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Oscillation Criteria for Delay Neutral Difference Equations with Positive and Negative Coefficients

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ABSTRACT: This paper is concerned with a class of neutral type difference equations with positive and negative coefficients of the form

$$\Delta\left(x_n - r_n x_{n-m}\right) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n = 0, 1, 2, ...,$$

where m, k and l are nonnegative integers, and $\{p_n\}$, $\{q_n\}$ as well as $\{r_n\}$ are nonnegative real sequences. Novel oscillation criteria for this equation are derived.

Contents

1. Introduction

Let x_n be the state variable of a stage dependent process where n belongs to a set of consecutive integers. In case the change $\Delta x_n = x_{n+1} - x_n$ depends on the past changes as well as the past state sizes, we may then write

$$\Delta x_n = F(\Delta x_{n-1}, \Delta x_{n-2}, ...; x_n, x_{n-1}, ...).$$

In this paper, we consider a special case in the form of a neutral difference equation with positive and negative coefficients

$$\Delta\left(x_n - r_n x_{n-m}\right) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n = 0, 1, 2, ..., \tag{1}$$

where we assume that

(i) m, k and l are three integers such that $0 \leq l < k$ and $m > 0$,
(ii) $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ are nonnegative sequences, and
(iii) the sequence $\{h_n\}$ defined by

$$h_n = p_n - q_{n-k+l}, \ n \geq k - l, \tag{2}$$

is nonnegative and has a positive subsequence.

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By a solution of (1), we mean a real sequence $\{x_n\}$ which is defined for $n \geq -\mu$ and satisfies (1) for $n \geq 0$, where $\mu = \max\{m, k\}$. Let $\Omega = \{-\mu, -\mu+1, \cdots, 1, 0\}$. It is easy to see that for any given sequence $\{\phi_n\}$ defined on Ω, there exists a solution $\{x_n\}$ of (1) with $x_n = \phi_n$ for $n \in \Omega$.

A solution $\{x_n\}$ of (1) is said to be eventually positive (or eventually negative) if $x_n > 0$ (or $x_n < 0$) for all large n. It is said to be oscillatory if it is neither eventually positive nor eventually negative.

Among the existing studies, the oscillation of Eq. (1) has been discussed by several authors, see for example [1-7]. In particular, Ladas [1] and Qian and Ladas [2] considered the case where $r_n \equiv 0$. Chen and Zhang [3] and Zhang and Wang [4] considered the case where $r_n \equiv r$ with $0 \leq r < 1$. Recently, Li and Cheng [5,6] and Tang et al. [7] considered the general case in which one of their main results is obtained under assumptions such as

$$r_n + \sum_{s=n-k+l}^{n-1} q_s = 1. \tag{3}$$

In this paper, we are concerned with oscillation criteria for (1) which do not make use of (3). Our approach is new and is based on the monotonicity of an associated sequence of a solution as well as a functional inequality satisfied by it. As a result, our results can be used to show oscillation when the previous results cannot.

In the sequel, empty sum will be taken to be zero as usual.

2. Oscillation Criteria

For any integer $t \in \{0, 1, \cdots, k-l\}$, let

$$R_n(t) = r_n + \sum_{s=n-t}^{n-1} q_s + \sum_{s=n}^{n-t+k-l-1} p_s. \tag{4}$$

In particular, if $t = 0$, then (4) is just

$$R_n(0) = r_n + \sum_{s=n}^{n+k-l-1} p_s,$$

and if $t = k-l$, then (4) is

$$R_n(k-l) = r_n + \sum_{s=n-k+l}^{n-1} q_s.$$

Let

$$\alpha(t) = \begin{cases} m, & \text{if } q_n \equiv 0 \text{ and } t = k-l \\ \max\{m, k\}, & \text{otherwise} \end{cases}$$

and

$$\beta(t) = \begin{cases} m, & \text{if } q_n \equiv 0 \text{ and } t = k-l \\ \min\{m, l+1\}, & \text{otherwise} \end{cases}.$$

Obviously, $\alpha(t) \geq m \geq \beta(t)$.

