Korteweg-de Vries Equation in Bounded Domains

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Contents

1 Introduction 30
2 Notations and results 31
3 Solvability of the problem (2.1)-(2.3) 32
4 Solvability of the KdV equation 33
5 Stability 36

1. Introduction

The goal of this paper is to prove the existence, uniqueness and the energy decay of global regular solutions of the KdV equation in a bounded domain approximating it by the Kuramoto-Sivashinsky equations.

In $Q = (0, 1) \times (0, T); x \in (0, 1), t \in (0, T)$ we consider the generalized Kuramoto-Sivashinsky equation,

$$u_t + uu_x + \mu u_{xxx} + \nu (u_{xx} + u_{xxxx}) = 0,$$

where $\mu, \nu$ are positive constants.

This equation, in the case $\mu = 0$, was derived independently by Sivashinsky [1] and Kuramoto [2] with the purpose to model amplitude and phase expansion of pattern formations in different physical situations, for example, in the theory of a flame propagation in turbulent flows of gaseous combustible mixtures, see Sivashinsky [1], and in the theory of turbulence of wave fronts in reaction-diffusion systems, Kuramoto [2]. The generalized KdV-KS equation (1.1) arises in modeling of long waves in a viscous fluid flowing down on an inclined plane. When $\nu = 0$, we have the KdV equation studied by various authors [6-12].

From the mathematical point of a view, the history of the KdV equation is much longer than the one of the KS equation. Well-posedness of the Cauchy problem for the KdV equation in various classes of solutions was studied in [6-9]. Solvability of mixed problems for the KdV equation and for the KdV equation with dissipation in bounded domains studied Bubnov [11], Hublov [12], see also

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In [10], Bui An Ton proved well-posedness of the mixed problem for the KdV equation in \((0, \infty) \times (0, T)\) approximating the KdV equation by the KS type equations. Mixed problems for some classes of third order equations studied Kozhanov [13] and Larkin [18]. The Cauchy problem for (1.1) was considered by Biagioni et al [6]. They proved the existence of a unique strong global solution and studied asymptotic behaviour of solutions as \(\nu\) tends to zero. This gave a solution to the Cauchy problem for the KdV equation as a limit of a sequence of solutions to the Cauchy problem for the KdV-KS equations. The Cauchy problem for the KS equation considered Tadmor [3] and Guo [5]. In [5], Guo studied also solvability of the mixed problem for the KS equation in bounded domains in one-dimensional and multi-dimensional cases. Cousin and Larkin [4] proved global well-posedness of the mixed problem for the KS equation in classes of regular solutions in bounded domains with moving boundaries. The exponential decay of \(L^2\) norms of solutions as \(t \to \infty\) was proved.

In the present paper we study asymptotics of solutions to a mixed problem for (1.1) when \(\nu\) tends to zero in order to prove therewith that solutions to a mixed problem for the KdV equation may be obtained as singular limits of solutions to a corresponding mixed problem for the KS equation. The passage to the limit as \(\nu\) tends to zero is singular because we loose one boundary condition in \(x = 0\).

We consider in the rectangle \(Q\) the mixed problem for (1.1) which is different from the one considered in [4,5,10]. In Section 2, we state our main results. In Section 3, exploiting the Faedo-Galerkin method with a special basis, we prove solvability of the mixed problem for (1.1) when \(\nu > 0\). In Section 4, we prove the existence and uniqueness of a strong solution to the mixed problem for the KdV equation letting \(\nu\) tend to zero. It must be noted that the Fourier transform, commonly used to solve the Cauchy problem, see [6-9], is not suitable in the case of the mixed problem. Instead, we use the Faedo-Galerkin method to solve the mixed problem for (1.1) and weighted estimates to pass to the limit as \(\nu\) tends to zero. In Section 5, we show that if \(\|u_0\|_{L^2(0,1)}\) is sufficiently small, then \(\|u(t)\|_{L^2(0,1)}\) decreases exponentially in time and no dissipativity on the boundaries of the domain is needed for this.

2. Notations and results

We use standard notations, see Lions-Magenes [16], some special cases will be given below. We denote

\[ Q = (0,1) \times (0,T), \quad (u,v)(t) = \int_0^1 u(x,t)v(x,t)dx, \quad \|u(t)\|^2 = (u,u)(t), \]

\[ D_j = \frac{\partial^j}{\partial x^j}, \quad \|u\|^2 = \|u\|^2_{L^2(Q)}, \quad H^m(D) \text{ denotes the Sobolev space } W^m_2(D). \]

We consider in \(Q\) the following problem,

\[ Lu = u_t + uu_x + u_{xxx} + \nu(u_{xx} + u_{xxxx}) = 0 \quad \text{in } Q, \tag{2.1} \]

\[ u(x,0) = u_0(x), \quad x \in (0,1), \tag{2.2} \]
Our result on solvability of (2.1)-(2.3) is the following.

