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Korteweg-de Vries Equation in Bounded Domains

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Contents

1. Introduction

The goal of this paper is to prove the existence, uniqueness and the energy decay of global regular solutions of the KdV equation in a bounded domain approximating it by the Kuramoto-Sivashinsky equations.

In $Q = (0,1) \times (0,T)$; $x \in (0,1)$, $t \in (0,T)$ we consider the generalized Kuramoto-Sivashinsky equation,

$$u_t + uu_x + \mu u_{xxx} + \nu(u_{xx} + u_{xxxx}) = 0, \tag{1.1}$$

where μ, ν are positive constants.

This equation, in the case $\mu = 0$, was derived independently by Sivashinsky [1] and Kuramoto [2] with the purpose to model amplitude and phase expansion of pattern formations in different physical situations, for example, in the theory of a flame propagation in turbulent flows of gaseous combustible mixtures, see Sivashinsky [1], and in the theory of turbulence of wave fronts in reaction-diffusion systems, Kuramoto [2]. The generalized KdV-KS equation (1.1) arises in modeling of long waves in a viscous fluid flowing down on an inclined plane. When $\nu = 0$, we have the KdV equation studied by various authors [6-12].

From the mathematical point of a view, the history of the KdV equation is much longer than the one of the KS equation. Well-posedness of the Cauchy problem for the KdV equation in various classes of solutions was studied in [6-9]. Solvability of mixed problems for the KdV equation and for the KdV equation with dissipation in bounded domains studied Bubnov [11], Hublov [12], see also

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[19]. In [10], Bui An Ton proved well-posedness of the mixed problem for the KdV equation in $(0,\infty)\times(0,T)$ approximating the KdV equation by the KS type equations. Mixed problems for some classes of third order equations studied Kozhanov [13] and Larkin [18]. The Cauchy problem for (1.1) was considered by Biagioni et al [6]. They proved the existence of a unique strong global solution and studied asymptotic behaviour of solutions as ν tends to zero. This gave a solution to the Cauchy problem for the KdV equation as a limit of a sequence of solutions to the Cauchy problem for the KdV-KS equations. The Cauchy problem for the KS equation considered Tadmor [3] and Guo [5]. In [5], Guo studied also solvability of the mixed problem for the KS equation in bounded domains in one-dimensional and multi-dimensional cases. Cousin and Larkin [4] proved global well-posedness of the mixed problem for the KS equation in classes of regular solutions in bounded domains with moving boundaries. The exponential decay of L^2- norms of solutions as $t\to\infty$ was proved.

In the present paper we study asymptotics of solutions to a mixed problem for (1.1) when ν tends to zero in order to prove therewith that solutions to a mixed problem for the KdV equation may be obtained as singular limits of solutions to a corresponding mixed problem for the KS equation. The passage to the limit as ν tends to zero is singular because we loose one boundary condition in $x=0$.

We consider in the rectangle Q the mixed problem for (1.1) which is different from the one considered in [4,5,10]. In Section 2, we state our main results. In Section 3, exploiting the Faedo-Galerkin method with a special basis, we prove solvability of the mixed problem for (1.1) when $\nu>0$. In Section 4, we prove the existence and uniqueness of a strong solution to the mixed problem for the KdV equation letting ν tend to zero. It must be noted that the Fourier transform, commonly used to solve the Cauchy problem, see [6-9], is not suitable in the case of the mixed problem. Instead, we use the Faedo-Galerkin method to solve the mixed problem for (1.1) and weighted estimates to pass to the limit as ν tends to zero. In Section 5, we show that if $\|u_0\|_{L^2(0,1)}$ is sufficiently small, then $\|u(t)\|_{L^2(0,1)}$ decreases exponentially in time and no dissipativity on the boundaries of the domain is needed for this.

2. Notations and results

We use standard notations, see Lions-Magenes [16], some special cases will be given below. We denote

$$Q=(0,1)\times(0,T),\ (u,v)(t)=\int_0^1 u(x,t)v(x,t)dx,\ \|u(t)\|^2=(u,u)(t),$$

$D_j=\frac{\partial^j}{\partial x^j}$, $\|u\|^2=\|u\|^2_{L^2(Q)}$, $H^m(D)$ denotes the Sobolev space $W_2^m(D)$.

We consider in Q the following problem,

$$Lu=u_t+uu_x+u_{xxx}+\nu(u_{xx}+u_{xxxx})=0 \text{ in } Q, \tag{2.1}$$

$$u(x,0)=u_0(x),\ x\in(0,1), \tag{2.2}$$

$$u(0,t) = u_{xx}(0,t) = u(1,t) = u_x(1,t) + \nu u_{xx}(1,t) = 0,\ t > 0. \tag{2.3}$$

Our result on solvability of (2.1)-(2.3) is the following.

