Bol. Soc. Paran. Mat.
(3s.) **v. 22** 2 (2004): 17–34.
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Indirect linear locally distributed damping of coupled systems

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ABSTRACT: The aim of this paper is to prove indirect internal stabilization results for different coupled systems with linear locally distributed damping (coupled wave equations, wave equations with different speeds of propagation). In our case, a linear local damping term appears only in the first equation whereas no damping term is applied to the second one (this is indirect stabilization, see [11]). Using the piecewise multiplier method we prove that the full system is stabilized and that the total energy of the solution of this system decays polynomially.

Key words: Wave equation, coupled system, piecewise multiplier method, internal stabilization, indirect damping, polynomial decay.

Contents

1. Introduction and main result

Let Ω be a non-empty bounded open set in $I\!R^N$ of class $\mathcal{C}^2$ and $\Gamma = \partial\Omega$ its boundary and let $a \in \mathcal{C}^0(\Omega)$ be a positive function in Ω.

We consider the following system of two coupled wave equations with homogeneous Dirichlet conditions on the boundary :

$$\left\{ \begin{array}{lll} u_1^{''} - \Delta u_1 + a(x)u_1^{'} + \alpha u_2 = 0 & \text{in} & \Omega \times I\!R^+ \\ u_2^{''} - \Delta u_2 + \alpha u_1 = 0 & \text{in} & \Omega \times I\!R^+ \\ u_1 = u_2 = 0 & \text{on} & \partial\Omega \times I\!R^+ \\ (u_1, u_1^{'})(0) = (u_1^0, u_1^1) & \text{in} & \Omega \\ (u_2, u_2^{'})(0) = (u_2^0, u_2^1) & \text{in} & \Omega \end{array} \right. \tag{1.1}$$

In this paper, we only consider a feedback which depends on the velocity in a linear way, i.e. $a(x)u_1^{'}$. One can remark that this local damping term appears only

2000 *Mathematics Subject Classification:* 35B40, 35L70

in the first equation. Our purpose is to prove that the full system is polynomially stable, i.e. that the energy of the solution decays polynomially for sufficiently smooth initial data. We refer to [3] for the proof that this system is not exponentially stable.

The problem of stabilization of the wave equation in a bounded domain using a locally distributed damping has been studied by several authors. When the feedback depends on the velocity in a linear way, Zuazua [14] proved thanks to the multiplier method that the energy decays exponentially when the damping region contains a neighborhood of $\partial\Omega$ or a neighborhood of $\Gamma(x^0) := \{x \in \partial\Omega, (x-x^0).\nu \geq 0\}$, where ν is the outward unit normal to Ω and $x^0 \in I\!\!R^N$. Martinez [12], using the piecewise multiplier method introduced by Liu [11], weakened the usual geometrical conditions on the localization of the damping.

The problem of stabilization of weakly coupled systems has also been studied by several authors. Under certain conditions imposed on the subset where the damping term is effective, Kapitonov [5] proves uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. Alabau and al. [3] studied the indirect internal stabilization of weakly coupled systems where the damping is effective in the whole domain. They prove that the behavior of the first equation is sufficient to stabilize the total system and to have polynomial decay for sufficiently smooth solutions. Alabau [1] proves indirect boundary stabilization (polynomial decay) of weakly coupled equations. She establishes a polynomial decay lemma for non-increasing and nonnegative function which satisfies an integral inequality.

Our purpose in this paper is to study the indirect internal stabilization of coupled systems with a local damping term applied only to the first equation and to prove that the full system is polynomially stabilized. We therefore generalize the result of [3] to the case of a non-coercive feedback operator in the case of wave equations.

We denote by $\mathcal{A}$ the unbounded operator in the energy space :
$\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ defined by :
$\mathcal{D}(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2$ and
$\mathcal{AU} = (-u_3, -u_4, -\Delta u_1 + \alpha u_2 + a(x)u_3, -\Delta u_2 + \alpha u_1)^T$ and $\mathcal{U} = (u_1, u_2, u_3, u_4)^T$.

The problem (1.1) can then be reformulated under the abstract form

$$\mathcal{U}' + \mathcal{AU} = 0$$

The existence and the regularity of the solution of (1.1) is given by the following theorem.

Theorem 1 *For all* $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{H}$, *the system* (1.1) *has a unique solution* U *such that* $U \in \mathcal{C}(I\!\!R^+, \mathcal{H})$.

Moreover if $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}^k)$ *for* $k \in I\!\!N^*$, *then the solution* U *is more regular and satisfies* $U \in \mathcal{C}^{k-j}(I\!\!R^+, \mathcal{D}(\mathcal{A}^j))$ *for* $j = 0, ..., k$.

Proof : To establish the well-posedness of our problem, we prove that $\mathcal{A}$ is a maximal monotone operator in the energy space i.e. $\mathcal{A}$ is monotone and $I + \mathcal{A}$ is onto. That is why we choose for $U = (u_1, u_2, u_3, u_4)^T$ and $V = (v_1, v_2, v_3, v_4)^T$ in $\mathcal{H}$ the appropriate scalar product on $\mathcal{H}$:

$$< U, V >_{\mathcal{H}} = \int_\Omega [\nabla u_1.\nabla v_1 + \nabla u_2.\nabla v_2 + \alpha(u_1 v_2 + u_2 v_1)]\ dx + \int_\Omega [u_3 v_3 + u_4 v_4]\ dx$$

The proof is standard and thus is left to the reader.

We define the partial energies :

$$E_i(u_i(t)) = \frac{1}{2}\int_\Omega (|u_i^{'}|^2 + |\nabla u_i|^2)\ dx \qquad i = 1, 2$$

and the full energy :

$$E(U(t)) = E_1(u_1(t)) + E_2(u_2(t)) + \alpha \int_\Omega u_1 u_2\ dx$$

associated to a solution $U = (u_1, u_2, u_1^{'}, u_2^{'})$ of (1.1).

Let us denote by C generic positive constants which do not depend on the initial data. α is supposed to be here a sufficiently small positive number. However the results are valid for sufficiently small negative α as well.

We make the following geometric assumptions. These assumptions have been introduced in [11] for the piecewise multiplier method (see also [12]).

• Let ω be an open subset of Ω.
We denote by ν the outward unit normal vector to its boundary.
If $\Omega_j \subset I\!R^N$ is a lipschitz domain, then we denote by ν_j the outward unit normal vector to its boundary $\partial\Omega_j$.

