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# Multi-parameter compact matrix quantum group with generators of norm one*

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ABSTRACT: Let $R \in gl(n^2)$ be a R-matrix determined by a matrix $A \in gl(n)$ and $A_R$ the corresponding FRT-bialgebra. The paper gives a sufficient condition for the quotient algebra of $A_R$ being a Hopf *-algebra. For a special class of Hopf *-algebra constructed from a Latin square, after being completed, a compact matrix quantum group with generators of norm one is given.

**Key words:** R-matrix, Hopf algebra, compact matrix quantum group

## Contents



## 1. Preliminary

The Yang-Baxter equation $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$ first came up in a paper by Yang [9] as a factorization condition of the scattering S-matrix in the many-body problems in one dimension and in work of Baxter on exactly solvable models in statistical mechanics. It also played an important role in the quantum inverses scattering method created by Faddeev, Sklyanin and Takhtadjian for the construction of quantum integrable systems. Attempts to find solution of Yang-Baxter equation, call it a R-matrix, in a systematic way have led to the theory of braided Hopf algebra, and moreover the theory of quantum group [1] [4]. Based on [1], Woronowicz exhibited C*-algebra structures on compact matrix quantum groups [7] [8]. Since then, the research on Hopf algebra was always going on with C*-algebra. In this paper we will use the method of C*-completeness to construct a compact quantum group with generators of norm one.

Let's review some facts and notations on the FRT bialgebra given by Faddeev, Reshetikhin and Takhtadjian.

**Definition 1.1** ([2] [3]) *Let $R \in End(\mathbb{C}^n \otimes \mathbb{C}^n)$ be a $R-$matrix. The corresponding FRT bialgebra $A_R$ of $R$ is defined as*

$$A_R = \mathbb{C}\langle t_{ij}, e | i, j = 1, 2, \cdots, n\rangle / \mathbb{C}\langle R \cdot T \otimes T - T \otimes T \cdot R\rangle$$

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*where e is the algebra unit and* $T = (t_{ij})_{n\times n}$.

It is well known that $A_R$ is a bialgebra with a product subject to

$$\sum_{k,l=1}^{n} R_{ij,kl} t_{ki'} t_{lj'} = \sum_{k,l=1}^{n} t_{ik} t_{jl} R_{kl,i'j'}, \tag{1.1}$$

and a coproduct $\Delta : A_R \to A_R \otimes A_R$ subject to

$$\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}, \qquad \Delta(e) = e \otimes e. \tag{1.2}$$

Also it has a counit $\varepsilon : A_R \to \mathbb{C}$ subject to

$$\varepsilon(t_{ij}) = \delta_{ij}, \qquad \varepsilon(e) = 1. \tag{1.3}$$

This paper will consider the $R$-matrix mentioned firstly by Manin [5]. For $A = (a_{ij}) \in gl(n)$, set

$$(R_A)_{ij,kl} = a_{ij} \delta_{il} \delta_{jk} \qquad i, j, k, l = 1, 2, \cdots, n.$$

By direct calculation, $R_A$ is a $R$-matrix and the relation (1.1) can be written as

$$a_{ki} t_{ij} t_{kl} = a_{lj} t_{kl} t_{ij}. \tag{1.4}$$

Using such a R-matrix, [6] constructs a new type of Hopf algebra which is neither commutative nor cocommutative. In detail, for $A = (a_{ij})$ satisfying

$$a_{ij} a_{ji} = a_{ii} = 1, \quad i, j = 1, 2, \cdots, n \tag{1.5}$$

$$\prod_{k=1}^{n} \left( \frac{a_{ki}}{a_{kj}} \right) = 1, \quad i, j = 1, 2, \cdots, n \tag{1.6}$$

and for $\sigma \in S_n$, where $S_n$ is the symmetric group on the set $\{1, 2, \cdots, n\}$, set

$$a(\sigma) = \begin{cases} \prod\limits_{\{(i,j)|1\le i<j\le n;\sigma(i)>\sigma(j)\}} a_{\sigma(i)\sigma(j)}, & \sigma \ne id \\ 1 & \sigma = id \end{cases}, \tag{1.7}$$

$$T_{ij} = \sum_{\sigma\in S_n:\sigma(i)=j} sgn(\sigma)\, a(\sigma) \left( \prod_{k=1}^{i-1} \frac{a_{j\sigma(k)}}{a_{ik}} \right) t_{1\sigma(1)} \cdots \widetilde{t_{ij}} \cdots t_{n\sigma(n)}, \tag{1.8}$$