Theorem 1 *Let* $\nu > 0$ *and* $u_0 \in H^4(0,1) \cap H_0^1(0,1)$; $u_{0xx}(0) = u_{0x}(1) + \nu u_{0xx}(1) = 0$.

Then there exists a unique solution to (2.1)-(2.3) from the class,

$$u \in C(0,T; H^2(0,1) \cap H_0^1(0,1)) \cap L^2(0,T; H^4(0,1) \cap H_0^1(0,1)),$$

$$u_t \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^2(0,1) \cap H_0^1(0,1)),$$

$$u_{tt} \in L^2(0,T; H^{-2}(0,1)).$$

When ν tends to zero, we obtain the following result.

Theorem 2 *Let* $u_0 \in H^4(0,1) \cap H_0^1(0,1)$, $u_{0x}(1) = 0$.
Then there exists a unique solution to the problem,

$$u_t + uu_x + u_{xxx} = 0, \quad in\ \ Q, \tag{2.4}$$

$$u(x,0) = u_0(x),\ x \in (0,1), \tag{2.5}$$

$$u(0,t) = u(1,t) = u_x(1,t) = 0 \tag{2.6}$$

from the class,

$$u \in L^\infty(0,T; H^3(0,1) \cap H_0^1(0,1)),\ u_t \in L^\infty(0,T; L^2(0,1)).$$

In reality, the sharper result is true.

Theorem 3 *Let* $u_0 \in H^3(0,1) \cap H_0^1(0,1)$, $u_{0x}(1) = 0$. *Then all the assertions of Theorem 2 are true.*

3. Solvability of the problem (2.1)-(2.3)

Lemma 1 *For every* $\nu > 0$ *there exist eigenfunctions of the following problem,*

$$\nu D_4 w_j = \lambda_j w_j,$$

$$w_j(0) = w_j(1) = w_{jxx}(0) = w_{jx}(1) + \nu w_{jxx}(1) = 0 \tag{3.1}$$

which create a basis in $H^4(0,1)$ *orthonormal in* $L^2(0,1)$.

Proof: It is easy to see that if $u, v \in H^4(0,1)$ and satisfy boundary conditions of Lemma 1, then

$$\nu(D_4u, v) = \nu(u, D_4v) \text{ and } \nu(D_4u, u) = \nu\|D_2u\|^2 + u_x^2(1).$$

This means that the operator corresponding to the problem above is selfadjoint and positive. Hence, assertions of Lemma 1 follow from the well-known facts, see Coddington and Levinson [15], Mikhailov [14]. □

We construct approximate solutions to (2.1)-(2.3) in the form,

$$u^N(x,t)=\sum_{j=1}^N g_j^N(t)w_j(x),$$

where $w_j(x)$ are defined in Lemma 1 and $g_j^N(t)$ are to be found as solutions to the Cauchy problem for the system of N ordinary differential equations,

$$(Lu^N,w_j)(t)=(u_t^N,w_j)(t)+(u^Nu_x^N,w_j)(t)$$

$$+(D_3u^N,w_j)(t)+\nu(u_{xx}^N,w_j)(t)+\nu(D_4u^N,w_j)(t)=0, \tag{3.2}$$

$$g_j^N(0)=(u_0,w_j),\ j=1,...,N. \tag{3.3}$$

System (3.2) is a normal nonlinear ODE system, hence, there exist on some interval $0,T_N)$ functions $g_1^N(t),...,g_N^N(t)$. To extend them to any $T<\infty$ and to pass to the limit as $N\to\infty$, we prove the following estimates:

$$\|u^N(t)\|^2+\int_0^t u_x^{N2}(0,s)ds+\nu\int_0^t\|u_{xx}^N(s)\|^2ds\ \leq\ C_1\|u_0\|^2, \tag{3.4}$$

where C_1 does not depend on $N,t\in(0,T),\ \nu>0.$

$$\nu\mid D_2u^N(1,t)\mid^2+\|D_2u^N(t)\|^2+\nu\int_0^t\|D_4u^N(s)\|^2ds$$

$$\leq C_2(\nu)(\nu\mid u_{0xx}(1)\mid^2+\|u_0\|^2_{H^2(0,1)}), \tag{3.5}$$

$$\|u_t^N(t)\|^2+\nu\int_0^t\|u_{sxx}^N(s)\|^2ds\leq C_3\|u_0\|^2_{H^4(0,1)\cap H_0^1(0,1)}, \tag{3.6}$$

where C_2, C_3 do not depend on $N,t\in(0,T)$.

