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The Navier-Stokes flow with linearly growing initial velocity in the whole space[1]

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ABSTRACT: In this paper, the uniqueness of the solutions to the Navier-Stokes equations in the whole space is constructed, provided that the velocity grows linearly at infinity. The velocity can be chosen as $Mx + u(x)$ for some constant matrix M and some function u. The perturbation u is taken in some homogeneous Besov spaces, which contain some nondecaying functions at space infinity, typically, some almost periodic functions. It is also proved that a locally-in-time solution exists, when M is essentially skew-symmetric which demonstrates the rotating fluid in 2- or 3-dimension.

Key words: Navier-Stokes equations, linearly growing data, mild solution, homogeneous Besov spaces, almost periodic function.

Contents

1. Introduction.

In this note we consider the Cauchy problem of the incompressible Navier-Stokes equations in the whole space $\mathbf{R}^n$ $(n \geq 2)$:

$$\text{(NS.1)} \qquad \begin{cases} U_t - \Delta U + (U,\nabla)U + \nabla \tilde{P} = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ \nabla \cdot U = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ U|_{t=0} = U_0 \qquad (\text{with } \nabla \cdot U_0 = 0) & \text{in } \mathbf{R}^n. \end{cases}$$

Here, $U := (U^1(x,t), \dots, U^n(x,t))$ and $\tilde{P} := \tilde{P}(x,t)$ represent, respectively, the unknown velocity vector field of the fluid and its pressure at a point $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and a time $t > 0$; U_0 is a given initial velocity. We have used standard notations about derivatives, i.e., $U_t := \partial_t U$, $(U,\nabla) := \sum_{i=1}^n U^i \partial_i$, $\partial_i := \frac{\partial}{\partial x_i}$, $\Delta := \sum_{i=1}^n \partial_i^2$, $\nabla \cdot U := \sum_{i=1}^n \partial_i U^i$ and $\nabla \tilde{P} := (\partial_1 \tilde{P}, \dots, \partial_n \tilde{P})$.

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Many authors already studied (NS.1). Especially, it is well-known that one can construct a smooth solution to (NS.1), at least when the initial velocity U_0 belongs to L^p for $p \in [n, \infty]$; see [22,16,13,7,8,14]. Here, $L^p := L^p(\mathbf{R}^n)$ for $1 \le p \le \infty$ is the usual Lebesgue space. Several researchers tried to prove the existence of unique classical solutions with more general initial velocity U_0 in some function spaces. In particular, in [25,26,8,1,32,9] this problem was investigated in Besov spaces. However, there are few results for growing initial velocity at space infinity except simple cases, see [27,20,2].

In this paper we select the initial velocity as

$$U_0(x) := Mx + u_0(x), \qquad x \in \mathbf{R}^n, \tag{1.1}$$

where u_0 is denoted by a function with $\nabla \cdot u_0 = 0$, and $M := (m_{ij})_{1 \le i,j \le n}$ stands for an $n \times n$ constant matrix satisfying $\operatorname{tr} M = 0$. Here, we have used the notation of $\operatorname{tr} M := \sum_{i=1}^n m_{ii}$. In [27] the readers can find the examples of M and the reason why we study this type initial velocity. Let $(U, \tilde{P})$ be a classical solution of (NS.1). Investigating (NS.1) with initial velocity (1.1), we notice the following simple substitution of solutions:

$$u(x,t) := U(x,t) - Mx \quad \text{and} \quad P(x,t) := \tilde{P}(x,t) - (\Pi x, x) \tag{1.2}$$

for $x \in \mathbf{R}^n$ and $t > 0$. Here, $\Pi := \frac{1}{2}(M_1^2 + M_2^2)$, and we divide M into the symmetric and antisymmetric (skew-symmetric) parts,

$$M_1 := \frac{1}{2}(M + {}^tM) \quad \text{and} \quad M_2 := \frac{1}{2}(M - {}^tM),$$

where ${}^tM$ stands for the transposed matrix of M. It is easy to see that (u, P) satisfies

$$\text{(NS.2)} \quad \begin{cases} u_t - \Delta u + (u, \nabla)u + (Mx, \nabla)u + Mu + \nabla P = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ \nabla \cdot u = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ u|_{t=0} = u_0 \quad (\text{with } \nabla \cdot u_0 = 0) & \text{in } \mathbf{R}^n. \end{cases}$$

Thanks to $(Mx, \nabla)Mx = \nabla(\Pi x, x)$ and $\operatorname{tr} M = 0$, (NS.2) follows from (1.2) and (NS.1), directly. We notice that the terms of Mu can be even more generalised. We thus consider that

$$\text{(NS.3)} \quad \begin{cases} u_t - \Delta u + (u, \nabla)u + (Mx, \nabla)u + Nu + \nabla P = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ \nabla \cdot u = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ u|_{t=0} = u_0 \quad (\text{with } \nabla \cdot u_0 = 0) & \text{in } \mathbf{R}^n. \end{cases}$$

with some constant matrix $N = (n_{ij})_{1 \le i,j \le n}$. Hereafter, we rather discuss (NS.3).

Before stating the main results, we introduce some function spaces used in this paper. We have already defined L^p for $1 \le p \le \infty$, and let $\| \cdot \|_p$ be a norm of L^p. Let $W^{1,p}$ be the Sobolev space whose norm is $\| \cdot \|_p + \|\nabla \cdot \|_p$. We sometimes suppress the notation of $(\mathbf{R}^n)$, and do not distinguish between spaces of vector and scalar-valued function, if no confusion seems to be likely. We will give the definition of the homogeneous Besov space $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbf{R}^n)$ in Section 2. The most important one in this paper is $\dot{B}^0_{\infty,1}$. We introduce its modification space:

$$\dot{\mathcal{B}}^0_{\infty,1} := \{f \in \mathcal{S}';\ \|f; \dot{B}^0_{\infty,1}\| < \infty \quad \text{and} \quad f = \sum_{j=-\infty}^{\infty} \phi_j * f \ \text{ in } \ \mathcal{S}'\}.$$

Here, ϕ_k is associated the Paley-Littlewood decomposition of unity, its definition will be given in Section 2. The details and examples of this space can be found in Section 2, and see also [3,35,34]. We define the space $X^{\mathcal{B}}$ by

$$X^{\mathcal{B}} := \{f \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbf{R}^n);\ \|\nabla f; \dot{B}^0_{\infty,1}\| + \|\nabla^2 f; \dot{B}^0_{\infty,1}\| < \infty\}.$$

We also denote by $[[f]] = \sum_{i=0}^{2} \|\nabla^i f; \dot{B}^0_{\infty,1}\|$ the norm of $X^{\mathcal{B}}$. Note that $X^{\mathcal{B}}$ equipped with the norm $[[\cdot]]$ is a Banach space. We also define $X^{\mathcal{B}}_\sigma$ by its solenoidal subspace.

Our main result is the uniqueness of the solutions to (NS.3) in $X^{\mathcal{B}}_\sigma$.

Theorem 1.1. *Let $n \geq 2$. Let $M = (m_{ij})$ and $N = (n_{ij})$ be $n \times n$ constant matrices satisfying $\operatorname{tr} M = 0$. Let u be a classical solution of (NS.3) on $(0, T_0)$ in the class $L^\infty([0, T_0]; X^{\mathcal{B}}_\sigma)$. Then $(u, \nabla P)$ is unique, provided that*

$$\partial_l P = \sum_{i,j=1}^{n} \partial_l R_i R_j u^i u^j + \sum_{i,j=1}^{n} m_{ij} R_l R_i u^j + \sum_{i,j=1}^{n} n_{ij} R_l R_i u^j \tag{1.3}$$

for all $l = 1, \dots, n$ and $t \in (0, T_0)$. Here, $R_i := \partial_i(-\Delta)^{1/2}$ is the Riesz transform.

Remarks 1.2. (i) The first term in the right hand side in (1.3) makes sense in $\dot{\mathcal{B}}^0_{\infty,1}$ by Lemma 1.4. We cannot expect to have a global smoothing effect in this problem. So, we do not know how to construct the solution expect the simple case, see Theorem 1.5.