$$T'_{ij} = \sum_{\sigma\in S_n:\sigma(j)=i} sgn(\sigma)\, a(\sigma)^{-1} \left( \prod_{k=1}^{j-1} \frac{a_{jk}}{a_{i\sigma(k)}} \right) t_{\sigma(1)1} \cdots \widetilde{t_{ij}} \cdots t_{\sigma(n)n}, \tag{1.9}$$

and

$$\det = \sum_{\sigma\in S_n} sgn(\sigma)\, a(\sigma)\, t_{1\sigma(1)} t_{2\sigma(2)} \cdots t_{n\sigma(n)},$$

where $\widetilde{t_{ij}}$ denote the element $t_{ij}$ being deleted in the corresponding equation.
**Theorem 1.1** [6] *Let $A_{R_A}$ be the FRT-bialgebra constructed from $R_A$. Then*
1) $T_{ij} = T'_{ij}$, $a_{ki} T_{ij} T_{kl} = a_{lj} T_{kl} T_{ij}$, and

$$\sum_{k=1}^{n} t_{ik} T_{jk} = \delta_{ij} \det .$$

2) *Let*

$$S(A) = A_{R_A} / \mathbb{C} \langle \det - e \rangle \tag{1.10}$$

*be the quotient algebra of $A_{R_A}$, $S(A)$ is a Hopf algebra with antipode $S$ satisfying $S(t_{ij}) = T_{ji}$ and $S(e) = e$. In particular, $S^2 = id$.*

## 2. Construction of *-structure on $S(A)$

**Definition 2.1** *A Hopf *-algebra is a Hopf algebra $H$ equipped with a *-algebra structure so that $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathbb{C}$ are *-homomorphisms, and for all $a \in H$, $S\left(S(a)^*\right)^* = a$.*

This section will give a sufficient condition for $S(A)$ to be a Hopf *-algebra. To do so, we choose specially matrix $A \in gl(n)$ satisfying (1.5), (1.6) and furthermore,

$$|a_{ij}| = 1, \quad i, j = 1, 2, \cdots, n. \tag{2.1}$$

**Lemma 2.1** *Let $S(A)$ be the Hopf algebra defined in Theorem 1.1. Set*

$$t^*_{ij} = T_{ij} = S(t_{ji}). \tag{2.2}$$

*1) $S_A$ can be made into a *-algebra. 2) For $1 \le i, j \le n$, $t_{ij} T_{ij}$ is in the center of $S(A)$.*
**Proof:** Since $a_{ij} a_{ji} = 1$ and $|a_{ij}| = 1$,

$$T_{kl} T_{ij} = \frac{a_{jl}}{a_{ik}} T_{ij} T_{kl} = \overline{\left(\frac{a_{ik}}{a_{jl}}\right)} T_{ij} T_{kl},$$

where the bar means the conjugate number, the *-operation can be extended to a conjugate anti-homomorphism from $A_{R_A}$ onto itself, which is still denoted by *. From the relation (1.8),

$$T^*_{ij} = \sum_{\sigma \in S_n : \sigma(i) = j} sgn(\sigma) \overline{a(\sigma)} \overline{\left( \prod_{k=1}^{i-1} \frac{a_{j\sigma(k)}}{a_{ik}} \right)} T_{n\sigma(n)} \cdots \widetilde{T_{ij}} \cdots T_{1\sigma(1)}.$$

Using Theorem 1.1,

$$T_{ji} = T'_{ji} = \sum_{\{\sigma \in S_n, \sigma(i) = j\}} sgn(\sigma) \overline{a(\sigma)} \overline{\left( \prod_{k=1}^{i-1} \frac{a_{j\sigma(k)}}{a_{ik}} \right)} t_{\sigma(1)1} \ldots \widetilde{t_{ji}} \ldots t_{\sigma(n),n},$$

therefore

$$\begin{aligned} S(T_{ji}) &= \sum_{\sigma\in S_n:\sigma(i)=j} sgn(\sigma)\overline{a\,(\sigma)}\overline{\left(\prod_{k=1}^{i-1}\frac{a_{j\sigma(k)}}{a_{ik}}\right)} S(t_{\sigma(n),n})\ldots\widetilde{S(t_{ji})}\ldots S(t_{\sigma(1),1}) \\ &= \sum_{\sigma\in S_n|\sigma(i)=j} sgn(\sigma)\overline{a\,(\sigma)}\overline{\left(\prod_{k=1}^{i-1}\frac{a_{j\sigma(k)}}{a_{ik}}\right)} T_{n,\sigma(n)}\ldots\widetilde{T_{ji}}\ldots T_{1,\sigma(1)} \\ &= T_{ij}^*. \end{aligned}$$