**Theorem 1** Let \( \nu > 0 \) and \( u_0 \in H^4(0,1) \cap H^1_0(0,1); w_{0xx}(0) = w_{0x}(1) + \nu w_{0xx}(1) = 0 \).

Then there exists a unique solution to (2.1)-(2.3) from the class

\[
\begin{align*}
  u &\in C(0,T; H^2(0,1) \cap H^1_0(0,1)), \\
  u_t &\in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^2(0,1) \cap H^1_0(0,1)), \\
  u_{tt} &\in L^2(0,T; H^{-2}(0,1)).
\end{align*}
\]

When \( \nu \) tends to zero, we obtain the following result.

**Theorem 2** Let \( u_0 \in H^4(0,1) \cap H^1_0(0,1), w_{0x}(1) = 0 \).

Then there exists a unique solution to the problem,

\[
\begin{align*}
  u_t + uu_x + u_{xxx} &= 0, \text{ in } Q, \\
  u(x,0) &= u_0(x), \; x \in (0,1), \\
  u(0,t) &= u(1,t) = u_x(1,t) = 0
\end{align*}
\]

from the class,

\[
\begin{align*}
  u &\in L^\infty(0,T; H^3(0,1) \cap H^1_0(0,1)), \\
  u_t &\in L^\infty(0,T; L^2(0,1)).
\end{align*}
\]

In reality, the sharper result is true.

**Theorem 3** Let \( u_0 \in H^3(0,1) \cap H^1_0(0,1), w_{0x}(1) = 0 \). Then all the assertions of Theorem 2 are true.

### 3. Solvability of the problem (2.1)-(2.3)

**Lemma 1** For every \( \nu > 0 \) there exist eigenfunctions of the following problem,

\[
\nu D_4 w_j = \lambda_j w_j,
\]

\[
w_j(0) = w_j(1) = w_{jxx}(0) = w_{jx}(1) + \nu w_{jxx}(1) = 0
\]

which create a basis in \( H^4(0,1) \) orthonormal in \( L^2(0,1) \).

**Proof:** It is easy to see that if \( u, v \in H^4(0,1) \) and satisfy boundary conditions of Lemma 1, then

\[
\nu(D_4 u, v) = \nu(u, D_4 v) \quad \text{and} \quad \nu(D_4 u, u) = \nu\|D_2 u\|^2 + u_x^2(1).
\]

This means that the operator corresponding to the problem above is selfadjoint and positive. Hence, assertions of Lemma 1 follow from the well-known facts, see Coddington and Levinson [15], Mikhailov [14].
We construct approximate solutions to (2.1)-(2.3) in the form,

\[ u^N(x,t) = \sum_{j=1}^{N} g^N_j(t)w_j(x), \]

where \( w_j(x) \) are defined in Lemma 1 and \( g^N_j(t) \) are to be found as solutions to the Cauchy problem for the system of \( N \) ordinary differential equations,

\[
(Lu^N, w_j)(t) = (u^N_t, w_j)(t) + (u^N u^N_x, w_j)(t)
+ (D^3u^N, w_j)(t) + \nu (D^4u^N, w_j)(t) = 0, \quad j = 1, ..., N. \tag{3.2}
\]

System (3.2) is a normal nonlinear ODE system, hence, there exist on some interval \( 0, T_N \) functions \( g^N_1(t), ..., g^N_N(t) \). To extend them to any \( T < \infty \) and to pass to the limit as \( N \to \infty \), we prove the following estimates:

\[
\|u^N(t)\|^2 + \int_0^t u^N_x^2(0,s) ds + \nu \int_0^t \|u^N_{xx}(s)\|^2 ds \leq C_1 \|u_0\|^2, \quad (3.4)
\]

where \( C_1 \) does not depend on \( N, t \in (0,T), \nu > 0 \).

\[
\nu |D^2u^N(1,t)|^2 + \|D^3u^N(t)\|^2 + \nu \int_0^t \|D^4u^N(s)\|^2 ds
\leq C_2(\nu \|u_0_{xx}(1)\|^2 + \|u_0\|_{H^2(0,1)}^2), \quad (3.5)
\]

\[
\|u^N_t(t)\|^2 + \nu \int_0^t \|u^N_{xxx}(s)\|^2 ds \leq C_3 \|u_0\|_{H^4(0,1)}^2, \quad (3.6)
\]

where \( C_2, C_3 \) do not depend on \( N, t \in (0,T) \).