(ii) The solution $(u, \nabla P)$ obtained satisfies (NS.3) in the $\dot{\mathcal{B}}^0_{\infty,1}$-sense. It has no ambiguity of the constants. We can still prove the uniqueness of the solutions with the data u in the class of $\cap_{i=0}^{2} \dot{B}^i_{\infty,1}$. However, a solution satisfying (NS.3) in $\dot{B}^0_{\infty,1}$-sense has an ambiguity of the constants. The author is unable to obtain similar results for other function spaces such as $W^{2,\infty}$ or $\cap_{i=0}^{2} \dot{B}^{s+i}_{\infty,\infty}$ for $s < 0$. We have to choose the function spaces to be homogeneous, since we need the boundedness of the Riesz transform in these spaces.

(iii) One can expect to establish a (C_0)-semigroup $\{e^{-t\mathcal{L}}; t \geq 0\}$ in $L^2_\sigma(\mathbf{R}^n)$ (or, more generally, in L^p_σ for $p \in (1, \infty)$), where $\mathcal{L}u = -\Delta u + (Mx, \nabla)u - Mu$. Then, one can also expect to get the solution of (NS.3) in these spaces, of course, they vanish at space infinity. The reader can find some results in this direction in the papers by T. Hishida [20], see also [11].

(iv) We may assume that m_{ij} and n_{ij} less than 1 by rescaling, but the essential difficulty remains unchanged. We still obtain the existence of a local solution if m_{ij}

and n_{ij} are not constants, even if $m_{ij} := m_{ij}(x,t)$ and $n_{ij} := n_{ij}(x,t)$ with suitable assumption, for example, $n_{ij} \in W^{1,\infty}([0,T]; W^{3,\infty})$ and $m_{ij} \in W^{1,\infty}([0,T]; W^{d,\infty})$ keeping $\nabla \cdot Mx = 0$ with $\partial_k m_{ij}(x,t) = O(|x|^{-1})$ as $|x| \to \infty$ for all i, j, k and t and for some sufficiently large d (roughly speaking, it is enough to choose $d = 2n+3$ in order to apply the Fourier Multiplier theory, see e.g. [35]). However, we can not expect that $(U, \nabla \tilde{P})$ solves (NS.1), where $(U, \nabla \tilde{P})$ is the transformation given by (1.2) of the nonconstant coefficient solution $(u, \nabla P)$; see the next corollary.

Due to the transformation (1.2), one can see that $(U, \nabla \tilde{P})$ solves (NS.1), provided that M is a constant matrix and $N = M$. Now we state a corollary:

Corollary 1.3. *Assume that M is a constant matrix satisfying* $\operatorname{tr} M = 0$. *Let $(U, \nabla \tilde{P})$ be a classical solution of* (NS.1) *on $[0, T_0]$ in the class of $U - Mx \in W^{1,\infty}([0,T_0]; X_\sigma^{\mathcal{B}})$ and*

$$\partial_l \tilde{P} := \sum_{i,j=1}^{n} \partial_l R_i R_j u^i u^j + 2 \sum_{i,j=1}^{n} m_{ij} R_l R_i u^j + \partial_l (\Pi x, x) \tag{1.4}$$

for all $l = 1, \ldots, n$ and $t \in (0, T_0]$; $u^i := U^i + \sum_{j=1}^{n} m_{ij} x_j$. Then $(U, \nabla \tilde{P})$ is unique.

We now recall the estimates for the quadratic terms with differential.

Lemma 1.4. *There exists a positive constant C such that*

$$\|f \cdot g; \dot{B}^1_{\infty,1}\| \le C(\|f; \dot{B}^1_{\infty,1}\| \; \|g; \dot{B}^0_{\infty,1}\| + \|f; \dot{B}^0_{\infty,1}\| \; \|g; \dot{B}^1_{\infty,1}\|)$$

for all $f, g \in \dot{\mathcal{B}}^0_{\infty,1} \cap \dot{B}^1_{\infty,1}$.

This lemma shows that the first terms in right hand side in (1.3) or (1.4) are well defined. We can prove Lemma 1.4 using by the equivalent norm:

$$\|v; \dot{B}^s_{p,q}\| \cong \Big[\int_0^\infty t^{-1-sq} \sup_{|y| \le t} \|\tau_y v + \tau_{-y} v - 2v\|_p^q dt \Big]^{1/q},$$

which is valid for $1 \le p, q \le \infty, 0 < s < 2$, where τ_y is the translation by $y \in \mathbf{R}^n$, that is, $\tau_y f(x) = f(x-y)$. This characterization of the Besov norm is obtained by [10]. The proof similar to this lemma can also be found in [19] so that we may skip the details of the proof.

We are now in position to give a typical example of M satisfying $\operatorname{tr} M = 0$:

$$n = 3, \qquad M' = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for} \quad a \in \mathbf{R}.$$

Note that above M' is an example of rotation, especially, it describes the Cauchy problem with the Coriolis force, see e.g., [2]. In this case we can solve (NS.1) by using another transformation instead of (1.2), so that it is enough to give the local existence theorem of the (NS.3) with $M = 0$ and $N = M'$; the reader can find the details in [2]. Of course, this problem is easier than ours, because there are no coefficient terms growing linearly.

We are also enable to consider irrotational flow so that

$$M'' = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \qquad \text{for} \quad a_1 + a_2 + a_3 = 0.$$

According to A. Majda [27], this example M'' demonstrates a jet, or the draining of the fluid. He showed that $U = Mx$ (with $\operatorname{tr} M = 0$) is an exact solution of (NS.1), provided that the pressure should be taken appropriately. In [15] Y. Giga and T. Kambe also investigated the axisymmetric irrotational flow ($a_1 = a_2 = -a_3/2$). They studied the stability of vortex, when the velocity field of the fluid U is expressed as $U = M''x + V$ with two-dimensional velocity field V, i.e., $V = (V^1, V^2, 0)$ so that the vorticity is a scalar function. It is obvious that the linear combination of a pure rotating like M' and irrotating M'' satisfies $\operatorname{tr} M = 0$, which illustrates the bathtub drain swirls. Thus, it is much meaningful to study the solutions of (NS.1) with (1.1).

We now state the local solvability of (NS.1), when the initial velocity is given by

$$U_0(x) := M'x + u_0(x), \qquad x \in \mathbf{R}^n \tag{1.5}$$

in 3-dimensional case. We sometimes use the following notation for the sake of simplicity: $(-x_2, x_1, 0) = e_3 \times x$, where $\times$ stands for the outer product in 3-dimension and $e_3 := (0, 0, 1)$. Before stating our results, we introduce the transformed equations, which is different from (1.2). It is usual way that we can transform (NS.1) with the initial velocity (1.5) into the Navier-Stokes equations with an additional Coriolis term (see e.g. [2]):

$$\text{(NS.4)} \qquad \begin{cases} \bar{u}_t - \Delta \bar{u} + (\bar{u}, \nabla)\bar{u} + a e_3 \times \bar{u} + \nabla \bar{P} = 0 & \text{in} \quad \mathbf{R}^n \times (0, T), \\ \nabla \cdot \bar{u} = 0 & \text{in} \quad \mathbf{R}^n \times (0, T), \\ \bar{u}|_{t=0} = u_0 \qquad (\text{with} \ \ \nabla \cdot u_0 = 0) & \text{in} \quad \mathbf{R}^n. \end{cases}$$

Here,

$$\bar{u}(y, t) := e^{-aJt} U(e^{aJt} y, t) - aJy \tag{1.6}$$

and

$$\bar{P}(y, t) := \tilde{P}(e^{aJt} y, t) + \frac{a^2}{2}(y_1^2 + y_2^2) \tag{1.7}$$

for $x = e^{aJt} y$, where

$$J := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e^{aJt} := \begin{pmatrix} \cos(at) & -\sin(at) & 0 \\ \sin(at) & \cos(at) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have used same notations of differentials to (NS.1), but we should note that $\partial_j = \partial/\partial y_j$. The transformations (1.6), (1.7) and their inverse establish a one-to-one correspondence between the vector fields $U(x,t)$ and $\bar{u}(y,t)$. We note that for $t=0$, $x=y$ and thus $u_0(y) = u_0(x)$.