This implies

$$t_{ij}^{**} = S\,(T_{ji}) = S\,(S\,(t_{ij})) = t_{ij}.$$

Notices that the last equation is from the relation $S^2 = id$. At last,

$$(\det)^* = \sum_{k=1}^{n} T_{ik}^* t_{ik}^* = \sum_{k=1}^{n} t_{ik} t_{ik}^* = \det.$$

Therefore $S\,(A)$ is a *-algebra.

2) Suppose that $1 \le i, j \le n$ and $\sigma \in S_n$ with $\sigma\,(i) = j$. $\forall 1 \le p, q \le n$,

$$\begin{aligned} (t_{1,\sigma(1)}\ldots\widetilde{t_{ij}}\ldots t_{n,\sigma(n)})t_{pq} &= (\prod_{k=1}^{n}\frac{a_{q\sigma(k)}}{a_{pk}})\frac{a_{pi}}{a_{qj}} t_{pq}(t_{1,\sigma(1)}\ldots\widetilde{t_{ij}}\ldots t_{n,\sigma(n)}) \\ &= \frac{a_{pi}}{a_{qj}} t_{pq}(t_{1,\sigma(1)}\ldots\widetilde{t_{ij}}\ldots t_{n,\sigma(n)}). \end{aligned}$$

Thus

$$t_{ij}^* t_{pq} = \frac{a_{pi}}{a_{qj}} t_{pq} t_{ij}^*,$$

and furthermore,

$$\begin{aligned} (t_{ij}t_{ij}^*)t_{pq} &= \frac{a_{pi}}{a_{qj}} t_{ij} t_{pq} t_{ij}^* \\ &= \frac{a_{pi}}{a_{qj}}\frac{a_{qj}}{a_{pi}} t_{pq} t_{ij} t_{ij}^* \\ &= t_{pq}(t_{ij}t_{ij}^*). \end{aligned}$$

Therefore, $t_{ij}t_{ij}^*$, as well as $t_{ij}^* t_{ij}$, is in the center of $S\,(A)$. □

**Theorem 2.2** *$S\,(A)$ is a Hopf *-algebra.*
**Proof:** Since for $1 \le i, j \le n$, $S\left(t_{ij}^*\right) = t_{ji} = S\,(t_{ij})^*$, $S\left(S\,(a)^*\right)^* = a\ (a \in H)$. Thus it suffices to prove the relation $\Delta\left(t_{ij}^*\right) = (\Delta\,(t_{ij}))^*$. Indeed, for the map

$(1,2) : a \otimes b \to b \otimes a,$

$$\begin{aligned} \Delta\left(t_{ij}^*\right) &= \Delta\left(S\left(t_{ji}\right)\right) = (1,2) \circ (S \otimes S) \circ \Delta\left(t_{ji}\right) \\ &= \sum_{k=1}^{n} S\left(t_{ki}\right) \otimes S\left(t_{jk}\right) \\ &= \sum_{k=1}^{n} t_{ik}^* \otimes t_{kj}^* \\ &= \left(\Delta\left(t_{ij}\right)\right)^*. \end{aligned}$$

Thus $S(A)$, equipped with the *-structure, is a Hopf *-algebra. □

## 3. Compact matrix quantum group with generators of norm one

This section will use the Latin square to construct a Hopf *-algebra, such that its C*-completeness is a compact matrix quantum group with generators of norm one.

**Definition 3.1** *Suppose that $G$ is a C*-algebra with unit $I$ and $U = (u_{ij})$ is an $n \times n$ matrix with entries in $G$. $(G, U)$ is called a compact matrix quantum group is the followings are satisfied:*

1) *$G$ is the smallest C*-algebra containing all matrix elements $u_{ij}$ of $U$;*

2) *there exists a *-algebra homomorphism $\rho : G \to G \otimes G$ such that*

$$\rho\left(u_{ij}\right) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj};$$

3) *there exists a linear anti-homomorphism $S : G' \to G'$ such that*

$$S(U)\,U = U S(U) = E,$$

*where $G'$ is the dense *-subalgebra of $G$ generated by all $u_{ij}$ and $E$ is an $n \times n$ matrix with entries $E_{ij} = \delta_{ij} I$.*