• Assumptions on the local damping :
$a \in \mathcal{C}^0(\Omega)$ and there exists a constant γ which satisfies : $\forall\ x \in \omega, \quad a(x) \geq \gamma > 0$
H_1

Geometric assumptions for Ω and ω :
We assume that there exist $\epsilon > 0$, domains $\Omega_j \subset \Omega$, $1 \leq j \leq J$ with a lipschitz boundary $\partial\Omega_j$ and points $x_j \in I\!R^N$ so that :

$$\begin{gathered} \Omega_i \cap \Omega_j = \emptyset \text{ if } i \neq j \\ \Omega \cap \mathcal{N}_\epsilon \left[\cup_j \Gamma_j(x_j) \cup (\Omega \ \setminus \cup_j \Omega_j)\right] \subset \omega \\ \text{with } \mathcal{N}_\epsilon(\theta) = \left\{x \in I\!R^N : inf_{y\in\theta}|x - y| < \epsilon\right\} \text{ where} \\ \theta \subset I\!R^N,\ \Gamma_j(x_j) = \{x \in \partial\Omega_j : (x - x_j).\nu_j(x) > 0\} \end{gathered}$$

We obtain then our main result :

Theorem 2 *For every initial data $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A})$, the full energy of the solution U of system* (1.1) *decays polynomially, i.e.*

$$\forall\ t > 0, \qquad E(U(t)) \leq \frac{C}{t}(E(U(0)) + E(U^{'}(0)))$$

Moreover if the initial data are more regular, i.e. $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}^n)$ *for a certain positive integer* n*, then the following inequality holds :*

$$\forall\ t > 0, \qquad E(U(t)) \leq \frac{C}{t^n} \sum_{p=0}^{p=n} E(U^p(0))$$

2. Proof of the main result

We first verify that (1.1) is a dissipative problem :

Lemma 2.1 *For every initial data in* $\mathcal{H}$*, the energy of the corresponding solution* U *of system (1.1) is dissipative, i.e.*

$$\forall\ 0 \leq S \leq T < +\infty, \qquad E(U(S)) - E(U(T)) = \int_S^T \int_\Omega a(x)|u_1^{'}|^2\ dx\ dt$$

center

Proof : This is a well-known result. Let $U^0 \in \mathcal{D}(\mathcal{A})$ and denote by U the solution of (1.1). Differentiating $E_1(u_1(t))$ and $E_2(u_2(t))$ and using the first and second equations of (1.1), we have :

$$E_1^{'}(u_1(t)) = -\int_\Omega u_1^{'}(a(x)u_1^{'} + \alpha u_2)\ dx \qquad \text{and} \qquad E_2^{'}(u_2(t)) = -\int_\Omega u_2^{'}\alpha u_1\ dx$$

which leads to : $\qquad E^{'}(U(t)) = -\displaystyle\int_\Omega a(x)|u_1^{'}|^2\ dx \leq 0$ for $a(x) > 0$.

Then integrating on $[S;T]$, we obtain :

$$E(U(S)) - E(U(T)) = \int_S^T \int_\Omega a(x)|u_1^{'}|^2 dx\ dt.$$

By density of $\mathcal{D}(\mathcal{A})$ in $\mathcal{H}$, this result holds for every $U^0 \in \mathcal{H}$.

One can prove the following elementary inequality between $E_1(u_1, t)$, $E_2(u_2, t)$ and $E(U, t)$ for α sufficiently small. Its proof is left to the reader :

Lemma 2.2 *For every initial data in* $\mathcal{H}$ *and for every* U *solution of the system* (1.1)*, we have the following estimates, provided that* α *is sufficiently small*

$$\exists\ c_0,\ c_1 > 0, \qquad c_0 E(U(t)) \leq E_1(u_1(t)) + E_2(u_2(t)) \leq c_1 E(U(t))$$

To prove Theorem 2, we first estimate $\displaystyle\int_S^T E_1(u_1(t))\ dt$ using piecewise multiplier techniques to obtain Proposition 1. Then we estimate easily $\displaystyle\int_S^T E_2(u_2(t))\ dt$. Finally, summing these two estimations, we use the result of Alabau [1] to conclude.

Proposition 2.1 *Assume the hypothesis H_1. Then there exists a constant $C > 0$, such that for all $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1)$ in $\mathcal{D}(\mathcal{A})$, the solution U of system* (1.1) *satisfies :*

$$\begin{aligned}\int_S^T E_1(u_1(t))\,dt \quad &\leq \quad C(E_1(u_1(S)) + E_1(u_1(T))) + C\int_S^T \int_\omega u_1^{'2}\,dx\,dt \quad (2.1)\\ &+C\int_S^T \int_\Omega a^2(x)u_1^{'2}\,dx\,dt + C\alpha\int_S^T \int_\Omega u_2^2\,dx\,dt.\end{aligned}$$

We need several intermediate steps to prove this estimate. The first proposition is based on the use of the piecewise multiplier method for a single wave locally damped equation.

The multiplier method was introduced by K. Liu [11] for a single wave damped equation when the damping term is locally distributed. P. Martinez [12] weakens the usual geometrical conditions on the localization of the damping.

2.1. STEP 1. We first prove an intermediate estimation concerning the partial energy $E_1(u_1(t))$:

Proposition 2.2 *Suppose H_1 and set $Q_i = \mathcal{N}_{\epsilon_i}[\cup_j\Gamma_j(x_j) \cup (\Omega \setminus \cup_j\Omega_j)]$ $\quad i = \{0,1,2\}$ $\quad$ with $\quad 0 < \epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon$. Then the following inequality holds*

$$\begin{aligned}\int_S^T E_1(u_1(t))\,dt \quad &\leq \quad C\left[E_1(u_1(S)) + E_1(u_1(T))\right] + C\int_S^T \int_\Omega a^2(x)u_1^{'2}\,dx\,dt \quad (2.2)\\ &+C\int_S^T \int_{\Omega\cap Q_1} (u_1^{'2} + |\nabla u_1|^2)\,dx\,dt + C\alpha\int_S^T \int_\Omega |u_2|^2\,dx\,dt.\end{aligned}$$

Proof : Let Θ be an open subset of Ω and h a vector field of class $\mathcal{C}^1$ from Θ to $I\!R^N$. We set $M(u_1) = 2h.\nabla u_1 + (N-1)u_1$. As usual, we use the multiplier $M(u_1)$ in the first equation (1.1) and integrate on $[S;T] \times \Theta$.

