrictions of u and v on $]a, c_{n+1,1}[$ and $]c_{n+1,1}, b[$ respectively, are eigenfunctions associated to $\lambda_1(\Lambda, \rho_{/]a,c_{n+1,1}[},]a, c_{n+1,1}[)$ and $\lambda_n(\Lambda, \rho_{/]c_{n+1,1},b[},]c_{n+1,1}, b[)$. By H.R.2 and H.R.4 we have $u = \alpha v$ in $]a, c_{n+1,1}[$ and $u = \beta v$ in $]c_{n+1,1}, b[$, on the other hand, u and v are $C^1(I)$ and $u'(c_{n+1,1}) \neq 0$, then $\alpha = \beta$. From the simplicity of $\lambda_{n+1}(\Lambda, \rho, I)$ and theorem of multiplicity [17] we conclude that

$$\lambda_{n+1}(\Lambda, \rho, I) < \lambda_{n+2}(\Lambda, \rho, I). \blacksquare$$

Proposition 3.5 *$\lambda_{n+1}(\Lambda, \rho, I)$ verifies the SMP with respect to the weight ρ and the domain I.*

Proof Let ρ_1, $\rho_2 \in M(I)$, such that $\rho_1(x) \leq \rho_2(x)$ with $\rho_1(x) < \rho_2(x)$ in some subset of nonzero measure. Denote $c_{n+1,i}$ and $c'_{n+1,i}$ for $1 \leq i \leq n$, the zeros of $\lambda_{n+1}(\Lambda, \rho_2, I)$ and $\lambda_{n+1}(\Lambda, \rho_2, I)$ respectively, three situations are presented

1. $c_{n+1,1} = c'_{n+1,1} = c$, one of the subsets is of nonzero measure,

   $$\{x \in I / \rho_1(x) < \rho_2(x)\} \cap]a, c[\quad \textit{and} \quad \{x \in I / \rho_1(x) < \rho_2(x)\} \cap]c, b[,$$

   by lemma 2 and (14), we have

   $$\begin{array}{rcl} \lambda_{n+1}(\Lambda, \rho_2, I) & = & \lambda_1(\Lambda, {\rho_2}_{/]a,c[},]a, c[) \\ & < & \lambda_1(\Lambda, {\rho_1}_{/]a,c[},]a, c[) \\ & = & \lambda_{n+1}(\Lambda, \rho_1, I) \end{array}$$

   or

   $$\begin{array}{rcl} \lambda_{n+1}(\Lambda, \rho_2, I) & = & \lambda_n(\Lambda, {\rho_2}_{/]c,b[},]c, b[) \\ & < & \lambda_n(\Lambda, {\rho_1}_{/]c,b[},]c, b[) \\ & = & \lambda_{n+1}(\Lambda, \rho_2, I). \end{array}$$

2. $c_{n+1,1} < c'_{n+1,1}$, also by lemma 2 and (14) we obtain

$$\begin{aligned}\lambda_{n+1}(\Lambda, \rho_2, I) &= \lambda_1(\Lambda, {\rho_2}_{/]a,c'_{n+1,1}[},]a, c'_{n+1,1}[) \\ &\leq \lambda_1(\Lambda, {\rho_1}_{/]a,c'_{n+1,1}[},]a, c'_{n+1,1}[) \\ &< \lambda_1(\Lambda, {\rho_1}_{/]a,c_{n+1,1}[},]a, c_{n+1,1}[) \\ &= \lambda_{n+1}(\Lambda, \rho_1, I).\end{aligned}$$

3. $c'_{n,1} < c_{n,1}$, from the same reason as before, we get

$$\begin{aligned}\lambda_{n+1}(\Lambda, \rho_2, I) &= \lambda_n(\Lambda, {\rho_2}_{/]c'_{n+1,1},b[},]c'_{n+1,1}, b[\\ &\leq \lambda_n(\Lambda, {\rho_1}_{/]c'_{n+1,1},b[},]c'_{n+1,1}[, b) \\ &< \lambda_n(\Lambda, {\rho_1}_{/]c_{n+1,1},b[},]c_{n+1,1}, b[) \\ &= \lambda_{n+1}(\Lambda, \rho_1, I).\end{aligned}$$

By similar argument as in proof of proposition 3, we prove the SMP with respect to the domain I. ■