Now suppose that $g = (1, n, n-1, \cdots, 2) \in S_n$. Then $g(i) \equiv (i-1) \pmod n$ and $sgn\left(g^i\right) = (-1)^{(i+1)(n+1)}$. Also assume that $A \in gl(n)$ satisfying

$$a_{11} = |a_{1i}| = 1; \tag{3.1}$$

$$a_{1i} \cdot a_{1,n+2-i} = 1; \tag{3.2}$$

$$a_{ij} = a_{1,g^{i-1}(j)} \quad (i \geq 2). \tag{3.3}$$

It is easy to see $a_{ij} = a_{g(i)g(j)}$ for $2 \leq i \leq n$ and each element of the $n$ elements in the first row appears one and only one time in an arbitrary row or in any column. Such a matrix is called a Latin square. Notice that if $n$ is an even number, $a_{1,\frac{n}{2}+1} = \pm 1$. Without loss of generality, one can suppose $a_{1,\frac{n}{2}+1} = -1$.

The matrix defined above has $\left[\frac{n-1}{2}\right]$ parameters and satisfies the relations (1.5),(1.6) and (2.1). Such a matrix do exist and the followings are examples of $n = 4$ and $n = 5$ respectively.

$$\begin{pmatrix} 1 & x & -1 & x^{-1} \\ x^{-1} & 1 & x & -1 \\ -1 & x^{-1} & 1 & x \\ x & -1 & x^{-1} & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & x & y & y^{-1} & x^{-1} \\ x^{-1} & 1 & x & y & y^{-1} \\ y^{-1} & x^{-1} & 1 & x & y \\ y & y^{-1} & x^{-1} & 1 & x \\ x & y & y^{-1} & x^{-1} & 1 \end{pmatrix}$$

where $x, y \in \mathbb{C}$ with $|x| = |y| = 1$.

**Lemma 3.1** *For* $g^{-1} = (1, 2, \cdots, n) \in S_n$, $a(g^{-r}) = (-1)^{(n+1)r}$, $(1 \leq r \leq n-1)$.

**Proof:** Since $a_{ij}a_{ji} = 1$, $\prod_{1 \leq i,j \leq r} a_{ij} = 1$, thus

$$\begin{aligned} a\left(g^{-r}\right) &= \prod_{k>r,\ l\leq r} a_{kl} \\ &= \prod_{k\geq 1,\ l\leq r} a_{kl} \left( \prod_{1\leq i,j\leq r} a_{ij} \right)^{-1} \\ &= \prod_{k\geq 1,\ l\leq r} a_{kl} \\ &= (-1)^{(n+1)r}. \end{aligned}$$

□

Now assume that $A \in gl(n)$ satisfying relations (3.1), (3.2) and (3.3) and that $S(A)$ is the Hopf *-algebra constructed in Theorem 1.1. We will follow [8] to construct a compact matrix quantum group generated by $\{t_{ij} : 1 \leq i, j \leq n\}$.

**Definition 3.2** *A *-representation* $\pi$ *of* $S(A)$ *on a Hilbert space* $H$ *is said admissible if* $\forall i, j = 1, 2, \cdots, k$,

$$\begin{aligned} \pi(\det) &= I, \\ \sum_{r=1}^{k} \pi(t_{ir})\, \pi(t_{jr})^* &= \sum_{r=1}^{k} \pi(T_{ir})^* \, \pi(T_{jr}) = \delta_{ij} I. \end{aligned}$$

For any $x \in S(A)$, set

$$\|x\| = \sup \|\pi(x)\|,$$

where $\pi$ runs over the set of all admissible representations of $S(A)$. It is easy to see $\|\cdot\|$ is a well defined C*-seminorm, and

$$N = \{x \in S(A) | \, \|x\| = 0\}$$

is a two sided ideal of $S(A)$. Such a seminorm can produce a C*-norm on the quotient algebra $S(A)/N$. Let $C(A)$ be the completion of $S(A)/N$ with respect to the C*-norm, then $C(A)$ is a compact matrix quantum group.

