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Nonexistence of Global Solutions to an Elliptic Equation with a Dynamical Boundary Condition

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ABSTRACT: We consider the equation $\Delta u = 0$ posed in $Q := (0, +\infty) \times \Omega, \Omega := \{x = (x', x_N)/x' \in R^{N-1}, x_N > 0\}$, with the dynamical boundary condition $B(t, x', 0)u_{tt} + A(t, x', 0)u_t - u_{x_N} \geq D(t, x', 0)|u|^q$ on $\Sigma := (0, \infty) \times R^{N-1} \times \{0\}$ and give conditions on the coefficient functions $A(t, x', 0), B(t, x', 0)$ and $D(t, x', 0)$ for the nonexistence of global solutions.

Contents

1. Introduction

In this paper, we consider the problem (**EI**)

$$\begin{cases} \Delta u &= 0, \quad \text{on } Q, \\ B\, u_{tt} + A\, u_t - u_{x_N} &\geq D\,|u|^q, \quad \text{on } \Sigma, \end{cases}$$

subject to the initial conditions

$$u(0, x', 0) = u_0(x', 0), \;\; u_t(0, x', 0) = u_1(x', 0), \;\; x' \in R^{N-1}, \tag{1}$$

where $\Delta = \partial_1^2 + ... + \partial_N^2$ is the Laplace operator with respect to $x = (x', x_N) \in \Omega, u_t = \partial u/\partial t, u_{x_N} = \partial u/\partial x_N$, and $u_0, u_1 \in L^2_{loc}(R^{N-1} \times \{0\})$, $D \in L^\infty_{loc}(\Sigma)$, $A, A_t, B, B_t, B_{tt} \in L^2_{loc}(\Sigma)$. The functions A, B and D are assumed nonnegative; D positive for large x, A and B don't vanish simultaneously.

Before discribing our result in detail, let us dwell on some literature related to equations with dynamical boundary conditions. These kind of problems have been studied for a long time; see [9], [8], [11], [5]. More information is contained in the book by Lions [12]; In chapter 11 of [12], the existence of weak solutions to the Laplace equation with various nonlinear dynamical boundary conditions of parabolic and hyperbolic type is studied. More recently, Kirane [10] considered blow up for three equations with dynamical boundary conditions of parabolic and

2000 *Mathematics Subject Classification:* 35J99, 35L20 hyperbolic types. Later, Escher [6] addressed the questions of local solvability and blow up for such problems. Andreucci and Gianni [3] discussed the global existence and blow up issues for a degenerate parabolic problem with nonlocal dynamical boundary conditions, this on one hand. On the other hand, Apushkinskaya and Nazarov [1] present a survey on the recent results on boundary value problems with boundary conditions described by second-order Venttsel operators. They paid a special attention to nonlinear problems for elliptic and parabolic equations. They stated a priori estimates and existence results in Sobolev and Hölder spaces.

In [2], Amann and Fila derived a Fijita's type result for the Laplace equation with a parabolic dynamical boundary condition with constant coefficients posed in a half-space; they were followed by Fila and Quittner [7] who discussed the same problem as in [2] but in a bounded domain.
In this paper, we generalize the results of [2] concerning blow up to inequalities with non constant coefficients rather than equations with constant coefficients. Observe also that our technique of proof is different of that used by Amann and Fila which parallels Fujita's one; we rather follow an idea from the papers of Baras and Pierre [4], Mitidieri and Pohozaev [13] which is based on a judicious choice of the test function in the weak formulation of the problem, and a scaling of the variables.

2. Preliminaries

The coefficient functions $A(t,x',0), B(t,x',0)$ and $D(t,x',0)$ appearing in the boundary condition are assumed to verify the hypotheses

H1. $\displaystyle\int_{R^{N-1}} u(0,x',0)\,B(0,x',0)\,dx' \geq 0;$

H2. $\displaystyle\int_{R^{N-1}} \Big(B_t(0,x',0)+A_t(0,x',0)\Big)u(0,x',0)\,dx' \leq 0;$

H3. $B_{tt}-A_t \leq 0$, in case we consider nonnegative solutions $u \geq 0$,
or $B_{tt}-A_t = 0$, in case the considered solutions u are of indefinite sign;

H4. $|2B_t - A|D^{-\frac{1}{q}} \leq C_1 t^{\sigma_1}|x'|^{\delta_1}, \quad |B|D^{-\frac{1}{q}} \leq C_2 t^{\sigma_2}|x'|^{\delta_2}$
for $|x'| >> 1, t>0$, where $(\delta_1-\delta_2)/(\sigma_1-\sigma_2+1) > 0$.