Estimates (3.4), (3.5), (3.6) imply that \( u^N(x,t) \) can be extended to all \( T \in (0,\infty) \) and that approximations \( (u^N) \) converge as \( N \to \infty \). Passing to the limit in (3.2), we prove the existence part of Theorem 1. Uniqueness can be proved by the standard methods, see [4]. Thus Theorem 1 is proved.

4. Solvability of the KdV equation

Theorem 1 guarantees well-posedness of the problem (2.1)-(2.3) for all \( \nu > 0 \). Our aim now is to pass to the limit as \( \nu \) tends to zero. For this purpose we need a priori estimates of solutions to (2.1)-(2.3) independent of \( \nu > 0 \). First we observe that estimate (3.4) does not depend on \( \nu \), but (3.5), (3.6) do depend.

Due to Theorem 1, for all \( \nu > 0 \) we have the integral identity,

\[
(u_{\nu t}, v)(t) + (u_{\nu}u_{\nu x}, v)(t) + (D^3u_{\nu}, v)(t)
+ \nu (D^2u_{\nu}, v)(t) + \nu (D^4u_{\nu}, v)(t) = 0 \quad (4.1)
\]
There exists at least one weak solution of the problem (2.4)-(2.6):

\[ u \text{ by } 0; \]

last inclusions imply that there exists a subsequence of \( u_{\nu t} \) and a function \( \nu \rightarrow 0 \) such that

\[ u_{\nu} \in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1_0(0, 1)) \subset L^2(0, T; C^{1/2}[0, 1]), \]
\[ u_{\nu t} \in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1_0(0, 1)) \subset L^2(0, T; C^{1/2}[0, 1]), \]
\[ \nu^{1/2} u_{\nu t} \in L^2(0, T; H^2(0, 1)); \nu^{1/2} u_{\nu t} \in L^2(0, T; H^2(0, 1)). \]

**Proof of Theorem 2**

Letting \( \nu \to 0 \), we have a sequence of functions \( u_{\nu} \) satisfying (4.1). The last inclusions imply that there exists a subsequence of \( u_{\nu} \), which we denote also by \( u_{\nu} \), and a function \( U \) such that

\[ u_{\nu} \to U \text{ strongly in } C(\bar{Q}) \]
\[ u_{\nu} \to U \text{ weakly } \star \text{ in } L^\infty(0, T; H^1_0(0, 1)) \]
\[ u_{\nu t} \to U_t \text{ weakly } \star \text{ in } L^\infty(0, T; L^2(0, 1)) \]
\[ u_{\nu t} \to U_t \text{ weakly in } L^2(0, T; H^1_0(0, 1)) \]
\[ \nu u_{\nu xx} \to 0 \text{ weakly } \star \text{ in } L^\infty(0, T; L^2(0, t))). \]

Using these convergences, we prove

**Theorem 4** There exists at least one weak solution of the problem (2.4)-(2.6):

\[ U \in C(0, T; H^1_0(0, 1)), U_t \in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1_0(0, 1)), \text{ satisfying the following identity,} \]

\[ (U_t, v)(t) + (U_{ux}, v)(t) + (U_x, v_{xx})(t) = 0, \]

where \( v(x, t) \) is an arbitrary function from \( W = \{ v \in L^2(0, T; H^2(0, 1) \cap H^1_0(0, 1)); v_x(0, t) = 0; t \in (0, T) \}. \)

**Proof:** Due to Theorem 1, for all \( \nu \in (0, 1/2) \) the following identity is valid

\[ \int_0^T \{(u_{\nu t}, v)(t) + (u_{\nu}, u_{\nu x})(t) + (D_3 u_{\nu}, v)(t) \]
\[ \nu(D_2 u_{\nu}, v)(t) + \nu(D_4 u_{\nu}, v)(t)\} dt = 0, \]

where \( v \) is an arbitrary function from \( L^2(0, T; L^2(0, 1)) \), in particular, we can take \( v \) an arbitrary function from \( W \). Then, taking into account boundary conditions (2.3), we can rewrite the last identity in the form,

\[ \int_0^T \{u_{\nu t}, v)(t) + (u_{\nu}, u_{\nu xx})(t) + (u_{xx}, v_{xx})(t) \]
\[ \nu(D_2 u_{\nu}, v)(t) + \nu(D_2 u_{\nu}, D_2 v)(t)\} dt = 0. \]

Passing to the limit as \( \nu \to 0 \), we obtain

\[ (U_t, v)(t) + (U_{ux}, v)(t) + (U_x, v_{xx})(t) = 0 \]
for a.e. \( t \in (0, T) \) and for all \( v \in W \). The boundary conditions \( U(0, t) = U(1, t) = 0 \) obviously are fulfilled and the boundary condition \( U_x(1, t) = 0 \) is fulfilled in a weak sense. It is clear that functions \( U \) and \( v \) have conjugate boundary conditions.