Theorem 1.5. *Assume that* $u_0 \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbf{R}^3)$ *with* $\nabla \cdot u_0 = 0$. *Then there exist* $\bar{T}_0 > 0$ *and* $u \in C([0,\bar{T}_0]; \dot{\mathcal{B}}^0_{\infty,1})$ *such that* $(\bar{u}, \nabla \bar{P})$ *is a unique classical solution to* (NS.4)*, provided that*

$$\nabla \bar{P} = \nabla \sum_{i,j} R_i R_j \bar{u}^i \bar{u}^j + a \left(R_1 R_1 \bar{u}^2 - R_1 R_2 \bar{u}^1, R_1 R_2 \bar{u}^2 - R_2 R_2 \bar{u}^1, 0 \right). \tag{1.8}$$

Remarks 1.6. (i) It is possible to derive the estimate of existence time by below:

$$\bar{T}_0 \geq C / \left(|a| + \| u_0; \dot{B}^0_{\infty,1} \| \right)^4$$

with some numerical constant C. This estimate follows from the way of the construction of mild solutions by iteration scheme.

(ii) We can also construct the locally-in-time solution, when $u_0 \in \dot{b}^{-\varepsilon}_{p,q}$ for $n < p \leq \infty$, $1 \leq q \leq \infty$ and $0 < \varepsilon < 1 - n/p$, where $\dot{b}^{-\varepsilon}_{p,q}$ is a small Besov space, see [32]. It can be still true that the solution is constructed for $u_0 \in \dot{B}^{-\varepsilon}_{p,q}$ except for the continuity with respect to time-variable at the initial time. These proofs are parallel to that in [32].

We are also able to obtain the local solvability of (NS.4) in 2-dimension, provided that we annihilate the third component of u. Moreover, it is proved that the solution can be extended globally.

Theorem 1.7. *Assume that* $u_0 \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbf{R}^2)$ *with* $\nabla \cdot u_0 = 0$. *Then there exists* $\bar{u} \in C([0,\infty); \dot{\mathcal{B}}^0_{\infty,1})$ *such that* $(\bar{u}, \nabla \bar{P})$ *is a unique classical globally-in-time solution to* (NS.4)*, provided that* $\nabla \bar{P}$ *is given by* (1.8)*.*

In [17] Y. Giga, S. Matsui and the author of this paper proved the global existence theory on the 2-dimensional (NS.1) with $U_0 \in L^\infty$. We can apply their method directly.

We next describe the outline of the proofs of Theorem 1.1, 1.5 and 1.7. Firstly, let us introduce the notion of a *mild solution* of (NS.3). A mild solution denotes by the classical solution of the abstract equation:

(ABS) $$u_t - \Delta u + \mathbf{P}(u,\nabla)u + \mathbf{P}(Mx,\nabla)u + \mathbf{P}Nu = 0$$

with $u(0) = u_0$, where $\mathbf{P}$ is the Helmholtz projection. The details will be given in Section 2, precisely. At that time, the biggest difficulty is to deal with the coefficient terms growing linearly at space infinity. To overcome this difficulty we give a priori estimate for the maximum principle as follows: let u be a classical solution of (ABS), then there is a positive constant C such that

$$\|u(t)\| \le C\|u_0\| + C\int_0^t (1 + [[u(s)]])\|u(s)\|ds.$$

Hence, the difference of two solutions is equal to zero in $\dot{\mathcal{B}}^0_{\infty,1}$ by the Gronwall inequality, which implies the uniqueness of solutions.

The proof of Theorem 1.5 is given by standard iteration argument, that is, the successive approximation. Since the projection $\mathbf{P}$ is a bounded operator in $\dot{\mathcal{B}}^0_{\infty,1}$, the term $e_3 \times \bar{u}$ can be regarded as the perturbation.

We prove Theorem 1.7 by deriving a priori estimate for $\bar{u}$ as follows:

$$\|\bar{u}(t)\| \le K \exp\{Ke^{Kt}\} \qquad \text{for} \quad t > 0$$

with some positive constant K depending only on u_0. Main idea of proof is based on the boundedness of rotation of $\bar{u}$, which comes from the maximum principle for the rotation equation.

We now refer to several results known in related to our situations. In [29] H. Okamoto showed that if $(U, \tilde{P})$ is a classical solution of (NS.1) satisfying the point-wise estimates as follows:

$$|U| \le C(|x| + 1), \quad |\nabla U| \le C \quad \text{and} \quad |\tilde{P}| \le C(|x| + 1)^{-1/2},$$

then $(U, \tilde{P})$ is unique. See also [24]. Since we do not know whether $\tilde{P}$ given by (1.4) satisfies above point-wise estimate, there seems to be no inclusion between his results and Corollary 1.3. J. Kato [21] also obtained some uniqueness theorem, but in his situation U must be bounded, then his results and ours are not comparable. The reader can find other results for the uniqueness of (NS.1) in [21].

The local existence theory for (NS.1) with $U_0 = M'x + u_0$ has already been investigated by A. Babin, A. Mahalov and B. Nicolaenko [2], when u_0 is a periodic function with suitable assumptions. In Theorem 1.5 we succeed to improve their results in the sense that we generalise the conditions of u_0, that is, our function space of the initial velocity includes theirs. In [20] T. Hishida considered the Navier-Stokes equations with the Coriolis and centrifugal force terms (that is (NS.3) with $M = M'$ and $N = -M'$) in an exterior domain. He established the semigroup theory based on L^2, and obtained the smooth solution, see Remark 1.2-(iii). If $N = -M$, then the problem is slightly easier than ours, because $\nabla\cdot\{(Mx, \nabla)u - Mu\} = 0$ if $\nabla \cdot u = 0$, see also the beginning of Section 2. His results and ours are not comparable.

Recently, Y. Giga and K. Yamada [18] constructed the solution of the Burgers type equations, when the initial velocity is *arbitrary* linearly growing at space infinity. Here, *arbitrary* means that $u_0(x)/|x|$ and $\nabla u_0(x)$ are bounded. In Corollary 1.3 we mention that one can show the uniqueness of solutions to the Navier-Stokes equations with *special shapes* linearly growing initial velocity. It is still open to solve (NS.1) with not only arbitrary but also special shapes linearly growing initial velocity except the case of $M = M'$. They established the maximum principle for the solution of the linearized problem. The basic strategy of the proof of Theorem 1.1 is essentially same as theirs. For other articles related to this topic the reader is referred to the above literature.

This paper is organised as follows. In Section 2 we shall introduce the notion of a mild solution, and the homogeneous Besov spaces including the examples of the initial data. In Section 3 we shall prepare several lemmas in order to prove the our results, Proposition 3.1 is the crucial step in this paper. In Section 4 we shall give the proofs of Theorem 1.1, 1.5 and 1.7.

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2. Mild Solution and Function Spaces.

In this section we introduce the notion of a mild solution, and the homogeneous Besov spaces. We also give same examples.

It seems to be a standard technique to operate with the Helmholtz projection $\mathbf{P}$ for searching properties of solutions to (NS.1), for instance, to construct the locally-in-time smooth solution; see [23,12,22,16]. However, it should be noted that even if $\nabla \cdot u = 0$, unfortunately,

$$\nabla \cdot \{(Mx, \nabla)u + Nu\} \neq 0$$

in general. Hence, we have to choose the function spaces of the initial data u_0 (and of the solution u in space variables) so that $\mathbf{P}$ is bounded.