In the different following estimates, we omit to write the differential elements to simplify the expressions.

- To evaluate $\displaystyle\int_S^T \int_\Theta (N-1)u_1(u_1^{''} - \Delta u_1 + a(x)u_1^{'} + \alpha u_2) = 0$, we integrate by parts to obtain :

$$\begin{aligned}&(N-1)\left[\int_\Theta u_1u_1^{'}\right]_S^T - (N-1)\int_S^T \int_\Theta u_1^{'2} + (N-1)\int_S^T \int_\Theta |\nabla u_1|^2 \quad (2.3)\\ &+(N-1)\int_S^T \int_\Theta u_1[a(x)u_1^{'} + \alpha u_2] = (N-1)\int_S^T \int_{\partial\Theta} \partial_\nu u_1\, u_1\end{aligned}$$

- Then we evaluate $\displaystyle\int_S^T \int_\Theta 2h \cdot \nabla u_1(u_1^{''} - \Delta u_1 + a(x)u_1^{'} + \alpha u_2) = 0$ and have

$\displaystyle\int_S^T\int_\Theta 2h\cdot\nabla u_1 u_1^{''} - \int_S^T\int_\Theta (2h\cdot\nabla u_1)\Delta u_1 + \int_S^T\int_\Theta 2h\cdot\nabla u_1 a(x) u_1^{'} + \alpha\int_S^T\int_\Theta 2h\cdot$
$\nabla u_1 u_2 = 0$

Hence after integrating by parts

$$
\begin{aligned}
0 \quad = \quad & \left[\int_\Theta 2u_1^{'} h\cdot\nabla u_1\right]_S^T - \int_S^T\int_\Theta 2u_1^{'} h\cdot\nabla u_1^{'} - \int_S^T\int_{\partial\Theta} 2\partial_\nu u_1 h\cdot\nabla u_1 \\
& + \int_S^T\int_\Theta \nabla u_1\cdot\nabla(2h\cdot\nabla u_1) + \int_S^T\int_\Theta 2h\cdot\nabla u_1 a(x) u_1^{'} + \alpha\int_S^T\int_\Theta 2h\cdot\nabla u_1 u_2
\end{aligned}
$$

Remark that $\displaystyle\int_S^T\int_\Theta 2u_1^{'} h\cdot\nabla u_1^{'} = \int_S^T\int_{\partial\Theta} h\cdot\nu u_1^{'2} - \int_S^T\int_\Theta (divh)\ u_1^{'2}$ and

$$\nabla u_1\cdot\nabla(2h\cdot\nabla u_1) = 2\sum_{i,k}\partial_k u_1\left[(\partial_k h_i)\partial_i u_1 + h_i\partial_k\partial_i u_1\right] = 2\sum_{i,k}\partial_k u_1\partial_i u_1\partial_k h_i + h\cdot\nabla(|\nabla u_1^2|)$$

Considering all the boundary integrals on the left-hand side, we deduce the following equality :

$$
\begin{aligned}
& \int_S^T\int_{\partial\Theta} 2\partial_\nu u_1 h\cdot\nabla u_1 + (h\cdot\nu)(u_1^{'2} - |\nabla u_1|^2) \\
= \quad & \left[\int_\Theta 2u_1^{'} h\cdot\nabla u_1\right]_S^T + \int_S^T\int_\Theta divh\ (u_1^{'2} - |\nabla u_1|^2) + 2\int_S^T\int_\Theta\sum_{i,k}\partial_i h_k\partial_i u_1\partial_k u_1 \\
& +2\int_S^T\int_\Theta h\cdot\nabla u_1\ a(x) u_1^{'} + \alpha\int_S^T\int_\Theta 2h\cdot\nabla u_1\ u_2
\end{aligned}
$$

The main problem is to estimate the boundary terms of previous equality. Usually, we choose $\Theta = \Omega$ and an adequate vector field h so that the estimate of the boundary integral is possible using the boundary condition : it is easy on the part of the boundary where $\{u = 0\}\cap\{m.\nu \leq 0\}$ and on the other part, we choose $h = 0$.
In our case, we estimate the boundary integral on each subset Ω_j. We use the vector field h_j as introduced in Martinez in [12].

We choose $\Theta = \Omega_j$ and $h_j(x) = \begin{cases}\Psi_j(x)m_j(x) & \text{if } x\in\Omega_j \\ 0 & \text{if } x\ \in\Omega\setminus\cup_j\Omega_j\end{cases}$ where the function Ψ_j is defined as follows : Since $\overline{\Omega_j\setminus Q_1}\cap\overline{Q_0} = \emptyset$, there exists a function $\Psi_j\in\mathcal{C}_0^\infty(I\!R^N)$ verifying :

$$
\begin{cases}
0\leq\Psi_j\leq 1 & \\
\Psi_j = 1 & \text{on } \overline{\Omega_j}\setminus Q_1 \\
\Psi_j = 0 & \text{on } Q_0
\end{cases}
$$

Hence :

$$
\begin{aligned}
&\int_S^T \int_{\partial\Omega_j} 2\partial_\nu u_1 \Psi_j m_j \cdot \nabla u_1 + (\Psi_j m_j \cdot \nu)(u_1^{'2} - |\nabla u_1|^2) \\
= \quad & \left[\int_{\Omega_j} 2u_1^{'} \Psi_j m_j \cdot \nabla u_1 \right]_S^T + \int_S^T \int_{\Omega_j} div(\Psi_j m_j)\ (u_1^{'2} - |\nabla u_1|^2) \qquad (2.4) \\
&+2 \int_S^T \int_{\Omega_j} \sum_{i,k} \partial_i (\Psi_j m_j)_k \partial_i u_1 \partial_k u_1 + 2 \int_S^T \int_{\Omega_j} \Psi_j m_j \cdot \nabla u_1\ a(x) u_1^{'} \\
&+\alpha \int_S^T \int_{\Omega_j} 2\Psi_j m_j \cdot \nabla u_1\ u_2
\end{aligned}
$$

First we show that the boundary integral in the above expression is negative. We remark that by construction: $\Psi_j = 0$ outside $((\partial\Omega_j \setminus \Gamma_j(x_j)) \cap \partial\Omega)$ and $u_1 = 0$ on $((\partial\Omega_j \setminus \Gamma_j(x_j)) \cap \partial\Omega)$.