**Remark 3.1** *Since $S(A)$ is a neither commutative nor cocommutative Hopf *-algebra, $C(A)$ is neither commutative nor cocommutative too.*
**Theorem 3.2** *As generators of $C(A)$, $\|t_{ij}\| = 1$, $(1 \le i, j \le n)$.*
**Proof:** First, there exist nontrivial admissible representations of $S(A)$. Indeed, set $P = \{t_{ij} : 1 \le i, j \le n\}$ and

$$P_r = \{t_{i,g^{-r}(i)} | i = 1, 2, \ldots, n\}.$$

Then $\{P_r : 1 \le r \le n\}$ is a partition of $P$. That is to say, $\bigcup_{1\le r\le n} P_r = P$ and $P_r \bigcap P_s = \emptyset$ if $r \not= s$.

Let $H$ be a separable Hilbert space. $\forall 1 \le i, j \le n$, set

$$\pi_r(t_{ij}) = \delta_{g^{-r}(i),j} I; \ \pi_r(e) = I.$$

Since $a_{ik} = a_{g^{-r}(i)} a_{g^{-r}(k)}$,

$$\pi_r(t_{ij})\pi_r(t_{kl}) = (a_{lj}/a_{ki})\pi_r(t_{kl})\pi_r(t_{ij}),$$

$\pi_r$ can be extended to an algebra homomorphism, which is still denoted by $\pi_r$, from $S(A)$ to $L(H)$, where $L(H)$ is the algebra of all linear bounded operator on $H$.

$$\begin{aligned}
\pi_r(T_{ij}) &= \sum_{\{\sigma\in S_n|\sigma(i)=j\}} sgn(\sigma)a(\sigma)(\prod_{k=1}^{i-1} a_{j\sigma(k)}/a_{ik})\pi_r(t_{1\sigma(1)})\ldots\widetilde{\pi_r(t_{ij})}\ldots\pi_r(t_{n,\sigma(n)}) \\
&= \delta_{g^{-r}(i),j} sgn(g^{-r})a(g^{-r})(\prod_{k=1}^{i-1} a_{g^{-r}(i)g^{-r}(k)}/a_{ik})I \\
&= \delta_{g^{-r}(i),j} I \\
&= \delta_{g^{-r}(i),j} \pi_r(t_{ij})^*.
\end{aligned}$$

Thus $(\pi_r, H)$ is a *-representation of $S(A)$. By direct calculation, $(\pi_r, H)$ is an admissible representation of $S(A)$ so that for $t_{ij} \in P_r$, $\|\pi_r(t_{ij})\| = 1$. This implies that for $\forall t_{ij} \in P$, there exists an admissible representation $(\rho, H)$ of $S(A)$ with

$$\|\rho(t_{ij})\| = 1.$$

According to the relation:

$$\sum_{r=1}^{n} t_{ik}t_{ik}^* = e,$$

one can know for each admissible representation $(\pi, H)$ of $S(A)$, $\|\pi(t_{ij})\| \le 1$. Therefore, as generators of $S(A)$,

$$\|t_{ij}\| = \sup_\pi \|\pi(t_{ij})\| = 1,$$

where $\pi$ runs over the set of all admissible representations of $S(A)$. □

**Corollary 3.3** *Let $(f, H)$ be a faithful irreducible $C^*$-representation of $C(A)$. Then either $f(t_{ij}) = 0$ or $f(t_{ij})$ is a unitary element in $L(H)$.*

**Proof:** Assume that $f(t_{ij}) \neq 0$, then $f\left(t_{ij}t_{ij}^*\right) = f(t_{ij}) f(t_{ij})^* \neq 0$. Using Theorem 2.1, $f\left(t_{ij}t_{ij}^*\right)$ is in the center of $f(C(A))$. Via Schur's lemma, there exists $C \neq 0$ so that

$$f(t_{ij}) f(t_{ij})^* = CI, \quad f(t_{ij})^* f(t_{ij}) = \overline{C}I.$$

Since $f\left(t_{ij}t_{ij}^*\right)$ is a positive element in $L(H)$, $C > 0$ and in particular, $\overline{C} = C$. This implies that $\frac{f(t_{ij})}{\|f(t_{ij})\|}$ is a unitary. Also, the faithfulness of $f$ implies that $f$ is isometric and thus $\|f(t_{ij})\| = 1$. Therefore $f(t_{ij})$ is a unitary and this completes the proof. □

**Remark 3.2** *Let $(f, H)$ be a faithful irreducible $C^*$-representation of $C(A)$. Since $\sum_{k=1}^{n} t_{ik}t_{ik}^* = e$, in each row of $(t_{ij})_{1\leq i,j\leq n}$, there is only one $t_{ij}$ such that $f(t_{ij})$ is a unitary, and others are all zero.*

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