Taking into account properties of \( U \), we can write

\[
(U_x, v_{xx})(t) = (F, v)(t),
\]

where

\[
F = -U_t - UU_x \in L^2(0, 1).
\]

It means that \( U \) is a weak solution to the following boundary value problem,

\[
U_{xxx} = F(x), \ x \in (0, 1),
\]

\[
U(0) = U(1) = U_x(1) = 0.
\]

Now we must prove that a weak solution is regular. To prove this fact, we use the following

**Lemma 2** A weak solution to (4.2)-(4.4) is uniquely defined.

On the other hand, it is easy to verify that the function

\[
U_0(x) = K_1x + K_2x^2 + \frac{1}{2} \int_0^x z^2F(z)dz - x \int_0^x zF(z)dz + \frac{x^2}{2} \int_0^x F(z)dz
\]

belongs to \( H^3(0, 1) \), \( U_0(0) = 0 \) for any \( F \in L^2(0, 1) \), and satisfies the equation,

\[
U_{0xxx} = F(x).
\]

Given \( F(x) \), the constants \( K_1, K_2 \) can be found to satisfy the boundary conditions,

\[
U_0(1) = U_0x(1) = 0.
\]

Multiplying (4.5) by any \( v \in W \) and integrating by parts, we come to the identity,

\[
(U_{0x}, v_{xx})(t) = (F, v)(t) \text{ for a.e. } t \in (0, T).
\]

Substracting this from (4.2), we get

\[
((U - U_0)_x, v_{xx})(t) = 0.
\]

By Lemma 2, \( U - U_0 = 0 \), hence, \( U = U_0 \) a.e. in \( (0, 1) \). It implies that \( U \in H^3(0, 1) \).

Returning to (4.2), we rewrite it as

\[
U_t + UU_x + U_{xxx} = 0 \text{ a.e. in } Q,
\]

\[
U(0) = U(1) = U_x(1) = 0,
\]

\[
U(x, 0) = u_0(x).
\]

This proves the existence part of Theorem 2.
Uniqueness

Let \( u_1, u_2 \) be two distinct solutions to (4.7). Then for \( z = u_1 - u_2 \) we have

\[
\begin{align*}
&z_t + \frac{1}{2}[(u_1 + u_2)z]_x + z_{xxx} = 0, \quad (4.8) \\
z(0) = z(1) = z_t(1) = 0, \quad (4.9) \\
z(x, 0) = 0. \quad (4.10)
\end{align*}
\]

Multiplying (4.8) by \( e^{\lambda x} z \), integrating over \((0,1)\), putting \( \lambda = 1 \) and taking into account properties of \( u \),

\[
\max_Q | u_1(x,t) + u_2(x,t) | \leq M \leq \infty,
\]

we obtain

\[
(e^x, z^2)(t) \leq C \int_0^t (e^x, z^2)(s)ds.
\]

By the Gronwall lemma, \((e^x, z^2)(t) = 0\), consequently, \( \| z(t) \| = 0 \) for all \( t \in (0,T) \). The proof of Theorem 2 is completed. \( \square \)

5. Stability

We have the following result.

**Theorem 5** There exist positive constants \( \lambda \in (0,1) \) and \( K \) such that if \( \| u_0 \| \leq 3/e \), then

\[
\| u(t) \|_{L^2(0,1)}^2 \leq K \| u_0 \|_{L^2(0,1)}^2 e^{-\chi t},
\]

where \( \chi = \frac{\lambda}{2e^x} \).

**Proof:** By Theorem 2 and by the arguments similar to those used by Browder [17], for all \( t > 0 \) \( u(x, t) \) is a strong solution to the following problem,

\[
\begin{align*}
&Lu = u_t + uu_x + u_{xxx} = 0 \text{ in } Q = (0,1) \times (0, \infty), \quad (5.1) \\
&u(x, 0) = u_0(x) \text{ in } (0,1), \quad (5.2) \\
&u(0, t) = u(1, t) = u_x(1, t) = 0, \quad t > 0. \quad (5.3)
\end{align*}
\]

Multiplying (5.1) by \( u \) and using (5.3), we get

\[
\frac{d}{dt} \| u(t) \|^2 + u_x^2(0, t) = 0.
\]

This implies

\[
\| u(t) \| \leq \| u_0 \| \text{ for all } t > 0. \quad (5.4)
\]

From the identity \((e^{\lambda x} u, Lu)(t) = 0\), for some \( \lambda \in (0,1) \) we obtain

\[
\frac{d}{dt} (e^{\lambda x} u^2)(t) + \frac{\lambda}{2e^x} (e^{\lambda x} u^2)(t) \leq 0.
\]

This implies the assertion of Theorem 5. \( \square \)
References


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