H5. The function D is strictly positive for x' outside a large ball.

Definition. By a solution of (EI) on Q subject to the conditions (1), we mean a function $u \in L^q_{loc}(\Sigma)$ such that

$$
\begin{aligned}
\int_\Sigma D(t,x',0)|u|^q\tilde{\varphi} \quad &\le -\int_{R^{N-1}} u_t(0,x',0)B(0,x',0)\tilde{\varphi}(0,x',0) \\
&\quad + \int_{R^{N-1}} B_t(0,x',0)u(0,x',0)\tilde{\varphi}(0,x',0) \\
&\quad + \int_{R^{N-1}} B(0,x',0)\tilde{\varphi}_t(0,x',0)u(0,x',0) \\
&\quad + \int_\Sigma u\Big(B_{tt}\tilde{\varphi} + 2B_t\tilde{\varphi}_t + B\tilde{\varphi}_{tt}\Big) \\
&\quad - \int_{R^{N-1}} u(0,x',0)A(0,x',0)\tilde{\varphi}(0,x',0) \\
&\quad - \int_\Sigma u\Big(A_t\tilde{\varphi} + A\tilde{\varphi}_t + \Delta\tilde{\varphi} - \tilde{\varphi}_{x_N}\Big) \qquad (2)
\end{aligned}
$$

for any test function $\tilde{\varphi} \in C_0^2(R_+ \times \Omega)$, $\tilde{\varphi}(T,x',x_N) = 0$ for T large enough.

For later use, we set

$$\tilde{\chi} := (\delta_1 - \delta_2)/(\sigma_1 - \sigma_2 + 1) = \chi/\mu > 0.$$

3. The Result

Now, we are in force to announce our main result.

Theorem. *Assume that conditions H1-H5 are satisfied, and*

- *either* (*i*) $\tilde{\chi}\sigma_1 + \delta_1 - \tilde{\chi} < 0$, *and* $q \ge (\tilde{\chi} + N - 1)/(\tilde{\chi}\sigma_1 + \delta_1 - \tilde{\chi})$;

- *or* (*ii*) $\tilde{\chi}\sigma_1 + \delta_1 - \tilde{\chi} > 0$ *and* $1 \le q \le -(\tilde{\chi} + N - 1)/(\tilde{\chi}\sigma_1 + \delta_1 - \tilde{\chi})$.

Then problem (EI)-(1) doesn't admit global non trivial solutions.

Proof. The proof is by contradiction. So, we assume that the solution is global. Let $\varphi_0 \in C_0^2(R)$, $\varphi_0 \ge 0, \varphi_0$ decreasing be such that

$$\varphi_0(\xi) = \begin{cases} 1 & \text{if} \quad 0 \le |\xi| \le 1, \\ \\ 0 & \text{if} \quad |\xi| \ge 2. \end{cases}$$

Next, let's define $\psi = \psi(t,x)$ as the solution of

$$\begin{cases} \Delta\psi & = 0, \qquad\qquad t \in R_+,\ x \in R^{N-1} \times R^+, \\ \psi(t,x',0) & = \varphi(t,x',0), \qquad t \in R_+,\ x' \in R^{N-1}. \end{cases}$$

We choose

$$\varphi(t,x') = \varphi_0^\lambda(\xi),$$

where $\xi = \theta^{-1}/(t^\chi + |x'|^\mu)$, θ a positive real and λ any real greater than p, such that the integrals

$$\int_{\mathrm{supp}\varphi_t} |\varphi_t|^p \varphi^{1-p} \quad \text{and} \quad \int_{\mathrm{supp}\varphi_{tt}} |\varphi_{tt}|^p \varphi^{1-p}$$

are finite.
The function ψ is given by the Poisson's formula

$$\psi(t, x', x_N) = \frac{2x_N}{\sigma_N} \int_{R^{N-1}} \frac{\varphi(t, y, 0)}{(|y - x'|^2 + x_N^2)^{N/2}}\, dy,$$

where σ_N/N is the volume of the unit ball; so, we have

$$\psi_{x_N}(t, x', 0) = \frac{2}{\sigma_N} \int_{R^{N-1}} \frac{\varphi(t, y, 0)}{|y - x'|^N}\, dy$$

so

$$-\varphi_{x_N}(t, x', 0) = -\psi_{x_N}(t, x', 0) < 0.$$

Multiplying Equation $(EI)_1$ by ψ and integrating, we obtain

$$\int_Q \Delta u \psi = 0,$$

which, in the light of the Green formula, yields

$$\int_\Sigma u_{x_N}(t, x', 0)\psi(t, x', 0) = \int_\Sigma u(t, x', 0)\psi_{x_N}(t, x', 0).$$

or

$$\int_\Sigma u_{x_N}(t, x', 0)\varphi(t, x', 0) = \int_\Sigma u(t, x', 0)\varphi_{x_N}(t, x', 0).$$

because on Σ, $\psi = \varphi$.

