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Mathematical Problems For A Dusty Gas Flow

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ABSTRACT: This brief review presents most important results of the authors dealing with mathematical models for the nonstationary two-phase flow of a dusty gas. We do not aim to analyze physical justifications and numerical simulations of dusty gas flows; our attention is focused on the qualitative properties of these models, such as local and global in time well-posedness, uniqueness and asymptotic behavior of solutions.

Keywords: Dusty gas, global solvability, stability.

Contents

1. General two-phase model

Historically, interest in two-phase flows dates from the 1930's due to the necessity to simulate composed and very complex processes of the solid fuel combustion in rockets nozzles. Later, in laboratory experiments, it was observed that adding dust to a gas flowing through a pipe results in reduction of a pressure gradient required to maintain the flow at its original rate, [9,19]. To explain this unexpected effect, precise mathematical analysis was more than welcome. In the next years it was noted that the two-phase modeling is appropriated not only for the "gas – particles" mixtures, but also for processes in nuclear reactors, in creation of new materials and technologies, among others, [20,14].

Probably, one of the first mathematical works in this field is Reference [15]. The basic idea/assumption in this type of models is that physical systems are notable for treating the phases as interpenetrating; that is, each phase has its own velocity, and in the flow region both velocities exist at every point. Such approach leads to so-called Single-Pressure two-phase models. This means that a flow pressure is

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distributed between phases according to its volume fraction. The corresponding PDE-system in this case can be written as follows:

$$(\rho_i)_t + (\rho_i u_i)_x = 0, \tag{1.1}$$

$$\rho_i\big((u_i)_t + u_i(u_i)_x\big) + \alpha_i p_x = F_i, \tag{1.2}$$

where

$$i = 1, 2; \quad p = p(\rho_1, \rho_2); \quad \alpha_1 + \alpha_2 = 1; \quad F_1 = -F_2.$$

Here $x \in \mathbb{R}$ and $t > 0$ are the Euler variables; ρ_i, u_i, α_i and F_i are the density, velocity, volume fraction and phase interaction force of the i-th phase correspondingly; and p is the pressure of a flow.

We drop here the entropy/energy conservation laws since corresponding equations do not influence the type of the system and only complicate calculations.

The main feature of system (1.1), (1.2) is that, in general, it possesses both real and complex characteristic values and therefore it is not hyperbolic. This circumstance makes the well-posedness of the initial/boundary-value problems doubtful and complicates stability and convergence of numerical algorithms, [20].

2. Dusty Gas Model

For real physical situations of a dusty gas (i. e., gas or liquid with a number of suspended small solid particles), one can use natural simplification of the general two-phase model (1.1), (1.2) by neglecting the volume fraction of a dispersive phase ($\alpha_2 = 0$). Other natural assumption is that the Reynolds number of the relative motion of dust versus gas is so small that the drag of the dust by the gas is proportional to their relative velocity. To study mathematical well-posedness, one can assume that dust particles are of uniform size, shape and mass. Furthermore, the heat conduction, sedimentation of particles and any external forces may be neglected. Under these assumptions, the system for one-dimensional dusty gas flow is written as follows, [10,16]:

$$\rho_t + (\rho u)_x = 0, \tag{2.1}$$

$$\rho(u_t + u u_x) + p_x = mK(v - u), \tag{2.2}$$

$$(mv)_t + (mv^2)_x = mK(u - v), \tag{2.3}$$

$$m_t + (mv)_x = 0. \tag{2.4}$$

Hereafter we consider the usual barotropic equation of state: $p = p(\rho)$ with $p' > 0$ and impose the initial data of the form

$$(\rho,\ u,\ v,\ m)(x, 0) = (\rho_0,\ u_0,\ v_0,\ m_0)(x),\ x \in \mathbb{R}. \tag{2.5}$$

The coefficient of the phase interaction $K > 0$ is usually calculated for a single particle and does not depend on a dust concentration. If spherical dust particles of radius r are used in the model, K is given by the Stokes drag formula as $6\pi\mu_0 r$, where μ_0 denotes the dynamic viscosity of a clean gas, [16,1].

Denoting $\overrightarrow{U} = (\rho,\ u,\ m,\ v)$, we write system (2.1)-(2.4) as $\overrightarrow{U_t} + A(\overrightarrow{U})\overrightarrow{U_x} = \overrightarrow{F}(\overrightarrow{U})$, with

$$A = \begin{bmatrix} u & \rho & 0 & 0 \\ \sqrt{p'}/\rho & u & 0 & 0 \\ 0 & 0 & v & m \\ 0 & 0 & 0 & v \end{bmatrix}$$

The eigenvalues of this matrix are real:

$$\lambda_{1,2} = u \pm \sqrt{p'}, \quad \lambda_{3,4} = v,$$

however, the third component of corresponding eigenvectors is equal to zero; therefore, system (2.1)-(2.4) is not hyperbolic.