Hereafter, we rather discuss the solution of the abstract equation (ABS) or the integral equation (INT). The abstract equation (ABS) is described by

$$u_t - \Delta u + \mathbf{P}(u, \nabla)u + \mathbf{P}(Mx, \nabla)u + \mathbf{P}Nu = 0$$

with $u(0) = u_0$. Also, the integral equation (INT) is described by

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u)(s)ds + \int_0^t e^{(t-s)\Delta}\mathbf{P}(Mx, \nabla)u(s)ds + \int_0^t e^{(t-s)\Delta}\mathbf{P}Nu(s)ds.$$

Note that (ABS) and (INT) are equivalent in some sense, and that they are formally equivalent to (NS.3). Here, $e^{t\Delta} = G_t*$ denotes a solution-operator of the heat equation, where $G_t(x) = (4\pi t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{4t})$ is the Gauss kernel, and $*$ means the convolution with respect to x; the Helmholtz projection $\mathbf{P}$ denotes the (orthogonal) projection, and it is written as an $n \times n$ matrix operator $\mathbf{P} = (\delta_{ij} + R_i R_j)_{1 \le i,j \le n}$, where δ_{ij} denotes Kronecker's delta, and R_i is the Riesz transform formally defined by $R_i = \partial_i(-\Delta)^{-1/2}$; $u \otimes u$ is a tensor, whose ij-component is $u^i u^j$.

We note that the operators ∇, $e^{t\Delta}$ and $\mathbf{P}$ commute in our situation. We have used that $(u, \nabla)u = \nabla \cdot (u \otimes u)$ since $\nabla \cdot u = 0$. We assume that u_0 is divergence-free. Then $\mathbf{P}u_0 = u_0$. Once one finds the solution u of (ABS), $(u, \nabla P)$ solves (NS.3) with suitable choice of P, for example, ∇P is given by (1.3). A solution of (ABS) or (INT) is often said to be *mild solution*. We also use this terminology.

In order to understand our results precisely, we recall the definition of homogeneous Besov spaces. Let $\phi_0 \in \mathcal{S}$ with $\mathrm{supp}\hat{\phi}_0 \subset \{1/2 \le |\xi| \le 2\}$ and let $\hat{\phi}_j(\xi) = \hat{\phi}_0(2^{-j}\xi)$ for $j \in \mathbf{Z}$ such that $\{\phi_j\}_{j=-\infty}^{\infty}$ satisfies $\sum_{j=-\infty}^{\infty} \hat{\phi}_j = 1$ except at the origin. Here, $\hat{f}$ stands for the Fourier transform of f and $\mathcal{S}$ is the space of the rapidly decreasing functions in the sense of L. Schwartz; $\mathcal{S}'$ is the topological dual of $\mathcal{S}$, which is the space of tempered distributions.

Definition 2.1. *Let $s \in \mathbf{R}$ and $p, q \in [1, \infty]$. The homogeneous Besov spaces $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbf{R}^n)$ are defined by*

$$\dot{B}^s_{p,q} = \{f \in \mathcal{Z}'; \|f; \dot{B}^s_{p,q}\| < \infty\},$$

where

$$\|f; \dot{B}^s_{p,q}\| = \begin{cases} \left[\sum_j 2^{sjq} \|\phi_j * f\|_p^q\right]^{1/q} & \text{if } \ q < \infty, \\ \sup_j 2^{sj} \|\phi_j * f\|_p & \text{if } \ q = \infty. \end{cases}$$

Here, $\mathcal{Z}'$ denotes the topological dual of

$$\mathcal{Z} = \{f \in \mathcal{S};\ \partial^\alpha \hat{f}(0) = 0 \ \text{ for all } \alpha \in \mathbf{N}_0^n\},$$

where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$.

Remark 2.2. Note that $\dot{B}^s_{p,q}$ is a Banach space. It is well-known that if the exponents satisfy

$$\text{either} \quad s < n/p \qquad \text{or} \quad s = n/p \ \text{ and } \ q = 1, \tag{2.1}$$

then $\dot{B}^s_{p,q}$ can be regarded as a subspace of $\mathcal{S}'$, see [5] and [26]. More precisely, if $f \in \dot{B}^s_{p,q}$ with exponents satisfying (2.1), then we can obtain that $\sum_{j \ge -N} \phi_j * f$ converges in $\mathcal{S}'$ for every N, and its limit has a canonical representation, i.e., $f = \sum_{j \in \mathbf{Z}} \phi_j * f$ in $\mathcal{Z}'$. Throughout this paper, we only treat the homogeneous Besov spaces whose exponents satisfy (2.1).

One can define the operator $\mathbf{P}$ in the homogeneous Besov spaces with exponents satisfying (2.1) in the sense of tempered distribution. It is a bounded operator in the homogeneous Besov space, although it is not bounded in L^∞. We are now in position to give examples of functions in the homogeneous Besov space $\dot{B}^0_{\infty,1}$.

Examples 2.3. (i) It is clear that the constant functions belong to $\dot{B}^0_{\infty,1}$, and its norm is equal to 0, since $\int \phi_j = 0$ for all j. This means that one cannot expect $\|f\|_\infty \le C\|f; \dot{B}^0_{\infty,1}\|$, in general.

(ii) If the first derivative of f and the primitive function of f belong to L^∞, then f belongs to $\dot{B}^0_{\infty,1}$. Of course, the trigonometric functions (e.g., $\alpha \sin(\lambda \cdot x)$) satisfy this condition, then they are contained in $\dot{B}^0_{\infty,1}$; see [33].

(iii) The $\dot{B}^0_{\infty,1}$ contains not only periodic functions but also several nondecaying functions at space infinity. Indeed, the following almost periodic function (in the sense of H. Bohr, see e.g. [4]) belongs to this space:

$$f \in \dot{B}^0_{\infty,1} \quad \text{if} \quad f(x) = \sum_{j=1}^\infty \alpha_j e^{\sqrt{-1}\lambda_j \cdot x} \quad \text{for} \quad \{\alpha_j\}_{j=1}^\infty \in l^1 \quad \text{and} \quad \lambda_j \in \mathbf{R}^n.$$

Such a representation of f is often said to be *generalised trigonometric series*. This notion is motivated by the following calculation: let $\lambda \in \mathbf{R}^n$ and $f_\lambda(x) = \sin(\lambda \cdot x)$, then $\|f_\lambda; \dot{B}^0_{\infty,1}\| \le C$ independent of λ. In general, not every almost periodic function does have such a representation.

(iv) Note that $\dot{B}^0_{\infty,1}$ is strictly smaller than L^∞. More precisely, $\dot{B}^0_{\infty,1} \subset BUC$, which is the space of bounded and uniformly continuous functions in $\mathbf{R}^n$. This fact follows from $\dot{B}^0_{\infty,1} = \dot{b}^0_{\infty,1}$ (which is so called a *small Besov space*; the reader can find the definition in [32]) and $\dot{b}^0_{\infty,1} \subset BUC$. Thus, a function which is not uniformly continuous does not belong to $\dot{B}^0_{\infty,1}$, for example, $f(x) = \sin(1/|x|^2)$ or $f(x) = \sin(e^{|x|})$; but, nevertheless, they are still bounded.

We shall define the $\dot{\mathcal{B}}^s_{p,q}$ spaces. They are essentially similar to the usual homogeneous Besov spaces $\dot{B}^s_{p,q}$, but it has no ambiguity of the constants.

Definition 2.4. *The space* $\dot{\mathcal{B}}^s_{p,q} := \dot{\mathcal{B}}^s_{p,q}(\mathbf{R}^n)$ *is defined by*

$$\dot{\mathcal{B}}^s_{p,q}(\mathbf{R}^n) := \{f \in \mathcal{S}';\ \|f; \dot{B}^s_{p,q}\| < \infty \quad \text{and} \quad f = \sum_{j=-\infty}^\infty \phi_j * f \quad \text{in } \mathcal{S}'\}.$$

Note that $\dot{\mathcal{B}}^s_{p,q}$ is a semi-normed space. If the exponents s, p, q satisfy (2.1), then $\dot{\mathcal{B}}^s_{p,q}$ is a Banach space and

$$\|f; \dot{B}^s_{p,q}\| = 0 \quad \text{if and only if} \quad f = 0 \quad \text{in } \mathcal{S}'.$$

The readers can find the details of these spaces in [34]. It is evident that $X^{\mathcal{B}}$ is a Banach space. In this paper we often use $\dot{\mathcal{B}}^0_{\infty,1}$, since $\|f\|_\infty \le \|f; \dot{B}^0_{\infty,1}\|$ holds for $f \in \dot{\mathcal{B}}^0_{\infty,1}$, contrary to the properties of usual homogeneous Besov space, see Examples 2.3-(i).