Now, using φ in (2) as a test function and splitting the expression into terms, we obtain

$$\begin{aligned}
\int_\Sigma B(t, x', 0)u_{tt}\varphi &= \int_{R^{N-1}} [u_t B\varphi]_0^T - \int_\Sigma (B\varphi)_t u_t \\
&= \int_{R^{N-1}} [u_t B\varphi]_0^T + \int_{R^{N-1}} [u(B\varphi)_t]_0^T + \int_\Sigma u(B\varphi)_{tt} \\
&= -\int_{R^{N-1}} u_t(0, x', 0)B(0, x', 0)\varphi(0, x', 0) \\
&+ \int_{R^{N-1}} B_t(0, x', 0)u(0, x', 0)\varphi(0, x', 0) \\
&+ \int_{R^{N-1}} B(0, x', 0)\varphi_t(0, x', 0)u(0, x', 0) \\
&+ \int_\Sigma u(B_{tt}\varphi + 2B_t\varphi_t + B\varphi_{tt}).
\end{aligned}$$

Also, we have

$$\begin{aligned}\int_\Sigma A(t,x',0)u_t\varphi &= \int_{R^{N-1}} [uA\varphi]_0^T - \int_\Sigma u(A\varphi)_t \\ &= -\int_{R^{N-1}} u(0,x')A(0,x',0)\varphi(0,x',0) - \int_\Sigma uA_t\varphi - \int_\Sigma uA\varphi_t.\end{aligned}$$

So if we assume $u \geq 0$, we may then write

$$\begin{aligned}\int_\Sigma D(t,x',0)|u|^q\varphi &\leq -\int_{R^{N-1}} u_t(0,x',0)B(0,x',0)\varphi(0,x',0) \\ &+ \int_{R^{N-1}} B_t(0,x',0)u(0,x',0)\varphi(0,x',0) \\ &+ \int_{R^{N-1}} B(0,x',0)\varphi_t(0,x',0)u(0,x',0) \\ &+ \int_\Sigma u(B_{tt}\varphi + 2B_t\varphi_t + B\varphi_{tt}) \\ &- \int_\Sigma u(0,x',0)A(0,x',0)\varphi(0,x',0) \\ &- \int_\Sigma uA_t\varphi - \int_\Sigma uA\varphi_t - \int_{R^{N-1}} u\psi_\sigma. \qquad (3)\end{aligned}$$

In the case where $B_{tt} - A_t = 0$, the solution u may change sign.

By $(H1)$ and $(H2)$, (3) becomes

$$\int_\Sigma D(t,x',0)|u|^q\varphi \leq \int_\Sigma u(2B_t - A)\varphi_t + \int_\Sigma uB\varphi_{tt}. \qquad (4)$$

Using the ε-Young inequality to estimate the right hand side of (4), we obtain

$$\begin{aligned}\int_\Sigma D(t,x',0)|u|^q\varphi &\leq \varepsilon\int_\Sigma D(t,x',0)|u|^q\varphi + C(\varepsilon)\int_\Sigma |2B_t - A|^p|\varphi_t|^pD^{1-p}\varphi^{1-p} \\ &+\varepsilon\int_\Sigma D(t,x',0)|u|^q\varphi + C(\varepsilon)\int_\Sigma |B|^p|\varphi_{tt}|^pD^{1-p}\varphi^{1-p},\end{aligned}$$

for some positive constants ε and $C(\varepsilon)$, and where $p + q = pq$.

For ε small enough, we obtain

$$\int_\Sigma D|u|^q\varphi \leq C\int_\Sigma |2B_t - A|^p|\varphi_t|^pD^{1-p}\varphi^{1-p} + C\int_\Sigma |B|^p|\varphi_{tt}|^pD^{1-p}\varphi^{1-p}. \qquad (5)$$

The right hand side of (5) is finite thanks to our choice of the test function.