Estimates (3.4), (3.5), (3.6) imply that $u^N(x,t)$ can be extended to all $T\in(0,\infty)$ and that approximations (u^N) converge as $N\to\infty$. Passing to the limit in (3.2), we prove the existence part of Theorem 1. Uniqueness can be proved by the standard methods, see [4]. Thus Theorem 1 is proved.

4. Solvability of the KdV equation

Theorem 1 guarantees well-posedness of the problem (2.1)-(2.3) for all $\nu>0$. Our aim now is to pass to the limit as ν tends to zero. For this purpose we need a priori estimates of solutions to (2.1)-(2.3) independent of $\nu>0$. First we observe that estimate (3.4) does not depend on ν, but (3.5), (3.6) do depend.

Due to Theorem 1, for all $\nu>0$ we have the integral identity,

$$(u_{\nu t},v)(t)+(u_\nu u_{\nu x},v)(t)+(D_3u_\nu,v)(t)$$

$$+\nu(D_2u_\nu,v)(t)+\nu(D_4u_\nu,v)(t)=0 \tag{4.1}$$

which is true for any $v \in L^2(0,1)$.

It can be shown that u_ν satisfy uniformly in $\nu > 0$ the following inclusions:

$$u_\nu \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H_0^1(0,1)) \subset L^2(0,T;C^{1/2}[0,1]),$$

$$u_{\nu t} \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H_0^1(0,1)) \subset L^2(0,T;C^{1/2}[0,1]),$$

$$\nu^{1/2} u_\nu \in L^2(0,T;H^2(0,1)); \ \nu^{1/2} u_{\nu t} \in L^2(0,T;H^2(0,1)).$$

Proof of Theorem 2

Proof: Letting $\nu \to 0$, we have a sequence of functions u_ν satisfying (4.1). The last inclusions imply that there exists a subsequence of u_ν, which we denote also by u_ν, and a function U such that

$$\begin{array}{l} u_\nu \to U \; strongly \text{ in } C(\bar{Q}) \\ u_\nu \to U \; weakly-star \text{ in } L^\infty(0,T;H_0^1(0,1)) \\ u_{\nu t} \to U_t \; weakly-star \text{ in } L^\infty(0,T;L^2(0,1)) \\ u_{\nu t} \to U_t \; weakly \text{ in } L^2(0,T;H_0^1(0,1)) \\ \nu u_{\nu xx} \to 0 \; weakly-star \text{ in } L^\infty(0,T;L^2(0,t))). \end{array}$$

Using these convergences, we prove

Theorem 4 *There exists at least one weak solution of the problem (2.4)-(2.6):* $U \in C(0,T;H_0^1(0,1))$, $U_t \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H_0^1(0,1))$, *satisfying the following identity,*

$$(U_t,v)(t) + (UU_x,v)(t) + (U_x,v_{xx})(t) = 0,$$

where $v(x,t)$ *is an arbitrary function from* $W = \{ v \in L^2(0,T;H^2(0,1) \cap H_0^1(0,1)); \ v_x(0,t) = 0; \ t \in (0,T)\}$.

Proof: Due to Theorem 1, for all $\nu \in (0,1/2)$ the following identity is valid

$$\int_0^T \{(u_{\nu t},v)(t) + (u_\nu u_{\nu x},v)(t) + (D_3 u_\nu,v)(t)$$
$$\nu(D_2 u_\nu,v)(t) + \nu(D_4 u_\nu,v)(t)\}\,dt = 0,$$

where v is an arbitrary function from $L^2(0,T;L^2(0,1))$, in particularly, we can take v an arbitrary function from W. Then, taking into account boundary conditions (2.3), we can rewrite the last identity in the form,

$$\int_0^T \{u_{\nu t},v)(t) + (u_\nu u_{\nu x},v)(t) + (u_{\nu x},v_{xx})(t)$$
$$\nu(D_2 u_\nu,v)(t) + \nu(D_2 u_\nu,D_2 v)(t)\}dt = 0.$$

Passing to the limit as $\nu \to 0$, we obtain

$$(U_t,v)(t) + (UU_x,v)(t) + (U_x,v_{xx})(t) = 0$$

for $a.e.\, t \in (0,T)$ and for all $v \in W$. The boundary conditions $U(0,t) = U(1,t) = 0$ obviously are fulfilled and the boundary condition $U_x(1,t) = 0$ is fulfilled in a weak sense. It is clear that functions U and v have conjugate boundary conditions. $\square$