Consequently, the initial data (or the data for almost all time) $u_0 \in X^{\mathcal{B}}_\sigma$ is chosen appropriately, provided that u_0 satisfies $\nabla \cdot u_0 = 0$ and has a representation as a generalised trigonometric series, i.e., $u_0(x) = \sum_{j=1}^{\infty} \alpha_j e^{\sqrt{-1}\lambda_j \cdot x}$ for some vector-valued sequences $\{\alpha_j\}_{j=1}^{\infty} \in l^1$ and $\{\lambda_j\}_{j=1}^{\infty} \in l^{\infty}$ with $\lambda_j \neq 0$ for all j. Of course, that sequence does not decay at space infinity, in general. Furthermore, also if $\lambda_j \neq 0$ does not hold, it is obvious that f belongs to $\dot{B}^0_{\infty,1}$, and also it belongs to $\cap_{i=0}^{2} \dot{B}^i_{\infty,1}$.

3. Maximum Principle.

In this section we shall prepare the lemmas to show the proof of our results. The Proposition 3.1 is a crucial point in this paper.

We now state the maximum principle lemma. It is characterised to have the linearly growing coefficient in the transport terms. Its L^∞-version (without $\mathbf{P}$ in (3.1), nor $(1+[[q_0]])\|u\|$ terms in the right hand side of (3.2)) has already proved by [18, Lemma 4.1]. We now recall that $[[f]] = \sum_{i=0}^{2} \|\nabla^i f; \dot{B}^0_{\infty,1}\|$. Here and hereafter, we set $\|f\| = \|f; \dot{B}^0_{\infty,1}\|$ for simplicity.

Proposition 3.1. *Suppose that $u_0 \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbf{R}^n)$. Assume that $T > 0$ is a finite time. Assume that $q(x,t) = q_0(x,t) + Qx$ for some q_0 and Q; q_0 is a vector-valued function enjoying a canonical representation in the class of $q_0 = (q_{01}, \ldots, q_{0n}) \in L^\infty([0,T]; X^{\mathcal{B}}_\sigma)$, and Q is a constant matrix $Q = (q_{ij})$. Let $v \in L^\infty([0,T]; \dot{\mathcal{B}}^0_{\infty,1})$. Assume that $u \in L^\infty([0,T]; \dot{\mathcal{B}}^0_{\infty,1})$ satisfying $\nabla \cdot u = 0$, and that u is a classical solution of*

$$\partial_t u - \Delta u + \mathbf{P}(q, \nabla) u + v = 0 \qquad \text{in} \quad \mathbf{R}^n \times (0,T) \tag{3.1}$$

with $u|_{t=0} = u_0$ in $\mathbf{R}^n$. Then there exists a positive constant C such that

$$\|u(t)\| \leq C \Big\{ \|u_0\| + \int_0^t \Big((1 + [[q_0(s)]]) \|u(s)\| + \|v(s)\| \Big) ds \Big\} \tag{3.2}$$

for all $t \in [0,T]$. Here, the constant C depends only on n, T, Q and ϕ_0.

Remark 3.2. Instead of $\dot{\mathcal{B}}^0_{\infty,1}$, we can get the similar estimates in $\dot{\mathcal{B}}^s_{\infty,1}$ for $s \in (-1, 0)$. The author, however, could not obtain the similar estimate in other spaces, for example, $\dot{\mathcal{B}}^s_{\infty,q}$ for $s < 0$ and $q \in (1, \infty]$. Of course, it seems to be difficult to apply this method for the inhomogeneous spaces (including L^∞) directly, because the Riesz transform is not bounded in those spaces.

Before proving this proposition, we prepare its scalar version. The following lemma then implies the proof of Proposition 3.1 shown later.

Lemma 3.3. *Suppose that $n \geq 1$ and that $a \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbf{R}^n)$ and $v \in L^\infty([0,T]; \dot{\mathcal{B}}^0_{\infty,1})$ are scalar functions for some finite T. Assume that $q(x,t) = q_0(x,t) + Qx$ for some vector-valued function $q_0 = (q_{01}, \ldots, q_{0n}) \in L^\infty([0,T]; X^{\mathcal{B}}_\sigma(\mathbf{R}^n))$ and some constant matrix $Q = (q_{ij})$. Assume that the scalar function $u \in L^\infty([0,T]; \dot{\mathcal{B}}^0_{\infty,1})$ is a classical solution of*

$$\partial_t u - \Delta u + (q, \nabla) u + v = 0 \qquad \text{in} \quad \mathbf{R}^n \times (0,T) \tag{3.3}$$

with $u|_{t=0} = a$ *in* $\mathbf{R}^n$. *Then there exists a positive constant* C *such that*

$$\|u(t)\| \le C\Big\{\|a\| + \int_0^t \big((1+[[q_0(s)]])\|u(s)\| + \|v(s)\|\big)ds\Big\} \tag{3.4}$$

for all $t \in [0,T]$. *Here, the constant* C *depends only on* n, T, Q *and* ϕ_0.

Proof: Let $k \in \mathbf{Z}$. We shall derive the estimates for L^∞-norm of $\phi_k * u$. We will divide the proof into two parts, the cases where $k \le 0$ and $k > 0$.

(Case of $k \le 0$). Let $k \le 0$. Convolve (3.3) with ϕ_k. For the sake of simplicity of notations we set $u_k := \phi_k * u$, $a_k := \phi_k * a$ and $v_k := \phi_k * v$. Then, u_k satisfies $u_k(0) = a_k$ and

$$\partial_t u_k - \Delta u_k + \phi_k * (q,\nabla)u + v_k = 0.$$

Firstly, we observe that

$$\begin{aligned}\phi_k * (q_0,\nabla)u(z) &= \sum_j \int (\partial_j\phi_k)(z-y)q_{0j}(y)u(y)dy \\ &= \sum_j \int (\partial_j\phi_k)(z-y)\big(q_{0j}(y) - q_{0j}(z) + q_{0j}(z)\big)u(y)dy \\ &= (q_0,\nabla)u_k(z) + \Phi_{k,q_0}u(z), \end{aligned} \tag{3.5}$$

since $\nabla \cdot q_0 = 0$. Here, we have defined

$$\Phi_{k,q_0}f(z) := \sum_j \int (\partial_j\phi_k)(z-y)\big(q_{0j}(y) - q_{0j}(z)\big)f(y)dy.$$

Since $(Qx,\nabla) = \sum_{i,j=1}^n q_{ij}x_j\partial_i$, we treat $\phi_k * (x_j\partial_i)u$ by integrating by parts:

$$\begin{aligned}\phi_k * (x_j\partial_i u)(z) &= \int \phi_k(z-y)y_j\partial_i u(y)dy \\ &= -\delta_{ij}\int \phi_k(z-y)u(y)dy + \int (\partial_i\phi_k)(z-y)y_j u(y)dy \\ &= -\delta_{ij}u_k(z) - \int (z_j-y_j)(\partial_i\phi_k)(z-y)u(y)dy + z_j\int(\partial_i\phi_k)(z-y)u(y)dy \\ &= -\delta_{ij}u_k(z) - \phi_k^{ij} * u(z) + z_j(\partial_i u_k)(z).\end{aligned}$$

Here, we define $\phi_k^{ij}(x) := \sum_{l=-1}^1 x_j(\partial_i\phi_{k+l})(x)$. We now recall that u has a canonical representation and $\hat\phi_k \cdot \hat\phi_l = 0$, if l satisfies $|l-k| \ge 2$ by the support of $\hat\phi_k$. Hence, we have $\phi_k^{ij} * u = \phi_k^{ij} * \tilde u_k$, where $\tilde u_k := \sum_{l=-1}^1 u_k$. We thus obtain that

$$\phi_k * (Qx,\nabla)u = -\bar q u_k - \sum_{i,j=1}^n q_{ij}\phi_k^{ij} * \tilde u_k + (Qx,\nabla)u_k.$$

Here, we have denoted by $\bar q := \operatorname{tr} Q$. Therefore, we now conclude that u_k satisfies

$$\partial_t u_k - \Delta u_k + (q,\nabla)u_k + \Phi_{k,q_0}u - \bar q u_k - \sum_{i,j=1}^n q_{ji}\phi_k^{ij} * \tilde u_k + v_k = 0.$$