At this stage, we introduce the scaled variables

$$t = \tau\,\theta^\chi \quad \text{and} \quad x' = \eta\,\theta^\mu,$$

so

$$\xi := (t^{\frac{1}{\chi}} + |x'|^{\frac{1}{\mu}})/\theta = \tau^{\frac{1}{\chi}} + |\eta|^{\frac{1}{\mu}}.$$

Observe that

$$\mathrm{supp}\varphi \subset \{(\tau,\eta)\in R^2\,:\,0\le \tau^{\frac{1}{\chi}}+|\eta|^{\frac{1}{\mu}}\le 2\} =: C_{1,2},$$

$$\mathrm{supp}\varphi_t \subset \{(\tau,\eta)\in R^2\,:\,1\le \tau^{\frac{1}{\chi}}+|\eta|^{\frac{1}{\mu}}\le 2\},\quad \mathrm{supp}\varphi_{tt}\subset C_{1,2}.$$

Now, using the scaled variables, we obtain the estimates

$$|2B_t-A|^pD^{1-p}\le C\theta^{p(\chi\sigma_1+\mu\delta_1)}\,\tau^{p\sigma_1}|\eta|^{p\delta_1},$$

$$\varphi_t=\theta^{-p\chi},\quad \varphi_{tt}=\theta^{-2p\chi},\quad dt\,dx'=\theta^{\chi+(N-1)\mu}d\tau\,d\eta.$$

So,

$$\int_\Sigma |2B_t-A|^pD^{1-p}|\varphi_t|^p\varphi^{1-p}\,dt\,dx'\le C_1\theta^{\Lambda_1}, \tag{6}$$

where

$$\Lambda_1=\chi\sigma_1p+\mu\delta_1p-\chi p+\chi+\mu(N-1).$$

Similarly, we obtain the estimate

$$\int_\Sigma |B|^p\,D^{1-p}\,|\varphi_{tt}|^p\,\varphi^{1-p}\le C_2\theta^{\Lambda_2}, \tag{7}$$

where

$$\Lambda_2=\chi\sigma_2p+\mu\delta_2p-2\chi p+\chi\mu(N-1).$$

Now, we choose $\Lambda_1\le 0$ and $\Lambda_2\le 0$, that is

$$\tilde\chi(\sigma_1p-p+1)+(\delta_1p+N-1)\le 0,$$

$$\tilde\chi(\sigma_2p-2p+1)+(\delta_2p+N-1)\le 0,$$

where we set $\tilde\chi=\chi/\mu$.
For an optimal choice of χ and μ, we take

$$\tilde\chi=(\delta_2-\delta_1)/(\sigma_1-\sigma_2+1).$$

As $\tilde\chi$ has to be positive, the inequality $(\delta_2-\delta_1)(\sigma_1-\sigma_2+1)>0$ is required.

Finally, we have

$$(\tilde\chi\sigma_1+\delta_1-\tilde\chi)p+(\tilde\chi+N-1)\le 0,\quad\text{with}\quad \tilde\chi=(\delta_2-\delta_1)/(\sigma_1-\sigma_2+1). \tag{8}$$

There are two possibilities
- either (i) $\tilde\chi\sigma_1+\delta_1-\tilde\chi<0 \implies p\ge(\tilde\chi+N-1)/(\tilde\chi\sigma_1+\delta_1-\tilde\chi)$;
- or (ii) $\tilde\chi\sigma_1+\delta_1-\tilde\chi>0 \implies 1\le p\le -(\tilde\chi+N-1)/(\tilde\chi\sigma_1+\delta_1-\tilde\chi)$.

In the case $\Lambda_1 < 0, \Lambda_2 < 0$, using (6), (7) and (8) we obtain

$$\int_\Sigma D|u|^q \leq 0 \quad \Longrightarrow \quad u = 0.$$

If Λ_1 or $\Lambda_2 = 0$, we have

$$\int_\Sigma D|u|^q \leq \text{ Const.} < \infty. \tag{9}$$

Now, estimating inequality (4) via the Hölder inequality and using (9), we may write

$$\begin{aligned}\int_\Sigma D|u|^q\varphi \quad &\leq \left(\int_{\{\theta\leq t^\chi+|x'|^\mu\leq 2\theta\}} D(t,x')|u|^q\varphi\right)^{\frac{1}{q}} \\ &\cdot\left(\left(\int_{\mathrm{supp}\varphi_t} |2B_t - A|^p \frac{|\varphi_t|^p}{(D\,\varphi)^{p-1}}\right)^{\frac{1}{p}} + \left(\int_{\mathrm{supp}\varphi_{tt}} |B|^p \frac{|\varphi_{tt}|^p}{(D\,\varphi)^{p-1}}\right)^{\frac{1}{p}}\right).\end{aligned}$$

As

$$\lim_{\theta\to\infty}\left(\int_{\{\theta\leq t^\chi+|x'|^\mu\leq 2\theta\}} D(t,x',0)|u|^q\varphi\right) = 0,$$

we deduce from the former estimate that

$$\int_\Sigma D|u|^q \leq 0 \quad \Longrightarrow \quad u = 0.$$

The proof is complete.