Lemma 3.8 *If* $(u, \lambda(\Lambda, \rho, I))$ *is a solution of* $(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,I)})$ *such that*

$$Z(u) = \{d_1,\, d_2,\, \cdots d_n\},$$

then

$$\lambda(\Lambda, \rho, I) = \lambda_{n+1}(\Lambda, \rho, I).$$

Proof: It is sufficient to prove that $d_i = c_{n+1,i}$ for all $1 \leq i \leq n$. If $c_{n+1,1} < d_1$ then, by lemma 2, (10), H.R.4 and H.R.5,

$$\begin{aligned}\lambda(\Lambda, \rho, I) &= \lambda_1(\Lambda, \rho_{/]a,d_1[},]a, d_1[) \\ &< \lambda_1(\Lambda, \rho_{/]a,c_{n+1,1}[},]a, c_{n+1,1}[) \\ &= \lambda_{n+1}(\Lambda, \rho, I) \\ &= \lambda_n(\Lambda, \rho_{/]c_{n+1,1},b[},]c_{n+1,1}, b[) \\ &< \lambda_n(\Lambda, \rho_{/]d_1,b[},]d_1, b[) \\ &= \lambda(\Lambda, \rho, I)\end{aligned}$$

contradiction, and if $d_1 < c_{n+1,1}$ again by lemma 2, (10), H.R.4 and H.R.5 we have

$$\begin{aligned}\lambda_{n+1}(\Lambda, \rho, I) &= \lambda_1(\Lambda, \rho_{/]a,c_{n+1}[},]a, c_{n+1}[) \\ &< \lambda_1(\Lambda, \rho_{/]a,d_1[},]a, d_1[) \\ &= \lambda(\Lambda, \rho, I) \\ &= \lambda_n(\Lambda, \rho_{/]d_1,b[},]d_1, b[) \\ &< \lambda_n(\Lambda, \rho_{/]c_{n+1,1},b[},]c_{n+1}, b[) \\ &= \lambda_{n+1}(\Lambda, \rho, I)\end{aligned}$$

contradiction, the proof is then complete. Theorem 1 is proved. ■

3.4. PROOF OF COROLLARY 2. Since for $F \in \mathcal{E}_k$, the compact $F \cap S_\Lambda \in \mathcal{C}_k$, by (9) we have:

$$\sup_{F \in \mathcal{E}_k} \min_{v \in F \cap S_\Lambda} \int_I \rho(x)|v|^p\, dx \leq \frac{1}{\lambda_k(\Lambda, \rho, I)}. \tag{19}$$

On the other hand, for a k dimensional subspace F of $W_0^{1,p}(I)$, the compact set $K = F \cap S_\Lambda \in \mathcal{C}_k$. Let $c_{k,i}$ for $1 \leq i \leq k-1$ be the zeros of $\lambda_k(\Lambda, \rho, I)$. Put $c_{k,0} = a$ and $c_{k,k} = b$, let u be an eigenfunction corresponding to $\lambda_k(\Lambda, \rho, I)$. Put $\phi_1(]c_{k,i}, c_{k,i+1}[)$, $0 \leq i \leq k-1$, the restrictions of u on $]c_{k,i}, c_{k,i+1}[$ respectively and $\tilde{\phi}_1(]c_{k,i}, c_{k,i+1}[)$ their extensions by zero on I. Then put

$$F_K = \langle \tilde{\phi}_1(]a, c_{k,1}[), \tilde{\phi}_1(]c_{k,1}, c_{k,2}[), \cdots, \tilde{\phi}_1(]c_{k,k-1}, b[) \rangle,$$

to conclude $F_K \cap S_\Lambda \in \mathcal{C}_k$. By an elementary computation as in proposition 3

$$\frac{1}{\lambda_k(\Lambda, \rho, I)} = \min_{F_K \cap S_\Lambda} \int_I \rho(x)|v|^p\, dx. \tag{20}$$