Lemma 2.1. Suppose there exists an integer $t \in \{0, 1, \cdots, k-l\}$ such that

$$R_n(t) = r_n + \sum_{s=n-t}^{n-1} q_s + \sum_{s=n}^{n-t+k-l-1} p_s \leq 1, \tag{5}$$

for all large n. Suppose further that $\{x_n\}$ is an eventually positive solution of the functional difference inequality

$$\Delta(x_n - r_n x_{n-m}) + p_n x_{n-k} - q_n x_{n-l} \leq 0. \tag{6}$$

Then the sequence $\{z_n\}$ defined by

$$z_n = x_n - r_n x_{n-m} - \sum_{s=n-t}^{n-1} q_s x_{s-l} - \sum_{s=n}^{n+k-l-t-1} p_s x_{s-k} \tag{7}$$

for all large n will satisfy $\Delta z_n \leq 0$ and $z_n > 0$ eventually.

Proof. Let $N_1 > \mu$ such that $x_n > 0$ for $n \geq N_1 - \mu$. Then, from (6) and (7), we have

$$\Delta z_n \leq -(p_{n-t+k-l} - q_{n-t}) x_{n-t-l} = -h_{n-t+k-l} x_{n-t-l} \leq 0, \quad n \geq N_1. \tag{8}$$

Hence z_n is nonincreasing for $n \geq N_1$. If $\{z_n\}$ is not eventually positive, then in view of our assumption on $\{h_n\}$, $z_n < 0$ for all large n. Thus there exist an integer $N_2 \geq N_1$ and a constant $c > 0$ such that $z_n < -c$ for $n \geq N_2$. Therefore, from (7), we have

$$x_n \leq -c + r_n x_{n-m} + \sum_{s=n-t}^{n-1} q_s x_{s-l} + \sum_{s=n}^{n+k-l-t-1} p_s x_{s-k}, \quad n \geq N_2. \tag{9}$$

If the solution $\{x_n\}$ is unbounded, then $\limsup_{n\to\infty} x_n = \infty$. Hence there exists a sequence of integers $\{n_i\}_{i=1}^{\infty}$ such that $n_i \geq N_2 + \alpha(t)$ for $i = 1, 2, 3, ...$, $\lim_{i\to\infty} n_i = \infty$, $\lim_{i\to\infty} x_{n_i} = \infty$ and

$$x_{n_i} = \max\{x_n | \ N_2 \leq n \leq n_i\}$$

for $i = 1, 2, 3, ...$. From (5), (9) and the above equalities, we have

$$x_{n_i} \leq -c + r_{n_i} x_{n_i-m} + \sum_{s=n_i-t}^{n_i-1} q_s x_{s-l} + \sum_{s=n_i}^{n_i+k-l-t-1} p_s x_{s-k} \leq -c + x_{n_i} < x_{n_i},$$

which is a contradiction.

If $\{x_n\}$ is bounded, then there exists a constant $\theta \geq 0$ such that $\limsup_{n\to\infty} x_n = \theta < \infty$. Choose a sequence of integers $\{n_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} n_i = \infty$ and $\lim_{i\to\infty} x_{n_i} = \theta$. It is easy to see that there exists a sequence of integers $\{\lambda_i\}_{i=1}^{\infty}$ such that $n_i - \alpha(t) \leq \lambda_i \leq n_i - \beta(t)$ and

$$x_{\lambda_i} = \max\{x_n |\ n_i - \alpha(t) \leq n \leq n_i - \beta(t)\}.$$

Then $\lim_{i\to\infty} \lambda_i = \infty$ and $\limsup_{n\to\infty} x_{\lambda_i} \leq \theta$. Hence, from (5) and (9), we have

$$x_{n_i} \leq -c + r_{n_i} x_{n_i - m} + \sum_{s=n_i-t}^{n_i-1} q_s x_{s-l} + \sum_{s=n_i}^{n_i+k-l-t-1} p_s x_{s-k} \leq -c + x_{\lambda_i}.$$

Taking the superior limit as $i \to \infty$ on both sides, we obtain $\theta \leq -c + \theta < \theta$, which is also a contradiction. The proof is complete.

Lemma 2.2. Suppose there exists an integer $t \in \{0, 1, \cdots, k-l\}$ such that

$$R_n(t) = r_n + \sum_{s=n-t}^{n-1} q_s + \sum_{s=n}^{n-t+k-l-1} p_s \geq 1 \tag{10}$$

for all large n. Suppose further that the following second-order difference inequality

$$\Delta^2 y_n + \frac{1}{\alpha(t)} h_{n-t+k-l} y_n \leq 0 \tag{11}$$

does not have any eventually positive solution. Then for any eventually positive solution $\{x_n\}$ of (6), the sequence $\{z_n\}$ defined by (7) satisfies $z_n < 0$ and $\Delta z_n \leq 0$ for all large n.