Notice that u_k is a complex-valued function. We divide u_k its real part and an imaginary part, i.e., $u_k = \Re u_k + \sqrt{-1}\Im u_k$. Hereafter, we discuss its real part only. We will mimic the proof of [18, Lemma 4.1], basically. We set $u_k^1(t) := \Re u_k(t)$. Then, u_k^1 satisfies

$$\partial_t u_k^1 - \Delta u_k^1 + (q, \nabla) u_k^1 + \Re\Phi_{k,q_0} u - \bar{q} u_k^1 - \sum_{i,j=1}^{n} q_{ji} \Re\phi_k^{ij} * \tilde{u}_k + \Re v_k = 0$$

with $u_k^1(0) = \Re a_k$. We set $u_k^2(t) := u_k^1(t) e^{-\bar{q}t}$, so u_k^2 satisfies $u_k^2(0) = \Re a_k$ and

$$\partial_t u_k^2 - \Delta u_k^2 + (q, \nabla) u_k^2 - e^{-\bar{q}t} \Big\{ \Re\Phi_{k,q_0} u + \sum_{i,j=1}^{n} q_{ji} \Re\phi_k^{ij} * \tilde{u}_k + \Re v_k \Big\} = 0.$$

It is easy to see that $\|\phi_k^{ij}\|_1 = C_0$ with some constant C_0 independent of k, which is clearly obtained by the dilation of $\phi_k(x) = 2^{kn}\phi_0(2^k x)$, which comes from $\hat{\phi}_k(\xi) = \hat{\phi}_0(2^{-k}\xi)$. Similarly, there is a positive constant C_1 such that

$$\begin{aligned} \|\Phi_{k,q_0} u\|_\infty &\le \sum_j \sup_x \Big| \int (\partial_j \phi_k)(y-x) \{q_{0j}(y) - q_{0j}(x)\} u(y) dy \Big| \\ &\le 2 \sum_j \|\partial_j \phi_k\|_1 \|q_{0j}\|_\infty \|u\|_\infty \le C_1 2^k \|q_0\| \; \|u\|, \end{aligned} \tag{3.6}$$

since q_0 and u have canonical representations in $\mathcal{S}'$, because they belong to $\dot{\mathcal{B}}^0_{\infty,1}$. Of course, the constant C_1 does not depend on k. Thus, we are now in position to transform again. Define

$$\begin{aligned} u_k^3(t) \; := \; u_k^2(t) \; - \; \|a_k\|_\infty \; - \; C' \int_0^t \; (\|\tilde{u}_k(s)\|_\infty + \|v_k(s)\|_\infty) \, ds \\ - \; C' 2^k \int_0^t \; \|q_0(s)\| \; \|u(s)\| \, ds \end{aligned}$$

with a certain constant $C' > 0$. In the sequel we will see that it is enough to choose $C' = (1 + C_1 + n^2 C_0 \max_{i,j} |q_{ij}|) \exp\{((-\bar{q}) \vee 0)T\}$, where $a \vee b = \max(a, b)$. So, the constant C' depends only on n, T, Q and ϕ_0. Note that $u_k^3(0) = \Re a_k - \|a_k\|_\infty \le 0$. We now confirm that $u_k^3(x,t) \le 0$ for all x and t. We use a contradiction argument, that is, we assume that for some $x \in \mathbf{R}^n$ and $t \in (0, T)$ we have $u_k^3(x,t) > 0$. For obtaining a finite maximum point $(x_0, t_0) \in \mathbf{R}^n \times (0, \infty)$, we use some modification arguments again. Let us put $u_k^4(t) := u_k^3(t) e^{-t}$, and set

$$u_k^\varepsilon(t) := u_k^4(t) - \varepsilon \log \langle x \rangle$$

for small $\varepsilon > 0$. Here, $<x>:= (1+|x|^2)^{1/2}$. Then, u_k^ε satisfies

$$\begin{aligned}\partial_t u_k^\varepsilon - \Delta u_k^\varepsilon + u_k^\varepsilon + (q,\nabla)u_k^\varepsilon + \varepsilon\Big(\log{<x>} - \Delta\log{<x>} + (q,\nabla)\log{<x>}\Big)\\ + e^{-t}\Big(C'2^k\|q_0(t)\|\,\|u(t)\| - e^{-\bar{q}t}\Re\Phi_{k,q_0}u + C'\|\tilde{u}_k(t)\|_\infty\\ - e^{-\bar{q}t}\sum_{i,j=1}^n q_{ji}\Re\phi_k^{ij}*\tilde{u}_k + C'\|v_k(t)\|_\infty - e^{-\bar{q}t}\Re v_k\Big) = 0. \qquad (3.7)\end{aligned}$$

Let $\alpha = \sup_{x,t} u_k^4(x,t) > 0$. By the definition of u_k^ε, for sufficiently small ε we have $\sup_{x,t} u_k^\varepsilon(x,t) > \alpha/2$. We notice that u_k^ε is negative at space infinity, since $u(t) \in \dot{\mathcal{B}}^0_{\infty,1} \subset L^\infty$. Then u_k^ε has a maximum point (x_0,t_0) as finite. We take ε small so that

$$\| -\Delta\log{<x>} + (q,\nabla)\log{<x>}\|_\infty < \frac{\alpha}{4\varepsilon}.$$

Then, since (x_0,t_0) is a maximum point of u_k^ε, we observe that

$$\begin{aligned}\partial_t u_k^\varepsilon - \Delta u_k^\varepsilon + u_k^\varepsilon + (q,\nabla)u_k^\varepsilon + \varepsilon\Big(\log{<x>} - \Delta\log{<x>} + (q,\nabla)\log{<x>}\Big)\\ + e^{-t}\Big(C'2^k\|q_0(t)\|\,\|u(t)\| - e^{-\bar{q}t}\Re\Phi_{k,q_0}u + C'\|\tilde{u}_k(t)\|_\infty\\ - e^{-\bar{q}t}\sum_{i,j=1}^n q_{ji}\Re\phi_k^{ij}*\tilde{u}_k + C'\|v_k(t)\|_\infty - e^{-\bar{q}t}\Re v_k\Big) > 0\end{aligned}$$

in $(x,t) \in B_\rho(x_0,t_0)$ for some small $\rho > 0$, since the choice of C' implies that $e^{-t}(\cdots) \ge 0$. This is a contradiction of (3.7), therefore, we conclude that $u_k^\varepsilon \le 0$. By sending ε to zero we have $u_k^4 \le 0$, we thus get $u_k^3 \le 0$. Back to the transformations we have that

$$\Re u_k(t) \le e^{\bar{q}T}\Big[\|a_k\|_\infty + C'\int_0^t \big\{\|\tilde{u}_k(s)\|_\infty + \|v_k(s)\|_\infty\big\}ds + C'2^k\int_0^t \|q_0(s)\|\,\|u(s)\|ds\Big].$$

For a symmetric argument we can have the estimate by below. Analogously, we can also get the same estimate for $\Im u_k$. Hence, we obtain that

$$\|u_k(t)\|_\infty \le C\Big[\|a_k\|_\infty + \int_0^t \big\{\|\tilde{u}_k(s)\|_\infty + \|v_k(s)\|_\infty\big\}ds + 2^k\int_0^t \|q_0(s)\|\,\|u(s)\|ds\Big].$$

(Case of $k > 0$). Let $k > 0$. Similarly to (Case of $k \le 0$), we take $\phi_k *$ into (3.3). Instead of (3.5), using the main theorem of integral and differential calculus twice, we thus observe that

$$\phi_k * (q_0,\nabla)u(z) = \tilde{\Phi}_{k,q_0}u(z) + (q_0,\nabla)u_k(z). \qquad (3.8)$$