Then combine (19) with (20) to get (10). ∎

3.5. REMARKS. The spectrum of the $\mathbf{A}_p$-Laplacian is entirely determined by the sequence $(\lambda_k(\Lambda, \rho, I))_{k \geq 1}$ if $\rho(x) \geq 0$ a.e in I. By the same way if $\rho(x) \leq 0$ a.e in I, then $-\rho \in M^+(I)$. Then, by theorem 2 the spectrum of the operator is constituted by a negative eigenvalues $(\lambda_{-k}(\Lambda, \rho, I))_{k \geq 1}$ so $\sigma_p(\mathcal{A}_p, \Lambda, \rho) = -\sigma_p^+(\mathcal{A}_p, \Lambda, -\rho)$. The main problem is when $\rho \in L^\infty(I)$ and ρ change sign, i.e $\rho \in M^+(I) \cap M^-(I)$. By theorem 2 and corollary 1, the spectrum of the $\mathbf{A}_p$-Laplacian is constituted by two sequence of eigenvalues one is an increasing positive sequence and the other is a decreasing negative sequence. The spectrum is given by

$$\sigma_p(\mathcal{A}_p, \Lambda, \rho) = \sigma_p^+(\mathcal{A}_p, \Lambda, \rho) \cup \sigma_p^-(\mathcal{A}_p, \Lambda, \rho). \tag{21}$$

4. Applications

4.1. P-LAPLACIAN SPECTRUM WITH INDEFINITE WEIGHT.

Definition 4.1 *The p-Laplacian spectrum with indefinite weight $\sigma_p(\Delta_p, m)$ is the set of all real numbers λ solutions of problem*

$$(\mathcal{V}.\mathcal{P}_{(m,I)}) \qquad \begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega \end{cases}$$

in a weak sense.

We know that the spectrum contains a sequence (resp. a double sequence) of eigenvalues if the weight m is positive (resp. positive and negative) somewhere. But any more information about the spectrum when $p \neq 2$ (the nonlinear problem). In the following, we will prove that the p-Laplacian spectrum with indefinite weight in one dimension is entirely given by a sequence (resp. a double sequence). For this, put $\Lambda \equiv 1$ and $\rho(x) = m(x) \in L^\infty(I)$ in theorem 2 to obtain,

$$\sigma_p(\Delta_p, m) = \sigma_p^+(\Delta_p, m) \cup \sigma_p^-(\Delta_p, m)$$

with $\sigma_p^+(\Delta_p, m) = \{\lambda_k(m, I),\ k = 1, 2, \cdots\}$. The eigenvalues are ordered as $0 < \lambda_1(m, I) < \lambda_2(m, I) < \lambda_3(m, I) < \cdots < \lambda_k(m, I) \to +\infty$. and

$$\sigma_p^-(\Delta_p, m) = \{\lambda_{-k}(m, I),\ k = 1, 2, \cdots\}.$$

The eigenvalues are ordered as $-\infty \leftarrow \lambda_{-k}(m, I) < \cdots < \lambda_{-2}(m, I) < \lambda_{-1}(m, I) < 0$. The problem is still open for $N > 1$.

4.2. p-LAPLACIAN SPECTRUM OF ORDER ONE.

Definition 1 *The p-Laplacian spectrum of order one $\sigma_1(\Delta_p, m)$ is the set of all surfaces $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^N$ solutions of problem*

$$\left(\mathcal{V}.\mathcal{P}_{(m,I)}\right) \quad \begin{cases} -\Delta_p u & = & \alpha m(x)|u|^{p-2}u + <\beta, |\nabla u|^{p-2}\nabla u> & in & \Omega, \\ u & = & 0 & on & \partial\Omega \end{cases} \tag{22}$$

in a weak sense, $\langle , \rangle$ is the scalar product in the euclidean space $\mathbb{R}^N$.

As the usual p-Laplacian spectrum we know that spectrum of order one of p-Laplacian operator contains a sequence (resp. a double sequence) of eigen-surfaces if the weight m is positive (resp. positive and negative) somewhere in I [3]. No more results are given for $p \neq 2$ (the nonlinear problem). The following proposition gives a solution for the problem.

Theorem 4.1 *For $N = 1$ and $p > 1$ we have for all $m \in M^+(I)$*

1.
$$\sigma_1^+(\Delta_p, m) = \cup_{n=1}^{\infty} G\left(\Gamma_n^p(m, \beta)\right) \tag{23}$$
resp.
$$\sigma_1^-(\Delta_p, m) = \cup_{n=1}^{\infty} G\left(\Gamma_{-n}^p(m, \beta)\right) \tag{24}$$
where G is the graph of the function $\Gamma_n^p(m, .)$ defined for all $\beta \in \mathbb{R}^N$ by
$$\frac{1}{\Gamma_n^p(m, \beta)} = \sup_{F \in \mathcal{F}_n} \min_{v \in F \cap S_\Lambda} \int_I e^{\beta . x} m(x)|v|^p\, dx \tag{25}$$
resp.
$$\frac{1}{\Gamma_{-n}^p(m, \beta)} = \inf_{F \in \mathcal{F}_n} \max_{v \in F \cap S_\Lambda} \int_I e^{\beta . x} m(x)|v|^p\, dx \tag{26}$$
where $\Lambda(x) = e^{\beta . x}$.