Proof. We will only consider the case where the conditions $q_n \equiv 0$ and $t = k-l$ do not hold simultaneously, since the other case can be dealt with in a similar fashion. Then $\alpha(t) = \max\{m, k\}$. From (6) and (7), (8) holds for all large n. Hence $\Delta z_n \leq 0$ for all large n.

Suppose to the contrary that $\{z_n\}$ is eventually positive, then there is an integer $T > \alpha(t)$ such that $x_n > 0$, $z_n > 0$ and $\Delta z_n \leq 0$ for any $n \geq T - \alpha(t)$.

Let $M = \min\left\{x_{T-\alpha(t)}, x_{T-\alpha(t)+1}, \cdots, x_T\right\} > 0$. Then, in view of (7) and (10), we have

$$\begin{aligned} x_T &= z_T + r_T x_{T-m} + \sum_{s=T-t}^{T-1} q_s x_{s-l} + \sum_{s=T}^{T+k-l-t-1} p_s x_{s-k} \\ &\geq M\left(r_T + \sum_{s=T-t}^{T-1} q_s + \sum_{s=T}^{T+k-l-t-1} p_s\right) \geq M. \end{aligned}$$

Hence

$$x_{T+1} \geq M\left(r_{T+1} + \sum_{s=T+1-l}^{T} q_s + \sum_{s=T+1}^{T+k-l-t} p_s\right) \geq M.$$

In general, we can obtain

$$x_n \geq M\left(r_n + \sum_{s=n-t}^{n-1} q_s + \sum_{s=n}^{n+k-l-t-1} p_s\right) \geq M, \quad T \leq n \leq T+\alpha(t).$$

By induction, we can further obtain

$$x_n \geq M, \quad T+(s-1)\,\alpha(t) \leq n \leq T+s\alpha(t), \quad s=1,2,\dots\,.$$

Hence

$$x_n \geq M \text{ for any } n \geq T-\alpha(t). \tag{12}$$

In view of the monotonicity of $\{z_n\}$, we may set $\lim_{n\to\infty} z_n = c$. If $c=0$, there exists an integer $T_1 > T$ such that $z_n < M/2$ holds for $n > T_1$. Hence, from (12), we have

$$x_n \geq \frac{1}{\alpha(t)} \sum_{s=T_1}^{n+\alpha(t)-1} z_s,\ T_1 \leq n \leq T_1+\alpha(t).$$

If $c>0$, then $z_n \geq c$ for $n \geq T$. In view of (7), (10) and (12), we have

$$x_n \geq c + r_n x_{n-m} + \sum_{s=n-t}^{n-1} q_s x_{s-l} + \sum_{s=n}^{n+k-l-t-1} p_s x_{s-k} \geq c+M,\ n \geq T_1.$$

In general, we get

$$x_n \geq sc+M, \quad n \geq T_1+(s-1)\,\alpha(t), \quad s=1,2,\dots\,.$$

Hence $\lim_{n\to\infty} x_n = \infty$. Thus there exists an integer $T_2 > T_1$ such that

$$x_n \geq \frac{1}{\alpha(t)} \sum_{s=T_2}^{n+\alpha(t)-1} z_s, \quad T_2 \leq n \leq T_2+\alpha(t).$$

In general, there exists an integer $N \geq T_2$ such that

$$x_n \geq \frac{1}{\alpha(t)} \sum_{s=N}^{n+\alpha(t)-1} z_s, \quad N \leq n \leq N+\alpha(t). \tag{13}$$

For $N+\alpha(t) \leq n \leq N+\alpha(t)+\beta(t)$, in view of the monotonicity of $\{z_n\}$, from (7),

(10) and (13),

$$\begin{aligned} x_n &= z_n + r_n x_{n-m} + \sum_{s=n-t}^{n-1} q_s x_{s-l} + \sum_{s=n}^{n+k-l-t-1} p_s x_{s-k} \\ &\geq z_n + \left(r_n + \sum_{s=n-t}^{n-1} q_s + \sum_{s=n}^{n+k-l-t-1} p_s \right) \frac{1}{\alpha(t)} \sum_{s=N}^{n-1} z_s \\ &\geq \frac{1}{\alpha(t)} \sum_{s=n}^{n+\alpha(t)-1} z_s + \frac{1}{\alpha(t)} \sum_{s=N}^{n-1} z_s \\ &= \frac{1}{\alpha(t)} \sum_{s=N}^{n+\alpha(t)-1} z_s. \end{aligned}$$