We deduce that the boundary integral term in (2.4) is equal to:

$$
\begin{aligned}
&\int_S^T \int_{((\partial\Omega_j \setminus \Gamma_j(x_j)) \cap \partial\Omega)} 2\partial_\nu u_1 \Psi_j m_j . \nabla u_1 + (\Psi_j m_j . \nu)(u_1^{'2} - |\nabla u_1|^2) \\
= \quad & \int_S^T \int_{((\partial\Omega_j \setminus \Gamma_j(x_j)) \cap \partial\Omega)} \Psi_j (m_j . \nu_j)(\partial \nu_j u_1)^2
\end{aligned}
$$

By definition of $\Gamma_j(x_j)$, we deduce that

$$
\int_S^T \int_{((\partial\Omega_j \setminus \Gamma_j(x_j)) \cap \partial\Omega)} \Psi_j (m_j . \nu_j)(\partial \nu_j u_1)^2 \leq 0.
$$

Hence, since $\Psi_j = 0$ on Q_0, using the previous inequality in (4) and summing the resulting inequalities on j, we obtain :

$$
\begin{aligned}
&\sum_j \left[\int_{\Omega_j} 2u_1^{'} \Psi_j m_j . \nabla u_1 \right]_S^T + \sum_j \int_S^T \int_{\Omega_j \setminus Q_1} div(\Psi_j m_j)\ (u_1^{'2} - |\nabla u_1|^2) \\
&+ \sum_j \left(\int_S^T \int_{\Omega_j \setminus Q_0} 2\Psi_j m_j \cdot \nabla u_1\ a(x) u_1^{'} + \alpha \int_S^T \int_{\Omega_j \setminus Q_0} 2\Psi_j m_j \cdot \nabla u_1\ u_2 \right) \\
\leq \quad & - \sum_j \left[\int_S^T \int_{\Omega_j \cap Q_1} div(\Psi_j m_j)\ (u_1^{'2} - |\nabla u_1|^2) + \int_S^T \int_{\Omega_j \cap Q_1} \sum_{i,k} 2\partial_i (\Psi_j m_j)_k \partial_i u_1 \partial_k u_1 \right] \\
& - \sum_j \int_S^T \int_{\Omega_j \setminus Q_1} 2 \sum_{i,k} \partial_i (\Psi_j m_j)_k \partial_i u_1 \partial_k u_1
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \sum_j \left[\int_{\Omega_j} 2u_1^{'}\Psi_j m_j \cdot \nabla u_1\right]_S^T + \sum_j \int_S^T \int_{\Omega_j\setminus Q_1} div(\Psi_j m_j)(u_1^{'2} - |\nabla u_1|^2) \\
& +2\sum_j \int_S^T \int_{\Omega_j\setminus Q_1} \partial_i(\Psi_j m_j)_k \partial_i u_1 \partial_k u_1 \\
& +\sum_j \left[\int_S^T \int_{\Omega_j\setminus Q_0} 2\Psi_j m_j \cdot \nabla u_1 \; a(x)u_1^{'} + \alpha \int_S^T \int_{\Omega_j\setminus Q_0} 2\Psi_j m_j \cdot \nabla u_1 \; u_2\right] \\
\leq \quad & -\sum_j \left[\int_S^T \int_{\Omega_j\cap Q_1} div(\Psi_j m_j)(u_1^{'2} - |\nabla u_1|^2) + \int_S^T \int_{\Omega_j\cap Q_1} \sum_{i,k} 2\partial_i(\Psi_j m_j)_k \partial_i u_1 \partial_k u_1\right] \\
\leq \quad & C\sum_j \int_S^T \int_{\Omega_j\cap Q_1} u_1^{'2} - |\nabla u_1|^2 + 2|\nabla u_1|^2 \\
= \quad & C\sum_j \int_S^T \int_{\Omega_j\cap Q_1} u_1^{'2} + |\nabla u_1|^2 \\
= \quad & C\int_S^T \int_{\Omega\cap Q_1} u_1^{'2} + |\nabla u_1|^2
\end{aligned}
$$

Using now the definition of h,

$$
\begin{aligned}
& \left[\int_{\Omega} 2u_1^{'} h \cdot \nabla u_1\right]_S^T + \int_S^T \int_{\Omega\setminus Q_1} N u_1^{'2} + (2-N)|\nabla u_1|^2 \\
& + \int_S^T \int_{\Omega} 2\; a(x)u_1^{'} h \cdot \nabla u_1 + \alpha \int_S^T \int_{\Omega} 2h \cdot \nabla u_1 \; u_2 \\
\leq \quad & C\int_S^T \int_{\Omega\cap Q_1} u_1^{'2} + |\nabla u_1|^2
\end{aligned}
\tag{2.5}
$$

Computing (2.3) with $\Theta = \Omega$ and (2.5), we have :

$$
\begin{aligned}
& \left[\int_{\Omega} M(u_1)u_1^{'}\right]_S^T + \int_S^T \int_{\Omega} M(u_1)a(x)u_1^{'} \\
& + \int_S^T \int_{\Omega\setminus Q_1} (-(N-1)+N)u_1^{'2} + ((N-1)+(2-N))|\nabla u_1|^2 \\
& +(N-1)\int_S^T \int_{\Omega\cap Q_1} |\nabla u_1|^2 - u_1^{'2} + \alpha \int_S^T \int_{\Omega} M(u_1)\; u_2 \\
\leq \quad & C\int_S^T \int_{\Omega\cap Q_1} u_1^{'2} + |\nabla u_1|^2
\end{aligned}
$$

Using then the definition of partial energy $E_1(u_1)$, we obtain the following inequality :

$$\begin{aligned}2\int_S^T E_1(u_1(t))\,dt &\le (C+1)\int_S^T\int_{\Omega\cap Q_1}(u_1^{'2}+|\nabla u_1|^2)\,dx\,dt-\left[\int_\Omega M(u_1)u_1'\,dt\right]_S^T\\ &-\int_S^T\int_\Omega M(u_1)a(x)u_1'\,dx\,dt-\alpha\int_S^T\int_\Omega M(u_1)\;u_2\,dx\,dt \qquad (2.6)\end{aligned}$$

We now estimate the right-hand side terms as follows :

◇ It is easy to verify that $||M(u_1)||_{L^2(\Omega)}\le ||2h\cdot\nabla u_1||_{L^2(\Omega)}$ and thus we have : $\left|\int_\Omega M(u_1)u_1'\,dx\right|\le CE_1(u_1(t))$ where C is a positive constant. We can then replace the second term of inequality (2.6) by $C[E_1(u_1(S))+E_1(u_1(T))]$.