As concerns systems with multiplied characteristics (in our case $\lambda_3 = \lambda_4$), there is rather great progress when the eigenvectors form a basis in a solutions space, that is the system is "non-strictly" hyperbolic, [13]. However, the precise description of behavior of solutions for the opposite situation (eigenvectors do not form a basis) was lacking, probably due to nonlinearity of the equations.

This situation can be illustrated by the following simple example. Consider the system of linear equations

$$\begin{aligned} u_t + u_x &= m, \\ v_t + v_x &= u, \\ m_t + m_x &= v_x. \end{aligned}$$

We observe that the differential operator of this system has the same structure as in (2.1)-(2.4) with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \lambda_{1,2,3} = 1.$$

Changing variables $\tilde{x} = x - t$ and dropping the tilde, we get

$$v_{ttt} = v_x. \tag{2.6}$$

It is easy to see that for $n = 1, 2, ...,$ the functions

$$v_n(x,t) = n^{-5k} \mathrm{e}^{n\sqrt{3}t} \cos(nt + 8n^3 x), \quad k \geq 1, \text{ integer},$$

are nontrivial solutions of (2.6). However, as in Hadamard's classical example of ill-posedness, $v_n(x, 0)$ tend to zero, together with all derivatives of order less then $k-1$, uniformly in x, while

$$\|v_n(x,t)\|_{C^{k-1}(\mathbb{R})} \longrightarrow \infty$$

as $n \to \infty$ for each point (x,t), $t > 0$. Thus, the Cauchy problem for equation (2.6) is ill-posed. For the same reason, related system (2.1)-(2.4) is expected to be ill-posed. This has been confirmed by the following result (1994):

Theorem 1 *For every finite $l \geq 1$ there are initial data from $C^l(\mathbb{R})$, such that classical solution does not exist for all $t > 0$.*

Proof: The proof can be found in [2].

3. Viscous Dusty Gas Model

To overcome the difficulties mentioned in previous sections, one uses either two-pressure models (which ensure hyperbolicity), [20,14], or models that take viscous forces into account, [16,1,3,4,5,12,8]. Our attention is focused on the second approach. Taking into account the viscosity of a carrier gas, we replace system (2.1)-(2.4) by its viscous multi-dimensional version:

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{3.1}$$

$$\rho\big(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) + \nabla p = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + mK(\mathbf{v}-\mathbf{u}), \tag{3.2}$$

$$m\big(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) = mK(\mathbf{u}-\mathbf{v}), \tag{3.3}$$

$$m_t + \nabla\cdot(m\mathbf{v}) = 0. \tag{3.4}$$

This system of a composite type in certain sense is “better” than hyperbolic one, but “worst” than the Navier-Stokes equations. Indeed, Theorem 2 below guarantees local existence and uniqueness of a strong solution to the Cauchy problem; however, there are serious difficulties in order to get a global-in-time solution, whereas such a solution can be rather easily obtained for the pure Navier-Stokes system by the method of continuation of a local one, [18].

For the viscous model (3.1)-(3.4) we have the following local result (1995):

Theorem 2 *Given initial functions*

$$\rho_0 \in C^\alpha(\mathbb{R}^n), \quad 0 < \rho^- \leq \rho_0(x) \leq \rho^+ < \infty, \quad \nabla\rho_0 \in L_q(\mathbb{R}^n), \quad q > n,$$
$$0 \leq m_0 \in W_q^1(\mathbb{R}^n), \quad u_0 \in W_q^{2-2/q}(\mathbb{R}^n), \quad v_0 \in W_q^2(\mathbb{R}^n),$$

one can find a real $T > 0$ such that for $(x,t) \in \Pi_T = \{x \in \mathbb{R}^n,\ 0 < t < T\}$ there exists a unique solution to problem(3.1)-(3.4), (2.5) with the following properties:

$$0 < \rho(x,t) \in W_{q,\infty}^{1,1}(\Pi_T), \quad 0 \leq m(x,t) \in L^\infty(0,T;\ W_q^1(\mathbb{R}^n)),$$
$$m_t(x,t) \in L^\infty(0,T;\ L_q(\mathbb{R}^n)), \quad (u,v)(x,t) \in W_q^{2,1}(\Pi_T).$$

Proof: For the proof see [3].