Remark 1. In the case where $B \neq 0$, the natural choice is $\chi = \mu = 2$ because the temporal and spatial derivatives which appear are of the same order; in this case $\tilde{\chi} = 1$ and hence $\sigma_1 + \delta_1 - 1 < 0$; the condition on q is then $q \leq N/(N+1-\sigma_1-\delta_1)$.

Remark 2. Amann and Fila showed that the problem

$$(\mathbf{AF}) \qquad \left\{\begin{array}{rll} \Delta u &= 0, & (0,\infty)\times\Omega, \\ \partial_t u + \partial_n u &= u^q, & (0,\infty)\times\partial\Omega, \\ u(0,x') &= \phi, & \partial\Omega. \end{array}\right.$$

admits, for each ϕ bounded and uniformly continuous on R^{N-1}, a unique maximal solution u_ϕ. Moreover, they showed that

- If $T_\phi < \infty$ then $\lim\limits_{t\to T_\phi} \|u_\phi\|_\infty = \infty$.
- If $q \leq N/(N-1)$ then every nonzero maximal solution blows up in finite time.
- If $q > N/(N-1)$ then there are solutions that exist globally as well as solutions that blow up in finite time.

In our notations, Problem (AF) corresponds to

$$B = 0, \;\; A = D = 1, \;\; \sigma_1 = \delta_1 = 0.$$

In this case, (8) reads

$$-p\tilde{\chi} + \tilde{\chi} + N - 1 \le 0 \implies \frac{q}{q-1} = p \ge 1 + \frac{N-1}{\tilde{\chi}}\ .$$

So if we choose $\tilde{\chi} = 1$, then $q \le N/(N-1)$ for $N \ge 2$; this corresponds to the result of Amann and Fila as expected.

References

1. Apushkinskaya, D. E. and A.I. Nazarov, A survey of results on nonlinear Venttsel problems. Appl. Math. 45 (2000), no. 1, 69–80.
2. Amann, H. and M. Fila, A Fujita-type theorem for the Laplace equation with a dynamical boundary condition. Acta Math. Univ. Comenian. (N.S.) 66 (1997), no. 2, 321–328.
3. Andreucci, D. and R. Gianni, Global existence and blow up in a parabolic problem with nonlocal dynamical boundary conditions. Adv. Differential Equations 1 (1996), no. 5, 729–752.
4. Baras, P. and M. Pierre, Problème paraboliques semi-linéaires avec données mesures. Applicable Analysis 18 (1984), 11-149.
5. Carslaw, H. S. and J. C. Jaeger, Conduction of heat in solids. Oxford University Press, 1959, xx+510 pp.
6. Escher, J., Nonlinear elliptic systems with dynamical boundary conditions. Math. Z. 210 (1992), no. 3, 413–439.
7. Fila, M. and P. Quittner, Large time behavior of solutions of a semilinear parabolic equation with a nonlinear dynamical boundary condition. Topics in nonlinear analysis, 251–272, Progr. Nonlinear Differential Equations Appl., 35, Birkhüser, Basel, 1999.
8. Friedman, A. and M. Shinbrot, The initial value problem for the linearized equations of water waves. J. Math. Mech. 17 (1967) 107–180.
9. Garipov, R. M., On the linear theory of gravity waves: the theorem of existence and uniqueness. Arch. Rational Mech. Anal. 24 (1967) 352–362.
10. Kirane, M., Blow-up for some equations with semilinear dynamical boundary conditions of parabolic and hyperbolic type. Hokkaido Math. J. 21 (1992), no. 2, 221–229.
11. Langer, R.E., A problem in diffusion or in the flow of heat for a solid in contact with a fluid, Tohoku Math. J., 35 (1932), 260-275.
12. Lions, J.-L., Quelques méthodes de résolution des problèmes aux limites non liné aires. Dunod; Gauthier-Villars, Paris 1969 xx+554 pp.
13. Mitidieri, E. and S. I. Pohozaev, Absence of positive solutions for systems of quasilinear elliptic equations and inequalities in R^N. (Russian) Dokl. Akad. Nauk 366 (1999), no. 1, 13–17.

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