◇ We estimate now the third term of (2.6) as follows :

$$\begin{aligned}\left|\int_S^T\int_\Omega M(u_1)a(x)u_1'\,dx\,dt\right| &\le \int_S^T\int_\Omega \frac{\tilde\delta}{2}M^2(u_1)\,dx\,dt+\frac{1}{2\tilde\delta}\int_S^T\int_\Omega (a(x)u_1')^2\,dx\,dt\\ &\le \delta\int_S^T E_1(u_1(t))\,dt+\frac{C}{\delta}\int_S^T\int_\Omega a^2(x)u_1^{'2}\,dx\,dt\end{aligned}$$

where C is a positive constant and α is supposed a sufficiently small positive number.

◇ Using the same argument, we estimate the last term of (2.6) by

$$\alpha\delta\int_S^T E_1(u_1(t))\,dt+C\alpha\int_S^T\int_\Omega |u_2|^2\,dx\,dt$$

Choosing now α and δ small enough, we conclude the proof of Proposition 2.

2.2. STEP 2. In this section, we want to get rid of $|\nabla u_1|^2$ in the estimate of Proposition 2.

Lemma 2.3 *There exists a positive constant C such that, for all $\delta>0$:*

$$\begin{aligned}\int_S^T\int_{\Omega\cap Q_1}|\nabla u_1|^2\,dx\,dt &\le C\,[E_1(u_1(S))+E_1(u_1(T))]+C\alpha\int_S^T\int_{\Omega\cap Q_2}|u_2|^2\,dx\,dt\\ &+C\int_S^T\int_{\Omega\cap Q_2}(u_1^{'2}+a^2(x)u_1^{'2}+u_1^2)\,dx\,dt\end{aligned}$$

Proof : Since $\overline{I\!R^N\setminus Q_2}\cap\overline{Q_1}=\emptyset$, there exists a function $\xi\in\mathcal{C}_0^\infty(I\!R^N)$ such that :

$$\begin{cases}0\le\xi\le 1 & \\ \xi=1 & \text{on}\quad Q_1\\ \xi=0 & \text{on}\quad I\!R^N\setminus Q_2\end{cases}$$

Multiplying the first equation of the initial system by ξu_1 and applying the Green's formula, we get :

$$\begin{aligned}\int_S^T\int_\Omega \nabla u_1\cdot\nabla(\xi u_1)\,dx\,dt &= -\left[\int_\Omega \xi u_1 u_1^{'}\,dx\right]_S^T+\int_S^T\int_\Omega \xi u_1^{'2}\,dx\,dt\\ &\quad+\int_S^T\int_\Omega \xi u_1 a(x)u_1^{'}\,dx\,dt-\alpha\int_S^T\int_\Omega u_2\xi u_1\,dx\,dt\end{aligned}$$

Note that $\displaystyle\int_S^T\int_\Omega \nabla u_1\cdot\nabla(\xi u_1)\,dx\,dt=\int_S^T\int_\Omega\left(\xi|\nabla u_1|^2-\frac{1}{2}u_1^2\Delta\xi\right)\,dx\,dt$ and so we have

$$\begin{aligned}\int_S^T\int_\Omega \xi|\nabla u_1|^2\,dx\,dt &= -\left[\int_\Omega \xi u_1 u_1^{'}\,dx\right]_S^T-\alpha\int_S^T\int_\Omega u_2\xi u_1\,dx\,dt\\ &\quad-\int_S^T\int_\Omega \xi u_1 a(x)u_1^{'}\,dx\,dt+\int_S^T\int_\Omega \xi u_1^{'2}\,dx\,dt\\ &\quad+\frac{1}{2}\int_S^T\int_\Omega u_1^2\Delta\xi\,dx\,dt\end{aligned}$$

Finally, using the definition of the function ξ, we obtain the following inequality :

$$\begin{aligned}\int_S^T\int_{\Omega\cap Q_1}|\nabla u_1|^2\,dx\,dt &\leq -\left[\int_\Omega \xi u_1 u_1^{'}\,dx\right]_S^T-\alpha\int_S^T\int_\Omega u_2\xi u_1\,dx\,dt \qquad (2.7)\\ &\quad+\int_S^T\int_\Omega\left(-\xi u_1 a(x)u_1^{'}+\xi u_1^{'2}+\frac{1}{2}u_1^2\Delta\xi\right)\,dx\,dt\end{aligned}$$

We need to estimate every term of the right-hand side of this inequality to prove Lemma 3.

◇ The first term of (2.7) can be easily estimated as follows :

$$\left|\left[\int_\Omega \xi u_1 u_1^{'}\right]_S^T\,dx\right|\leq C[E_1(u_1(S))+E_1(u_1(T))]$$

◇ For the second term of (2.7), using the definition of ξ, we obtain :

$$\begin{aligned}-\alpha\int_S^T\int_\Omega u_2\xi u_1\,dx\,dt &\leq \alpha\int_S^T\int_\Omega \xi\left(\frac{u_1^2}{2}+\frac{u_2^2}{2}\right)\,dx\,dt\\ &\leq C\int_S^T\int_{\Omega\cap Q_2}u_1^2\,dx\,dt+\alpha C\int_S^T\int_{\Omega\cap Q_2}u_2^2\,dx\,dt\end{aligned}$$

$\diamond$ For the last term :

$$\begin{aligned}
& \int_S^T \int_\Omega \left(-\xi u_1 a(x) u_1' + \xi u_1'^2 + \frac{1}{2} u_1^2 \Delta \xi \right) \, dx \, dt \\
= & \int_S^T \int_{\Omega \cap Q_2} \left(-\xi u_1 a(x) u_1' + \xi u_1'^2 + \frac{1}{2} u_1^2 \Delta \xi \right) \, dx \, dt \\
\leq & \; C \int_S^T \int_{\Omega \cap Q_2} \left(u_1^2 + (a(x) u_1')^2 + u_1'^2 \right) \, dx \, dt
\end{aligned}$$

Hence, using these estimates in (2.7), we get the inequality announced in Lemma 3.