Here, $\tilde{\Phi}_{j,q_0}$ stands for

$$\tilde{\Phi}_{k,q_0} f(z) = \sum_{j,l,m} \int (y_l - z_l)(y_m - z_m)(\partial_j \phi_k)(z-y) \times \int_0^1 \left(\int_0^1 t(\partial_l \partial_m q_{0j})(ts(y-z)+z)dt \right) ds f(y)dy + \sum_{j,l} (\partial_l q_{0j})(z) \int (y_l - z_l)(\partial_j \phi_k)(z-y) f(y)dy.$$

Thus, we now have that for $k > 0$

$$\partial_t u_k - \Delta u_k + (q, \nabla) u_k + \tilde{\Phi}_{k,q_0} u - \bar{q} u_k - \sum_{i,j=1}^n q_{ji} \phi_k^{ij} * \tilde{u}_k + v_k = 0.$$

We note that there is a constant C_2 depending only on n and ϕ_0 such that

$$\begin{aligned} \|\tilde{\Phi}_{k,q_0} u\|_\infty &\le C 2^{-k} \|\nabla^2 q_0\|_\infty \|u\|_\infty + C \|\nabla q_0\|_\infty \|\tilde{u}_k\|_\infty \\ &\le C_2 2^{-k} \|\nabla^2 q_0\| \ \|u\| + C_2 \|\nabla q_0\| \ \|\tilde{u}_k\|_\infty. \end{aligned} \tag{3.9}$$

By the same arguments for the case $k \le 0$, we are able to obtain that

$$e^{-\bar{q}t} \Re u_k(t) - \|a_k\|_\infty - C'' \int_0^t \Big\{ (1 + \|\nabla q_0\|) \|\tilde{u}_k(s)\|_\infty + \|v_k(s)\|_\infty \Big\} ds - C'' 2^{-k} \int_0^t \|\nabla^2 q_0(s)\| \ \|u(s)\| ds \le 0$$

for the constant $C'' = (1 + C_2 + n^2 C_0 \max_{i,j} |q_{ij}|) \exp \Big\{ \big((-\bar{q}) \vee 0\big) T \Big\}$. Of course, we may get similar estimates for $\Im u_k$. Finally, we sum up with respect to k, then we have

$$\begin{aligned} \|u(t)\| &\le C \Big\{ \|a\| + \int_0^t \Big[(1 + \|\nabla q_0(s)\|) \|u(s)\| + \|v(s)\| \Big] ds \\ &\quad + \sum_{k \le 0} 2^k \int_0^t \|q_0(s)\| \ \|u(s)\| ds + \sum_{k > 0} 2^{-k} \int_0^t \|\nabla^2 q_0(s)\| \ \|u(s)\| ds \\ &\le C \Big\{ \|a\| + \int_0^t \Big[(1 + [[q_0(s)]]) \|u(s)\| + \|v(s)\| \Big] ds. \end{aligned}$$

Therefore, we completed the proof. □

Remark 3.4. It should be noted that if $\bar{q} = \operatorname{tr} Q = 0$, then the constant C of (3.4) can be taken independently of T.

We now give the proof of Proposition 3.1.

Proof: [Proof of Proposition 3.1.] We estimate the l-th component of (3.1). To begin with, by the definition of $\mathbf{P}$ the quadratic term reads as follows:

$$(\mathbf{P}(q,\nabla)u)^l = (q,\nabla)u^l + \sum_{m=1}^{n} R_l R_m (q,\nabla)u^m.$$

Note that $R_m \partial_j f = R_j \partial_m f$. Since $\nabla \cdot u = 0$, we now calculate that

$$\begin{aligned}
\sum_m R_l R_m (Qx,\nabla)u^m &= \sum_m R_l R_m \Big(\sum_{i,j} q_{ji} x_i \partial_j \Big) u^m \\
&= \sum_{i,j} q_{ji} \sum_m R_l R_m x_i \partial_j u^m = \sum_{i,j} q_{ji} \sum_m \mathcal{F}^{-1} \left\{ \sqrt{-1} \frac{\xi_l \xi_m}{|\xi|^2} \partial_i \mathcal{F}(\partial_j u^m) \right\} \\
&= \sum_{i,j} q_{ji} \sum_m \mathcal{F}^{-1} \left\{ (-\sqrt{-1}) \Big[\frac{\delta_{il}\xi_m}{|\xi|^2} + \frac{\delta_{im}\xi_l}{|\xi|^2} - \frac{2\xi_m \xi_l \xi_i}{|\xi|^4} - \sqrt{-1} x_i \frac{\xi_l \xi_m}{|\xi|^2} \Big] \mathcal{F}(\partial_j u^m) \right\} \\
&= \sum_{i,j} q_{ji} R_l R_j u^i.
\end{aligned}$$

Then, we have that

$$\partial_t u^l - \Delta u^l + (q,\nabla)u^l + \sum_{i,j} q_{ji} R_l R_j u^i + \sum_m R_l R_m (q_0,\nabla)u^m + v^l = 0.$$

Now we apply Lemma 3.3 and obtain that

$$\begin{aligned}
\|u^l(t)\| \le C \Big[\|u_0^l\| + \int_0^t \Big\{ (1 + [[q_0(s)]]) \|u^l(s)\| \\
+ \| \sum_{i,j} q_{ji} R_l R_j u^i(s) + \sum_m R_l R_m (q_0(s),\nabla)u^m(s) + v^l(s) \| \Big\} ds \Big].
\end{aligned}$$

It remains to estimate the $R_l R_m (q_0,\nabla)u^m$ terms. If $k \le 0$, we may compute the estimates essentially similar as in (3.5):

$$\begin{aligned}
\sum_m \phi_k * \{ R_l R_m (q_0,\nabla)u^m \} &= \sum_m R_l R_m \phi_k * \{ (q_0,\nabla)u^m \} \\
&= \sum_m R_l R_m \Phi_{k,q_0} u^m + \sum_m R_l R_m (q_0,\nabla)u_k^m \\
&= \sum_m R_l R_m \Phi_{k,q_0} u^m + \sum_{j,m} R_l R_j (\partial_m q_{0j}) u_k^m
\end{aligned}$$

for all $l = \{1,\ldots,n\}$. If $k > 0$, in the similar way to derive (3.8) we have

$$\sum_m \phi_k * \{ R_l R_m (q_0,\nabla)u^m \} = \sum_m R_l R_m \tilde{\Phi}_{k,q_0} u^m + \sum_{j,m} R_l R_j (\partial_m q_{0j}) u_k^m.$$

Since the Riesz transform R_i is a bounded operator on $\dot{B}^0_{\infty,1}$ (of course, also on $\dot{\mathcal{B}}^0_{\infty,1}$), and q_0 and u enjoy canonical representations, we deduce that

$$\|R_l R_j(q_0,\nabla)u^m\| \leq C[[q_0]]\ \|u\|$$

with some constant $C > 0$ by using the same arguments in (3.6) and (3.9). Therefore, this completed the proof of Proposition 3.1. □

4. Proofs of theorems.

In this section we shall give the proofs of Theorem 1.1, 1.5 and 1.7. First let us recall some notations of norms: $\|\cdot\| := \|\cdot;\dot{B}^0_{\infty,1}\|$, since we only deal with the homogeneous Besov space with this exponents, and $[[\cdot]] := \sum_{i=0}^{2}\|\nabla^i\cdot\|$. We now prove Theorem 1.1.

Proof: [Proof of Theorem 1.1.] Let $n \geq 2$. Assume that the initial data u_0 belongs to $X^{\mathcal{B}}_\sigma(\mathbf{R}^n)$. If there are two classical solutions u and v of (ABS) on $(0,T)$ in the class of $L^\infty([0,T];X^{\mathcal{B}}_\sigma)$ with initial data given by u_0. Since $(u,\nabla P)$ solves (NS.3), provided that ∇P has a representation of (1.3), we only deal with the solutions of (ABS). We define their difference by $w = u - v$, then w satisfies

$$\partial_t w - \Delta w + \mathbf{P}(w,\nabla)u + \mathbf{P}(v,\nabla)w + \mathbf{P}(Mx,\nabla)w + \mathbf{P}Nw = 0$$

with $w(0) = 0$. We now apply the Proposition 3.1 to obtain

$$\|w(t)\| \leq C\int_0^t (1 + [[u(s)]] + [[v(s)]])\|w(s)\|ds$$

for almost all $t \in [0,T]$. Applying the Gronwall inequality, we can see that $w = 0$. This implies that the solution u is always unique as long as the solution exists in this space. The proof of Theorem 1.1 is now complete. □

We shall give the proof of Theorem 1.5 by using a successive iteration. This method seems to be standard when we construct a solution to (NS.1) with U_0 in L^p for $p \geq n$ (see [22,16,13]), in L^∞ (see [7,8,14]) and in the Besov spaces (see [26,8,1,32]). Since we handle $\dot{\mathcal{B}}^0_{\infty,1}$, we can also get the continuity of approximate sequence in time up to initial time as well as the solution has non-ambiguity of constant.