Taking into account properties of U, we can write

$$(U_x, v_{xx})(t) = (F, v)(t), \tag{4.2}$$

where

$$F = -U_t - UU_x \in L^2(0,1).$$

It means that U is a weak solution to the following boundary value problem,

$$U_{xxx} = F(x),\ x \in (0,1), \tag{4.3}$$

$$U(0) = U(1) = U_x(1) = 0. \tag{4.4}$$

Now we must prove that a weak solution is regular. To prove this fact, we use the following

Lemma 2 *A weak solution to (4.2)-(4.4) is uniquely defined.*

On the other hand, it is easy to verify that the function

$$U_0(x) = K_1 x + K_2 x^2 + \frac{1}{2}\int_0^x z^2 F(z)dz - x\int_0^x zF(z)dz + \frac{x^2}{2}\int_0^x F(z)dz$$

belongs to $H^3(0,1), U_0(0) = 0$ for any $F \in L^2(0,1)$, and satisfies the equation,

$$U_{0xxx} = F(x). \tag{4.5}$$

Given $F(x)$, the constants K_1, K_2 can be found to satisfy the boundary conditions,

$$U_0(1) = U_{0x}(1) = 0. \tag{4.6}$$

Multiplying (4.5) by any $v \in W$ and integrating by parts, we come to the identity,

$$(U_{0x}, v_{xx})(t) = (F, v)(t) \text{ for } a.e.\, t \in (0,T).$$

Substracting this from (4.2), we get

$$((U - U_0)_x, v_{xx})(t) = 0.$$

By Lemma 2, $U - U_0 = 0$, hence, $U = U_0$ $a.e.$ in $(0,1)$, It implies that $U \in H^3(0,1)$.

Returning to (4.2), we rewrite it as

$$U_t + UU_x + U_{xxx} = 0 \text{ } a.e. \text{ in } Q,$$

$$U(0) = U(1) = U_x(1) = 0,$$

$$U(x,0) = u_0(x). \tag{4.7}$$

This proves the existence part of Theorem 2.

Uniqueness

Let u_1, u_2 be two distinct solutions to (4.7). Then for $z = u_1 - u_2$ we have

$$z_t + \frac{1}{2}[(u_1 + u_2)z]_x + z_{xxx} = 0, \tag{4.8}$$

$$z(0) = z(1) = z_x(1) = 0, \tag{4.9}$$

$$z(x, 0) = 0. \tag{4.10}$$

Multiplying (4.8) by $e^{\lambda x}z$, integrating over (0,1), putting $\lambda = 1$ and taking into account properties of U,

$$\max_{\bar{Q}} \mid u_1(x,t) + u_2(x,t) \mid \leq M \leq \infty,$$

we obtain

$$(e^x, z^2)(t) \leq C \int_0^t (e^x, z^2)(s)ds.$$

By the Gronwall lemma, $(e^x, z^2)(t) = 0$, consequently, $\|z(t)\| = 0$ for all $t \in (0, T)$. The proof of Theorem 2 is completed. □

5. Stability

We have the following result.

Theorem 5 *There exist positive constants $\lambda \in (0,1)$ and K such that if $\|u_0\| \leq 3/e$, then*

$$\|u(t)\|^2_{L^2(0,1)} \leq K\|u_0\|^2_{L^2(0,1)}e^{-\chi t},$$

where $\chi = \frac{\lambda}{2e^\lambda}$.

Proof: By Theorem 2 and by the arguments similar to those used by Browder [17], for all $t > 0$ $u(x,t)$ is a strong solution to the following problem,

$$Lu = u_t + uu_x + u_{xxx} = 0 \text{ in } Q = (0,1) \times (0, \infty), \tag{5.1}$$

$$u(x, 0) = u_0(x) \text{ in } (0, 1), \tag{5.2}$$

$$u(0,t) = u(1,t) = u_x(1,t) = 0, \; t > 0. \tag{5.3}$$

Multiplying (5.1) by u and using (5.3), we get

$$\frac{d}{dt}\|u(t)\|^2 + u_x^2(0,t) = 0.$$

This implies

$$\|u(t)\| \leq \|u_0\| \text{ for all } t > 0. \tag{5.4}$$

From the identity $(e^{\lambda x}u, Lu)(t) = 0$, for some $\lambda \in (0,1)$ we obtain

$$\frac{d}{dt}(e^{\lambda x}, u^2)(t) + \frac{\lambda}{2e^\lambda}(e^{\lambda x}, u^2)(t) \leq 0.$$

This implies the assertion of Theorem 5. □

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