2.3. STEP 3. Now we want to get rid of the new term $\int_S^T \int_{\Omega \cap Q_2} u_1^2 \, dx \, dt$ introduced in Lemma 3.

Lemma 2.4 *There exists a positive constant C such that, for all $\eta > 0$:*

$$\begin{aligned}
\int_S^T \int_{\Omega \cap Q_2} u_1^2 \, dx \, dt \quad \leq \quad & C \left[E_1(u_1(S)) + E_1(u_1(T)) \right] + C\eta \int_S^T E_1(u_1(t)) \, dt \\
& + \frac{C}{\eta} \int_S^T \int_\Omega (a(x) u_1')^2 \, dx \, dt + \frac{C}{\eta} \int_S^T \int_\omega u_1'^2 \, dx \, dt \\
& + \frac{C\alpha}{2} \int_S^T \int_\Omega u_2^2 \, dx \, dt
\end{aligned}$$

Proof : Since $\overline{I\!R^N \setminus \omega} \cap \overline{Q_2} = \emptyset$, there exists a function $\beta \in \mathcal{C}_0^\infty(I\!R^N)$ such that :

$$\begin{cases} 0 \leq \beta \leq 1 & \\ \beta = 1 & \text{on} \quad Q_2 \\ \beta = 0 & \text{on} \quad I\!R^N \setminus \omega \end{cases}$$

Multiplying the first equation of initial system (1.1) by z, we have :

$$\int_S^T \int_\Omega z u_1'' \, dx \, dt - \int_S^T \int_\Omega z \Delta u_1 \, dx \, dt + \int_S^T \int_\Omega a(x) u_1' z \, dx \, dt + \int_S^T \int_\Omega \alpha z u_2 \, dx \, dt = 0$$

Then, using the boundary conditions and the system verified by z, we have :

$$\begin{aligned}
\int_S^T \int_\Omega \beta u_1^2 \, dx \, dt \quad = \quad & \left[\int_\Omega z u_1' \, dx \right]_S^T - \int_S^T \int_\Omega z' u_1' \, dx \, dt \\
& + \int_S^T \int_\Omega a(x) u_1' z \, dx \, dt + \alpha \int_S^T \int_\Omega z u_2 \, dx \, dt
\end{aligned} \tag{2.8}$$

We first give some well-known results which will be used to estimate the different right-hand side terms of inequality (2.8).

For each t, we consider the solution z of the following elliptic problem :

$$\begin{cases} \Delta z = \beta(x) u_1 & \text{in} \quad \Omega \\ z = 0 & \text{on} \quad \partial\Omega \end{cases}$$

Hence using the Green's formula, we have :

$$\begin{aligned} \int_\Omega |\nabla z|^2 \, dx &= -\int_\Omega \beta u_1 z \, dx \\ &\leq C \left(\int_\Omega |u_1|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |z|^2 \, dx \right)^{\frac{1}{2}} \end{aligned}$$

Then using Poincaré's inequality, we have : $\left(\int_\Omega |z|^2 \, dx \right)^{\frac{1}{2}} \leq C \left(\int_\Omega |u_1|^2 \, dx \right)^{\frac{1}{2}}$.

In a similar way with the derived system, we obtain : $\int_\Omega |z^{'}|^2 \, dx \leq C \int_\Omega \beta u_1^{'2} \, dx$.

We are now ready to estimate the different right-hand side terms of inequality (2.8) as follows.

Thanks to $\left(\int_\Omega |z|^2 \, dx \right)^{\frac{1}{2}} \leq C \left(\int_\Omega |u_1|^2 \, dx \right)^{\frac{1}{2}}$, we get :

$$\left| \left[\int_\Omega z u_1^{'} \, dx \right]_S^T \right| \leq c \left(E_1(u_1(T)) + E_1(u_1(S)) \right)$$

Thanks to $\int_\Omega |z^{'}|^2 \, dx \leq C \int_\Omega \beta u_1^{'2} \, dx$ and Young's inequality, we have for every $\eta > 0$:

$$\begin{aligned} \left| \int_S^T \int_\Omega z^{'} u_1^{'} \, dx \, dt \right| &\leq \frac{C}{2\eta} \int_S^T \int_\Omega \beta u_1^{'2} \, dx \, dt + \frac{C\eta}{2} \int_S^T \int_\Omega u_1^{'2} \, dx \, dt \\ &\leq \frac{C}{\eta} \int_S^T \int_\omega u_1^{'2} \, dx \, dt + \eta C \int_S^T \int_\Omega u_1^{'2} \, dx \, dt \\ &\leq \frac{C}{\eta} \int_S^T \int_\omega u_1^{'2} \, dx \, dt + \eta C \int_S^T E_1(u_1(t)) \, dt \end{aligned}$$

For the third term of (2.8), we have :

$$\begin{aligned} \int_S^T \int_\Omega a(x) z u_1^{'} \, dx \, dt &\leq \frac{C}{\eta} \int_S^T \int_\Omega (a(x) u_1^{'})^2 \, dx \, dt + \frac{\eta}{2} \int_S^T \int_\Omega z^2 \, dx \, dt \\ &\leq \frac{C}{\eta} \int_S^T \int_\Omega (a(x) u_1^{'})^2 \, dx \, dt + C\eta \int_S^T E_1(u_1(t)) \, dt \end{aligned}$$

At last using Poincaré's inequality, we have :

$$\begin{aligned}\int_S^T\int_\Omega zu_2\,dx\,dt &\le \frac{\alpha C}{2}\int_S^T\int_\Omega u_1^2\,dx\,dt+\frac{\alpha}{2}\int_S^T\int_\Omega u_2^2\,dx\,dt\\ &\quad \frac{\alpha}{2}\int_S^T\int_\Omega u_2^2\,dx\,dt+C\alpha\int_S^T E_1(u_1(t))\,dt\end{aligned}$$

Choosing α small enough and using these estimates in (2.8), we obtain :

$$\begin{aligned}\int_S^T\int_\Omega \beta u_1^2\,dx\,dt &\le C\left[E_1(u_1(S))+E_1(u_1(T))\right]+C\eta\int_S^T E_1(u_1(t))\,dt\\ &\quad +\frac{C}{\eta}\int_S^T\int_\Omega (a(x)u_1^{'})^2\,dx\,dt+\frac{C}{\eta}\int_S^T\int_\omega u_1^{'2}\,dx\,dt\\ &\quad +\frac{\alpha}{2}\int_S^T\int_\Omega u_2^2\,dx\,dt\end{aligned}$$

Since $\int_S^T\int_\Omega \beta u_1^2\,dx\,dt \ge \int_S^T\int_{\Omega\cap Q_2} u_1^2\,dx\,dt$, we obtain the announced inequality of Lemma 4.