By induction, we obtain

$$x_n \geq \frac{1}{\alpha(t)} \sum_{s=N}^{n+\alpha(t)-1} z_s, \quad N + \alpha(t) + (j-1)\,\beta(t) \leq n \leq N + \alpha(t) + j\beta(t),$$

for $j = 1, 2, \ldots$. Hence

$$x_n \geq \frac{1}{\alpha(t)} \sum_{s=N}^{n+\alpha(t)-1} z_s, \; n \geq N. \tag{14}$$

Let $y_n = \sum_{s=N}^{n-1} z_s$. Then $y_n > 0$ for all large n, and $\Delta y_n = z_n$ and $\Delta^2 y_n = \Delta z_n$ for $n \geq N$. From (8) and (14), in view of $\alpha(t) \geq t + l$, we have

$$\Delta z_n \leq -h_{n-t+k-l} x_{n-t-l} \leq -\frac{h_{n-t+k-l}}{\alpha(t)} \sum_{s=N}^{n-t-l+\alpha(t)-1} z_s \leq -\frac{h_{n-t+k-l}}{\alpha(t)} \sum_{s=N}^{n-1} z_s,$$

and so

$$\Delta^2 y_n + \frac{1}{\alpha(t)} h_{n-t+k-l} y_n \leq 0, \quad n \geq N + \alpha(t),$$

i.e., $\{y_n\}$ is an eventually positive solution of (11). This is contrary to our assumption. The proof is complete.

Theorem 1 *Assume there exist two integers $t_1, t \in \{0, 1, \cdots, k-l\}$ such that*

$$R_n(t_1) = r_n + \sum_{s=n-t_1}^{n-1} q_s + \sum_{s=n}^{n-t_1+k-l-1} p_s \leq 1 \tag{15}$$

and

$$R_n(t) = r_n + \sum_{s=n-t}^{n-1} q_s + \sum_{s=n}^{n-t+k-l-1} p_s \geq 1 \tag{16}$$

for all large n. Further assume that the functional inequality (11) does not have any eventually positive solution. Then every solution of (1) oscillates.

Indeed, from Lemma 2.1 and (15), for every eventually positive solution $\{x_n\}$ of (1), the sequence $\{z_n\}$ defined by (7) is eventually positive. But in view of (11), (16) and Lemma 2.2, the sequence $\{z_n\}$ is eventually negative, thus we obtain a contradiction. Thus (1) cannot have any eventually positive, nor any eventually negative, solutions.

We remark that there are many sufficient conditions which will guarantee the nonexistence of eventually positive solutions of (11). For instance, the following result is taken from [7]: If $\{d_n\}_{n=0}^{\infty}$ is a nonnegative sequence and

$$\liminf_{n\to\infty} n\sum_{s=n}^{\infty} d_s > \frac{1}{4}, \tag{17}$$

then the following functional inequality

$$\Delta^2 y_n + d_n y_n \le 0,\ \ n = 0, 1, 2, ... \tag{18}$$

does not have any eventually positive solutions.

Consequently, if (15) and (16) hold for t_1, t and all large n and if

$$\liminf_{n\to\infty} n\sum_{s=n}^{\infty} (p_{s+k-l} - q_s) > \frac{\alpha(t)}{4}, \tag{19}$$

then every solution of (1) oscillates.

Corollary 2.1. Assume there exists an integer $t \in \{1, \cdots, k-l-1\}$ such that (16) holds for all large n, and

$$\liminf_{n\to\infty} n\sum_{s=n}^{\infty} (p_{s+k-l} - q_s) > \frac{\max\{m,k\}}{4},$$

holds. Further assume that $\{q_n/\left(p_{n+k-l} - q_n\right)\}$ and $\{p_n/\left(p_n - q_{n-k+l}\right)\}$ are nondecreasing and there exist two nonnegative constants δ_1 and δ_2 such that

$$r_{n-k}\left(p_n - q_{n-k+l}\right) \le \delta_1\left(p_{n-m} - q_{n-k+l-m}\right), \tag{20}$$

$$q_{n-k}\left(p_n - q_{n-k+l}\right) \le \delta_2\left(p_{n-l} - q_{n-k}\right), \tag{21}$$

$$p_n\left(p_{n+t+l} - q_{n+t-k+2l}\right) \le \delta_2\left(p_n - q_{n-k+l}\right), \tag{22}$$

for all large n and $\delta_1 + (k-l)\,\delta_2 = 1$. Then every solution of Eq. (1) oscillates.