2. *For all $\beta \in \mathbb{R}$, $\lim_{n \to +\infty} \Gamma_n^p(m, \beta) = +\infty$ (resp. $\lim_{n \to +\infty} \Gamma_{-n}^p(m, \beta) = -\infty$).*

3. *The sequence $\Gamma_n^p(m, \beta)$ (resp. $\Gamma_{-n}^p(m, \beta)$) is such that*
$$\begin{cases} \Gamma_1^p(m, \beta) < \Gamma_2^p(m, \beta) < \cdots < \Gamma_n^p(m, \beta) < \cdots \to +\infty \quad for\ \beta \in \mathbb{R} \\ If\ \Gamma_n^p(m, \beta) < \alpha < \Gamma_{n+1}^p(m, \beta) \quad then\ (\alpha, \beta) \notin \sigma_1^+(\Delta_p, m). \end{cases}$$
resp.
$$\begin{cases} \Gamma_{-1}^p(m, \beta) > \Gamma_{-2}^p(m, \beta) > \cdots > \Gamma_{-n}^p(m, \beta) > \cdots \to -\infty \quad for\ \beta \in \mathbb{R} \\ If\ \Gamma_{-n}^p(m, \beta) < \alpha < \Gamma_{-(n+1)}^p(m, \beta) \quad then\ (\alpha, \beta) \notin \sigma_1^-(\Delta_p, m). \end{cases}$$

4. *If* $(\alpha,\beta) \in \Gamma_n^p(m,\beta) \cup \Gamma_{-n}^p(m,\beta)$ *then* (α,β) *is simple and each eigenfunction corresponding to it has exactly* $n-1$ *simple zeros in* I.

5. *For all* $n \geq 1$, $\Gamma_n^p(m,\beta)$ *(resp.* $\Gamma_{-n}^p(m,\beta)$*) verifies the strict monotonicity property (PSM) with respect to the weight* m *and the domain* I.

Proof Consider the following problem

$$\left(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,I)}\right) \qquad \begin{cases} \left(-\Lambda(x)|u'|^{p-2}u'\right)' = \alpha\rho(x)|u|^{p-2}u \\ u(a) = u(b) = 0. \end{cases}$$

Put $\Lambda(x) = e^{\beta.x}$ and $\rho(x) = e^{\beta.x}m(x)$. It is obvious that $\Lambda \in C^1(I) \cap C(\bar{I})$ and non negative in $\bar{I}$, $\rho \in L^\infty(I)$. Problem $\left(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,I)}\right)$ is then equivalent to the problem $\left(\mathcal{V}.\mathcal{P}_{(m,I)}\right)$ (equation 22). So, making use of theorem 2 to obtain the results mentionned in theorem 3. ■

4.3. SPECTRUM OF ORDER ONE WITH WEIGHTS OF THE P-LAPLAPCIAN OPERATOR. In this section we introduce a new notion about the spectrum. We call spectrum of order one with weights (m_1, m_2) of the p-Laplapcian operator the set $\sigma_1(\Delta_p, m_1, m_2)$ of all curves $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ solutions of problem

$$\left(\mathcal{V}.\mathcal{P}_{(m_1,m_2)}\right) \begin{cases} \left(-\Lambda(x)|u'|^{p-2}u'\right)' = \alpha m_1(x)|u|^{p-2}u + \beta m_2(x)|u'|^{p-2}u' \quad in \quad I \\ u(a) = u(b) = 0. \end{cases} \tag{27}$$

with $m_1 \in L^\infty(I)$ and $m_2 \in C(\bar{I})$.

To solve this problem we put $\sigma(x) = \int_a^x m_2(t)\,dt$ and consider the problem $\left(\mathcal{V}.\mathcal{P}_{(\Lambda,\rho,I)}\right)$, with $\rho(x) = m_1(x)e^{\sigma(x)}$ and $\Lambda(x) = e^{\sigma(x)}$. The problem (27) is then equivalent to problem (1), so making use of theorem 2 to conclude the same results as in theorem 3. ■

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