2.4. STEP 4. We can now conclude the proof of Proposition 1.
While $Q_1\subset Q_2$, we estimate $\int_S^T\int_{\Omega\cap Q_1} u_1^{'2}\,dx\,dt$ by $\int_S^T\int_{\Omega\cap Q_2} u_1^{'2}\,dx\,dt$. Using both lemmas 3 and 4 in Proposition 2, we easily obtain :

$$\begin{aligned}\int_S^T E_1(u_1(t))\,dt &\le C\left[E_1(u_1(S))+E_1(u_1(T))\right]+\frac{C}{\eta}\int_S^T\int_\omega u_1^{'2}\,dx\,dt\\ &\quad +\frac{C}{\eta}\int_S^T\int_\Omega a^2(x)u_1^{'2}\,dx\,dt+C\alpha\int_S^T\int_\Omega u_2^2\,dx\,dt\\ &\quad +C\eta\int_S^T E_1(u_1(t))\,dt\end{aligned}$$

Finally, for $c\eta<1$, we have proved the claim announced in Proposition 1, that is :

$$\begin{aligned}\int_S^T E_1(u_1(t))\,dt &\le C\left[E_1(u_1(S))+E_1(u_1(T))\right]+\frac{C}{\eta}\int_S^T\int_\omega u_1^{'2}\,dx\,dt\\ &\quad +C\int_S^T\int_\Omega a^2(x)u_1^{'2}\,dx\,dt+C\alpha\int_S^T\int_\Omega u_2^2\,dx\,dt\end{aligned}$$

2.5. STEP 5. Let us now estimate the three integral terms on the right hand side of Proposition 1 to obtain a first inequality between the partial energy $E_1(u_1(t))$ and the full energy $E(U(t))$.

The estimation of $C\int_S^T \int_\omega u_1^{'2}\ dx\ dt$ is easy because we are on the subset where the damping is effective : on ω, $a(x) \geq \gamma > 0$. Then :

$$\gamma \int_\omega u_1^{'2}\ dx \leq \int_\omega a(x) u_1^{'2}\ dx \leq \int_\Omega a(x) u_1^{'2}\ dx = -E^{'}(U(t))$$

Integrating on $[S;T]$, we obtain :

$$\gamma \int_S^T \int_\omega u_1^{'2}\ dx\ dt \leq \int_S^T -E^{'}(U(t))\ dt = [E(U(t))]_T^S \leq E(U(S))$$

Thus we have : $C\int_S^T \int_\omega u_1^{'2}\ dx\ dt \leq \dfrac{C}{\gamma} E(U(S))$.

We estimate the second term as follows :

$$\begin{aligned} C\int_S^T \int_\Omega a^2(x) u_1^{'2}\ dx\ dt &\leq CM \int_S^T \int_\Omega a(x) u_1^{'2}\ dx\ dt \\ &= C\int_S^T -E^{'}(U(t))\ dt \leq CE(U(S)). \end{aligned} \tag{2.9}$$

To estimate the term $C\int_S^T \int_\Omega u_2^2\ dx\ dt$, we multiply the first equation of the initial system (1.1) by u_2 and the second equation by u_1. We compute then and integrate on $\Omega \times [S;T]$ and thanks to the initial data we obtain the following equality :

$$\alpha \int_S^T \int_\Omega u_2^2\, dx\, dt = \alpha \int_S^T \int_\Omega u_1^2\, dx\, dt + \int_S^T \int_\Omega (u_1 u_2^{''} - u_2 u_1^{''})\, dx\, dt - \int_S^T \int_\Omega a(x) u_1^{'} u_2\, dx\, dt.$$

We estimate as usual the right hand side terms :

$$\begin{aligned} \int_S^T \int_\Omega u_2^2\ dx\ dt &\leq \alpha \int_S^T \int_\Omega u_1^2\ dx\ dt + CE(U(S)) \\ &+ \int_S^T \int_\Omega \frac{1}{2\alpha} (a(x) u_1^{'})^2\ dx\ dt + \int_S^T \int_\Omega \frac{\alpha}{2} u_2^2\ dx\ dt \end{aligned}$$

Let us send now the term with u_2 to the left hand side to obtain :

$$\begin{aligned} \frac{\alpha}{2} \int_S^T \int_\Omega u_2^2\ dx\ dt &\leq C\alpha \int_S^T \int_\Omega u_1^2\ dx\ dt + CE(U(S)) \\ &+ \frac{C}{\alpha} \int_S^T \int_\Omega |a(x) u_1^{'}|^2\ dx\ dt \end{aligned} \tag{2.10}$$

Hence using (2.9) we obtain :

$$\frac{\alpha}{2} \int_S^T \int_\Omega u_2^2\ dx\ dt \leq C\alpha \int_S^T E_1(u_1(t))\ dt + CE(U(S)) \tag{2.11}$$

Computing (2.10) in Proposition 1, we get that :

$$(1-C\alpha)\int_S^T E_1(u_1(t))\,dt \le C\,[E_1(u_1(S))+E_1(u_1(T))]+\frac{C}{\gamma}E(U(S))+CE(U(S))$$

Hence using Lemma 2 and for α small enough we obtain the announced result :

$$\int_S^T E_1(u_1(t))\,dt \quad\le\quad CE(U(S)) \tag{2.12}$$

2.6. STEP 6. Let us now give an estimation of the second partial energy

$$\int_S^T E_2(u_2(t))\,dt.$$

It is easy to obtain this estimation because the damping term doesn't appear on the second equation of the initial system.