Proof: [Proof of Theorem 1.5.] Assume that an initial data u_0 belongs to $\dot{\mathcal{B}}^0_{\infty,1}(\mathbf{R}^3)$ with $\nabla\cdot u_0 = 0$. Hereafter, we rather discuss the solution of the integral equation:

$$\bar{u}(t) = e^{t\Delta}u_0 - \int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbf{P}(\bar{u}\otimes\bar{u})(s)ds + a\int_0^t e^{(t-s)\Delta}\mathbf{P}e_3\times\bar{u}(s)ds, \quad (4.1)$$

which is formally equivalent to (NS.4), that is same argument to Section 2. We call the solution of (4.1) *a mild solution of* $(NS.4)$.

We define the successive approximation by starting at $\bar{u}_1(t) := e^{t\Delta}u_0$, and

$$\bar{u}_{j+1}(t) := e^{t\Delta}u_0 - \int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbf{P}(\bar{u}_j\otimes\bar{u}_j)(s)ds - a\int_0^t e^{(t-s)\Delta}\mathbf{P}e_3\times\bar{u}_j(s)ds \quad (4.2)$$

for $j \geq 1$. We shall estimate (4.2) in the $\|\cdot\|$-norm.

Let $T \in (0,1)$ and $K_0 = \|u_0\|$. Define $K_j = K_j(T)$ and $K_j' = K_j'(T)$ by

$$K_j := \sup_{0\leq t\leq T}\|\bar{u}_j(t)\| \qquad \text{and} \qquad K_j' = \sup_{0<t\leq T} t^{1/2}\|\nabla\bar{u}_j(t)\|.$$

Then we have

$$\begin{aligned}
&\|\bar{u}_{j+1}(t)\| \\
&\leq \|e^{t\Delta}u_0\| + \int_0^t \|\nabla\cdot e^{(t-s)\Delta}\mathbf{P}(\bar{u}_j\otimes\bar{u}_j)(s)\|ds + |a|\int_0^t \|e^{(t-s)\Delta}\mathbf{P}e_3\times\bar{u}_j(s)\|ds \\
&\leq \|G_t\|_1\|u_0\| + C\int_0^t \|G_{t-s}\|_1\|\bar{u}_j(s)\|\;\|\nabla\bar{u}_j(s)\|ds + C|a|\int_0^t \|G_{t-s}\|_1\|e_3\times\bar{u}_j(s)\|ds
\end{aligned}$$

by Young's inequality and the boundedness of $\mathbf{P}$ in the $\|\cdot\|$-norm. Taking $\sup_{0\leq t\leq T}$ in both hands, by $\|G_t\|_1 = 1$ and Lemma 1.4 we obtain

$$K_{j+1} \leq K_0 + C_1(T^{1/2}K_jK_j' + |a|TK_j).$$

Similarly, taking ∇ to (4.2) and estimating it in the $\|\cdot\|$-norm, we thus have

$$K_{j+1}' \leq \tilde{C}K_0 + C_2(T^{1/2}K_jK_j' + |a|TK_j).$$

We now take $T_1 \leq 1$ small so that $\max_j 2C_j(1+|a|)T_1^{1/2}(2\tilde{C}K_0+1) < 1$. Therefore,

$$\sup_j K_j(T) \leq 2K_0 \qquad \text{and} \qquad \sup_j K_j'(T) \leq 2\tilde{C}K_0$$

for any $T \leq T_1$. We can get the other properties (for example, the continuity with respect to time, uniqueness and so on) in usual way, then we skip the details. □

Finally, we shall prove the Theorem 1.7. Before giving the proof of Theorem 1.7, we prepare an another estimate (from Lemma 1.4) for bilinear terms.

Lemma 4.1. *There exists a positive constant C such that*

$$\|\nabla\cdot e^{t\Delta}\mathbf{P}(f\otimes f)\| \leq C(1+N+2^{-N}t^{-1/2})\|f\|\;\|\mathrm{rot}\,f\|_\infty + C2^{-N}\|f\|^2$$

for all $t>0$, nonnegative integer N and $f \in C^1$ with $\nabla\cdot f = 0$.

Note that $\nabla\cdot(f\otimes f) = (f,\nabla)f = \mathrm{rot}\,f\times f + \frac{1}{2}\nabla|f|^2$, since $\nabla\cdot f = 0$. This lemma holds in not only 2-D but also higher dimensional cases. The boundedness of $\{\nabla \cdot e^{t\Delta}\mathbf{P}\}$ in L^p for every $p \in [1,\infty]$ has already been obtained by [14]. Its L^p-version (instead of $\dot{B}^0_{\infty,1}$-norm) is also proved by [17], we obtain Lemma 4.1 by modification of theirs. Furthermore, one can find similar estimates in [31,33], then we omit the proof.

Proof: [Proof of Theorem 1.7.] By Theorem 1.5 we have already obtained the locally-in-time mild solution of (NS.4), and also the existence time is estimated by below as Remark 1.6-(i). The basic strategy is same as that in [17]. We shall establish an a priori estimate: there exists a positive constant K (depending only on $|a|$, $\|u_0\|$ and $\|\mathrm{rot}\, u_0\|_\infty$) such that

$$\|\bar{u}(t)\| \le K \exp(Ke^{Kt}) \tag{4.3}$$

for all $t > 0$, which is similar to that of [17, Theorem 2], the details are shown by [31] which is more closed to our situation.

Especially, the uniform bound of the vorticity has an important role to get (4.3). Fortunately, taking rotation in (NS.4), the vorticity equation is same as that of (NS.1):

$$\text{(Vor)} \qquad w_t - \Delta w + (u, \nabla) w = 0,$$

where $w = \mathrm{rot}\, \bar{u}$, since $\mathrm{rot}\,(-\bar{u}^2, \bar{u}^1) = \nabla \cdot \bar{u} = 0$. We can apply the maximal principle for (Vor) to get $\|w(t)\|_\infty \le \|\mathrm{rot}\, u_0\|_\infty$ for $t > 0$. We may suppose $\nabla u_0 \in L^\infty$, because for any $t_0 > 0$ the solution $\nabla \bar{u}(t_0) \in L^\infty$ from its construction, we thus retake the initial time as t_0. Hence, we can derive the a priori estimate. Combining with the uniqueness and Remark 1.6-(i), (4.3) yields that the solution can be extended globally.

Finally, we will derive (4.3). Similarly as the proof of Theorem 1.5, taking the $\|\cdot\|$-norm into (4.1), we thus have

$$\begin{aligned}\|\bar{u}(t)\| &\le \|u_0\| + \int_0^t \|\nabla \cdot e^{(t-s)\Delta}\mathbf{P}(\bar{u} \otimes \bar{u})(s)\| ds + |a| \int_0^t \|\bar{u}(s)\| ds \\ &\le \|u_0\| + C' \int_0^t (t-s)^{-1/2} \|\bar{u}(s)\| ds + C' \int_0^t \|\bar{u}(s)\| \log(\|u(s)\| + 1) ds.\end{aligned}$$

We have used Lemma 4.1 with $N \sim \log(\|\bar{u}(s)\| + 1)$, which setting is similar to [6,17]. Here, the constant $C' = C(1 + \|\mathrm{rot}\, u_0\|_\infty + |a|)$ with numerical constant C given by Lemma 4.1. We now appeal to the Gronwall inequality [17, Lemma 4] to obtain (4.3). Therefore, the proof of Theorem 1.7 is now complete. □

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