The question now is: does global regular solution exist at least in 1-D case? First, it should be noted that such a solution is forbidden for an arbitrary data due to well-known example of inviscid Burgers equation: solution of the Cauchy problem $v_t + vv_x = 0$, $v(x,0) = v_0(x)$ with $v_0' < 0$ blows up at a finite time. On the other hand, the presence of a linear damping term in this type of equations together with sufficiently small initial data yield the existence of strong and even classical solutions for all $t > 0$. Indeed, the equation

$$v(x,t) = v_0\big(x + v(x,t)(1 - e^{Kt})/K\big)e^{-Kt}$$

gives an implicit solution to the damped Cauchy problem

$$\begin{aligned}&v_t + vv_x + Kv = 0, \quad K > 0,\ t > 0 \text{ (equation (3.3) with } u \equiv 0),\\ &v(x,0) = v_0(x), \quad x \in \mathbb{R}.\end{aligned}$$

If the initial function is sufficiently small (more precisely, if $|v_0'(x)| \le K$), then the classical solution to this problem is globally defined by the Implicit Function theorem, because

$$1 + v_0'(\cdot)(1 - e^{-Kt})/K \neq 0 \quad \forall t > 0.$$

However, if $|v_0'| > K$, then there exists $t^* = t^*(K)$ where the solution blows up.

Thus, one can expect global solvability for system (3.1)-(3.4), even for one-dimensional case, only if the initial data are small in some sense.

To obtain a global solution (at least by some of the known methods), one has to obtain a priori estimates independent on arbitrary $t > 0$. It seems not trivial. In fact, using multiplication technique (see, for instance, [5]), one can get the first estimate (we consider here only 1-D case):

$$\begin{aligned}&\frac{1}{2}\big(\|\sqrt{\rho}\,u\|^2 + \|\sqrt{m}\,v\|^2\big)(t) + \|\sqrt{\phi(\rho)}\|^2(t)\\ &+ \int_0^t \big(\mu\|u_x\|^2 + K\|\sqrt{m}(u-v)\|^2\big)(\tau)d\tau \le C_0\end{aligned}$$

which is completed by the mass conservation law:

$$\int_{\mathbb{R}} m(x,t)\,dx = \int_{\mathbb{R}} m_0(x)\,dx \quad (\forall t > 0).$$

Thus, we have: $m \in L^1(\mathbb{R})$, $\sqrt{m}v \in L^2(\mathbb{R})$ and $\sqrt{m}(u-v) \in L^2(\mathbb{R})$. These estimates are not sufficient to prove the existence of global regular solutions by any of classical methods. Regularizations of model (3.1)-(3.4) by viscosity of the dust phase (both in third and even in forth equations) also does not help by the same reason.

In the next section we propose to model a dusty gas flow by the Kuramoto-Sivashinsky equation which permits to dismiss this gap.

4. Kuramoto-Sivashinsky Model

In this section we study a two-phase model of a dusty gas governed by the Kuramoto-Sivashinsky equation. This equation has a dissipative term of the fourth order and is widely used in the theory of viscous turbulent flows and in studies of flame fronts propagation [11,17]. The latter ones provide classical example of a dusty gas.

For $T > 0$, let $Q = \{(x,t) : \ x \in \Omega, \ t \in (0,T)\}$ where $\Omega \subseteq \mathbb{R}$ is either the interval $\Omega = (0,1)$ in the case of mixed problem, or the line $\Omega = \mathbb{R}$ in the case of the Cauchy problem. In Q we consider the following problem:

$$u_t + uu_x + \mu u_{xx} + \nu u_{xxxx} + \alpha u = mK(v-u), \tag{4.1}$$

$$v_t + vv_x = K(u-v), \tag{4.2}$$

$$m_t + (mv)_x = 0, \tag{4.3}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad m(x,0) = m_0(x) \geq 0, \tag{4.4}$$

$$\left.\begin{array}{l} u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0, \\ v(0,t) = v(1,t) = 0, \end{array}\right\} \text{ if } \Omega = (0,1). \tag{4.5}$$

Here u and v are velocities of the medium and solid particles respectively; m is the local concentration of particles; μ, ν and α are positive constant coefficients of viscosity and friction and $K > 0$ is the constant coefficient of the phase interaction.

The questions of global solvability and asymptotic behavior have been answered for this model by the following theorem (2002):

Theorem 3 *Both initial ($\Omega = \mathbb{R}$) and initial-boundary ($\Omega = (0,1)$) value problems possess global regular solutions provided small initial data. If the total mass of the dust is sufficiently small, then the solution decays exponentially.*

Let us describe these affirmations more precisely.