Multiplying the second equation of system (1.1) by u_2 , integrating on $\Omega\times[S;T]$ and using Green's formula, we have :

$$\int_S^T\int_\Omega |\nabla u_2|^2\,dx\,dt=\int_S^T\int_\Omega u_2^{'2}\,dx\,dt-\alpha\int_S^T\int_\Omega u_1u_2\,dx\,dt-\left[\int_\Omega u_2^{'}u_2\,dx\right]_S^T$$

Thus,

$$\int_S^T\int_\Omega \frac{|\nabla u_2|^2+u_2^{'2}}{2}\,dx\,dt=\int_S^T\int_\Omega u_2^{'2}\,dx\,dt-\frac{\alpha}{2}\int_S^T\int_\Omega u_1u_2\,dx\,dt-\frac{1}{2}\left[\int_\Omega u_2^{'}u_2\,dx\right]_S^T$$

Using then the definition of $E_2(u_2(t))$ and for α small enough, we get that :

$$\begin{aligned}\int_S^T E_2(u_2(t))\,dt \quad&\le\quad C\int_S^T\int_\Omega u_2^{'2}\,dx\,dt+\alpha\int_S^T\int_\Omega\frac{u_1^2}{4}\,dx\,dt\\&\qquad+\alpha\int_S^T\int_\Omega\frac{u_2^2}{4}\,dx\,dt-\frac{1}{2}\left[\int_\Omega u_2^{'}u_2\,dx\right]_S^T\end{aligned}\tag{2.13}$$

Let us estimate the right-hand side terms :

Using the fact that the full energy is non-increasing, we obtain easily that :

$$\left[\int_\Omega u_2^{'}u_2\,dx\right]_S^T\le CE(U(S)).$$

For the second and third terms, we have :

$$C\alpha\int_S^T\int_\Omega u_1^2\,dx\,dt\le C\alpha\int_S^T E_1(u_1)\,dt$$

and

$$C\alpha\int_S^T\int_\Omega u_2^2\,dx\,dt \le C\alpha\int_S^T E_2(u_2)\,dt.$$

The only term whose estimation is not easy is the last term $\displaystyle\int_S^T\int_\Omega u_2^{'2}\,dx\,dt$: Using inequality (2.10) with the derivatives, we obtain :

$$C\int_S^T\int_\Omega u_2^{'2}\,dx\,dt \le C\int_S^T\int_\Omega u_1^{'2}\,dx\,dt + \frac{C}{\alpha}E(U^{'}(S)) + \frac{C}{\alpha^2}\int_S^T\int_\Omega |a(x)u_1^{''}|^2\,dx\,dt$$

The second and third terms of the right-hand side are easily estimated by $\dfrac{C}{\alpha^2}E(U^{'}(S))$.

For the first term, we use (2.12) to obtain : $\displaystyle C\int_S^T\int_\Omega u_1^{'2}\,dx\,dt \le CE(U(S))$, so that, we have

$$\int_S^T\int_\Omega |u_2^{'}|^2\,dx\,dt \le CE(U(S)) + \frac{C}{\alpha^2}E(U^{'}(S))$$

Using this last estimate in (2.13), we obtain for sufficiently small α :

$$\begin{aligned}\int_S^T E_2(u_2(t))\,dt \quad &\le \quad CE(U(S)) + \frac{C}{\alpha^2}E(U^{'}(S))\\ &\quad + C\alpha\int_S^T E_1(u_1(t))\,dt + C\int_S^T E_1(u_1(t))\,dt\end{aligned}$$

The constant of the last term does not depend on α, so we need to use (2.12) to estimate this term and finally we obtain :

$$\int_S^T E_2(u_2(t))\,dt \le CE(U(S)) + \frac{C}{\alpha^2}E(U^{'}(S)) + C\alpha\int_S^T E_1(u_1(t))\,dt$$

2.7. STEP 7. We can now complete the proof of Theorem 2. We add both estimates of the partial energies $E_1(u_1(t))$ and $E_2(u_2(t))$ obtained in steps 5 and 6, and so we have :

$$\int_S^T (E_1(u_1(t)) + E_2(u_2(t)))\,dt \le CE(U(S)) + \tilde{C}E(U^{'}(S)) + C\alpha\int_S^T E_1(u_1(t))\,dt$$

For α small enough we deduce that :

$$\int_S^T (E_1(u_1(t)) + E_2(u_2(t)))\,dt \le CE(U(S)) + \tilde{C}E(U^{'}(S))$$

Hence :

$$\int_S^T E(U(t))\,dt \quad \le \quad CE(U(S)) + \tilde{C}E(U^{'}(S)) \tag{2.14}$$

Using then the following result of Alabau [1], we prove the polynomial energy decay of the solution of the system (1.1) :
If E is a non-increasing function which verifies (13) for all $U \in \mathcal{D}(\mathcal{A})$, then the full energy of the solution U of system (1.1) decays polynomially, i.e.

$$\forall\ t > 0, \qquad E(U(t)) \leq \frac{C}{t}(E(U(0)) + E(U^{'}(0)))$$

Moreover if the initial data are in $\mathcal{D}(\mathcal{A}^n)$ for a certain positive integer n, then the following inequality holds:

$$\forall\ t > 0, \qquad E(U(t)) \leq \frac{C}{t^n}\sum_{p=0}^{p=n} E(U^p(0))$$

This completes the proof of Theorem 2.
Remark : We obtain the same result for the system of two coupled wave equations with different speeds of propagation with locally distributed damping.

Let Ω be a non-empty bounded set in $I\!R^N$ of class $\mathcal{C}^2$ and $\Gamma = \partial\Omega$ its boundary.

$$\begin{cases} u_1^{''} - c_1\Delta u_1 + a(x)u_1^{'} + \alpha u_2 = 0 & \text{in} \quad \Omega \times I\!R^+ \\ u_2^{''} - c_2\Delta u_2 + \alpha u_1 = 0 & \text{in} \quad \Omega \times I\!R^+ \\ u_1 = u_2 = 0 & \text{on} \quad \partial\Omega \times I\!R^+ \\ (u_1, u_1^{'})(0) = (u_1^0, u_1^1) & \text{in} \quad \Omega \\ (u_2, u_2^{'})(0) = (u_2^0, u_2^1) & \text{in} \quad \Omega \end{cases} \tag{2.15}$$

where $a \in \mathcal{C}^0(\Omega)$ is a positive function in Ω, c_1 and c_2 are two different constants in $I\!R^*$.

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