Proof. Suppose to the contrary that Eq.(1) has an eventually positive solution $\{x_n\}$ and let $\{z_n\}$ be defined by (7). From Lemma 2.2, we have $z_n < 0$ and $\Delta z_n \le 0$

eventually. In view of (20), (21) and (22),

$$\begin{aligned}
\Delta z_n &= -h_{n-t+k-l}x_{n-t-l} \\
&= -h_{n-t+k-l}\left(z_{n-t-l}+r_{n-t-l}x_{n-t-l-m}\right) \\
&\quad -h_{n-t+k-l}\left(\sum_{s=n-2t-l}^{n-t-l-1} q_s x_{s-l} + \sum_{s=n-t-l}^{n-2t+k-2l-1} p_s x_{s-k}\right) \\
&\geq -h_{n-t+k-l}z_{n-t-l} - \delta_1 h_{n-t+k-l-m}x_{n-t-l-m} \\
&\quad -h_{n-t+k-l}\sum_{s=n-2t-l}^{n-t-l-1}\frac{q_s h_{s+k-l}}{h_{s+k-l}}x_{s-l} - h_{n-t+k-l}\sum_{s=n-t-l}^{n-2t+k-2l-1}\frac{p_s h_s}{h_s}x_{s-k} \\
&\geq -h_{n-t+k-l}z_{n-t-l} - \delta_1 h_{n-t+k-l-m}x_{n-t-l-m} \\
&\quad -\frac{q_{n-t-l}h_{n-t+k-l}}{h_{n-t+k-2l}}\sum_{s=n-2t-l}^{n-t-l-1}\left(-\Delta z_{s+t}\right) \\
&\quad -\frac{p_{n-2t+k-2l}h_{n-t+k-l}}{h_{n-2t+k-2l}}\sum_{s=n-t-l}^{n-2t+k-2l-1}\left(-\Delta z_{s+t-k+l}\right) \\
&\geq -h_{n-t+k-l}z_{n-t-l} + \delta_1\Delta z_{n-m} + \delta_2\sum_{s=n-2t-l}^{n-t-l-1}\Delta z_{s+t} \\
&\quad +\delta_2\sum_{s=n-t-l}^{n-2t+k-2l-1}\Delta z_{s+t-k+l} \\
&= -h_{n-t+k-l}z_{n-t-l} + \delta_1\Delta z_{n-m} + \delta_2\left(z_{n-l}-z_{n-t-l}\right) + \delta_2\left(z_{n-t-l}-z_{n-k}\right) \\
&\geq -h_{n-t+k-l}z_{n-k} + \delta_1\Delta z_{n-m} + \delta_2 z_{n-l} - \delta_2 z_{n-k} \\
&= -\left(h_{n-t+k-l}+\delta_2\right)z_{n-k} + \delta_1\Delta z_{n-m} + \delta_2 z_{n-l}.
\end{aligned}$$

Thus

$$\Delta\left(z_n-\delta_1 z_{n-m}\right) + \left(h_{n-t+k-l}+\delta_2\right)z_{n-k} - \delta_2 z_{n-l} \geq 0,$$

which implies $\{-z_n\}$ is an eventually positive solution of the inequality

$$\Delta\left(x_n-\delta_1 x_{n-m}\right) + \left(h_{n-t+k-l}+\delta_2\right)x_{n-k} - \delta_2 x_{n-l} \leq 0. \tag{23}$$

In view of Lemma 2.1-2.2 and $\delta_1 + (k-l)\,\delta_2 = 1$, (23) has no eventually positive solution, thus we will obtain a contraction. In fact, if $\{x_n\}$ is an eventually positive solution of (23) and let $r_n = \delta_1$, $p_n = h_{n+k-l}+\delta_2$ and $q_n = \delta_2$, then

$$R_n(k-l) = r_n + \sum_{s=n-k+l}^{n-1} q_s = \delta_1 + (k-l)\delta_2 = 1 \leq 1,$$

$$R_n(0) = r_n + \sum_{s=n}^{n+k-l-1} p_s = \delta_1 + (k-l)\delta_2 + \sum_{s=n}^{n+k-l-1} h_{s+k-l} \geq 1,$$

and from the known condition,

$$\liminf_{n\to\infty} n\sum_{s=n}^{\infty}(p_{s+k-l}-q_s) > \frac{\max\{m,k\}}{4} = \frac{\alpha(0)}{4}.$$

Hence from Lemma 2.1, $z_n < 0$ defined by (7) is negative. But from Lemma 2.2, $z_n > 0$. We obtain a contradiction. The proof is complete.