First, we define a real λ as follows:

$$\lambda = \left[\|v_0'\|^2 + K(\|u_0\|^2 + \|\sqrt{m_0}v_0\|^2)/2\alpha + K(\|u_0\|^2 + \|\sqrt{m_0}v_0\|^2)/2\nu\right]^{1/2} + \left[\|v_0''\|^2 + K(\|u_0\|^2 + \|\sqrt{m_0}v_0\|^2)/\nu\right]^{1/2}. \tag{4.6}$$

Then, we assume that

$$\alpha > 3K \int_\Omega m_0(x)\,dx \quad \text{and} \quad \nu > K \int_\Omega m_0(x)\,dx. \tag{4.7}$$

Lemma 1 *Let $\Omega = (0,1)$, $0 < \mu < \min\{\alpha, \nu\}$, $K > 0$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $m_0 \in H^1(\Omega)$. If $\lambda < K/7$, then for all $T > 0$ the problem (4.1)-(4.5) has a unique strong solution:*

$$\begin{aligned} & u \in L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^4(\Omega)), \\ & u_t \in L^2(0,T;L^2(\Omega)), \\ & v \in L^\infty(0,T;H^2(\Omega)), \\ & v_t \in L^\infty(0,T;H^1(\Omega)), \\ & m \in L^\infty_{loc}(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), \\ & m \geq 0, \\ & m_t \in L^\infty_{loc}(0,T;L^2(\Omega)). \end{aligned} \tag{4.8}$$

If conditions (4.7) hold, then there exists a real $\theta > 0$ such that

$$\|u\|^2(t) + \|\sqrt{m}v\|^2(t) \le \left[\|u_0\|^2 + \|\sqrt{m_0}v_0\|^2\right] e^{-\theta t}. \tag{4.9}$$

Lemma 2 *Let $\Omega = \mathbb{R}$, $0 < \mu < \min\{\alpha, \nu\}$, $K > 0$, $u_0 \in H^2(\Omega)$, $v_0 \in H^2(\Omega)$ and $m_0 \in H^1(\Omega) \cap L^1(\Omega)$. If $\lambda < K/7$, then the Cauchy problem (4.1)-(4.4) has a unique strong solution satisfying (4.8) for all $T > 0$. If (4.7) are valid, then (4.9) holds.*

Proof: Proofs of these two lemmas and consequently of Theorem 3 can be found in [6,7].

5. Simplified Viscous Model

In this section we consider a natural simplification of a viscous dusty gas model which possesses global solvability and stability of the associated energy.

The main assumption here is that the number of the dust particles is so small that the phase interaction term can be written as $KM(u - v)$, where $M = \int m\, dx$ is the averaged concentration of the particles. We consider a one-dimensional flow in the bounded region: $x \in (0, 1)$. In this case the equations are:

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0,\\
&\rho(u_t + uu_x) + p_x = \mu u_{xx} + K(v - u)\int_0^1 m(x,t)\, dx,\\
&v_t + vv_x = K(u - v),\\
&m_t + (mv)_x = 0.
\end{aligned}$$

The system is completed by the polytropic state equation: $p = \rho^\gamma,\ \gamma > 1$.

When $t = 0$, we impose the initial data

$$(\rho,\ u,\ v,\ m)(x, 0) = (\rho_0,\ u_0,\ v_0,\ m_0)(x),\ x \in (0, 1),$$

satisfying

$$(\rho_0 - \rho^0,\ u_0,\ v_0)(x) \in H^2(0, 1) \cap H_0^1(0, 1)$$

and

$$0 \le m_0 \in H^1(0, 1),\ m_0 \not\equiv 0.$$

To describe the flow in neighborhoods of the walls, we choose one of the following boundary conditions:

$$(u,\ v)(0, t) = (u,\ v)(1, t) = 0,$$

or

$$u(0, t) = u(1, t) = 0.$$

Define:

$$E(t) = \|\rho - \rho^0\|_{H^2(0,1)}(t) + \|u\|_{H^2(0,1)}(t) + \|v\|_{H^2(0,1)}(t)$$

and

$$M_0 = \int_0^1 m_0(x)\,dx.$$

Global well-posedness of this model is guaranteed by the following result (2003):

Theorem 4 *There exist positive constants ϵ and C such that if*

$$E(0) < \min\left\{\frac{\epsilon}{2},\ \frac{\epsilon}{2\sqrt{C}},\ \frac{K}{7}\right\},$$

then the problem has a unique strong solution:

$$(\rho - \rho^0,\ u,\ v)(x,t) \in C(0,\infty; H^2(0,1) \cap H_0^1(0,1)) \cap C^1(0,\infty; H^1(0,1)),$$
$$0 \le m(x,t) \in C(0,\infty; H^1(0,1)) \cap C^1(0,\infty; L^2(0,1)).$$

Furthermore, if $M_0 < 1/K$, then there exist real $\theta > 0$ and $N = N(E(0)) > 0$ such that for all $t > 0$

$$\|(\rho - \rho^0,\ u,\ v)\|^2(t) \le N \exp\{-\theta t\}.$$

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