Corollary 2.2. Assume that (16) holds for $t = 0$ and all large n, and

$$\liminf_{n\to\infty} n\sum_{s=n}^{\infty}(p_{s+k-l}-q_s) > \frac{\max\{m,k\}}{4}$$

holds. Further assume that $\{p_n/(p_n - q_{n-k+l})\}$ is nondecreasing and there exist two nonnegative constants δ_1 and δ_2 such that $\delta_3 \geq p_n - q_{n-k+l}$ eventually and for all large n,

$$r_{n-k}(p_n - q_{n-k+l}) \leq \delta_1 (p_{n-m} - q_{n-k+l-m}), \tag{24}$$

$$p_n(p_{n+l} - q_{n-k+2l}) \leq \delta_3(p_n - q_{n-k+l}), \tag{25}$$

and $\delta_1 + (k-l)\delta_3 = 1$ and $h_n < \delta_3$ eventually. Then every solution of Eq.(1) oscillates.

Indeed, by arguments similar to those in the proof of Corollary 2.1, we obtain,

$$\Delta z_n \geq (\delta_3 - h_{n+k-l}) z_{n-l} + \delta_1 \Delta z_{n-m} - \delta_3 z_{n-k}.$$

Hence $\{-z_n\}$ is an eventually positive solution of the inequality

$$\Delta(x_n - \delta_1 x_{n-m}) + \delta_3 x_{n-k} - (\delta_3 - h_{n+k-l}) x_{n-l} \leq 0,$$

which yields a contradiction by Lemma 2.1 and 2.2.

Similarly, we can obtain the following result.

Corollary 2.3. Assume that (16) holds eventually and (19) holds for $t = k - l$. Further assume that $\{q_n/(p_{n+k-l} - q_n)\}$ is nondecreasing and there exist two nonnegative constants δ_1 and δ_2 such that for all large n,

$$r_{n-k}(p_n - q_{n-k+l}) \leq \delta_1 (p_{n-m} - q_{n-k+l-m}), \tag{26}$$

$$q_{n-k}(p_n - q_{n-k+l}) \leq \delta_2 (p_{n-l} - q_{n-k}), \tag{27}$$

and $\delta_1 + (k-l)\delta_2 = 1$. Then every solution of Eq.(1) oscillates.

Theorem 2 *Suppose there exists an integer $t \in \{0, 1, \cdots, k-l\}$ such that (3) holds eventually. Suppose further that*

$$\liminf_{n\to\infty} n\sum_{s=n}^{\infty}(p_{s+k-l}-q_s) > \frac{max(m,t+l)}{4}$$

holds and that

$$r_{n-k}(p_n - q_{n-k+l}) \geq p_{n-m} - q_{n-k+l-m} \tag{28}$$

for all large n. Then every solution of Eq.(1) is oscillatory.

Proof. Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1). Then, in view of Lemma 2.1, $z_n > 0$ and $\Delta z_n \le 0$ for all large n. In view of (1), we have

$$\begin{aligned}\Delta z_n &= -h_{n-t+k-l} x_{n-t-l} \\ &= -h_{n-t+k-l}\left(z_{n-t-l} + r_{n-t-l} x_{n-t-l-m}\right) \\ &\quad -h_{n-t+k-l}\left(\sum_{s=n-2t-l}^{n-t-l-1} q_s x_{s-l} + \sum_{s=n-t-l}^{n-2t+k-2l-1} p_s x_{s-k}\right)\end{aligned}$$

Hence

$$\Delta z_n + h_{n-t+k-l} z_{n-t-l} + r_{n-t-l} h_{n-t+k-l} x_{n-t-l-m} \le 0 \tag{29}$$

for all large n. From (1), we have

$$\Delta z_{n-m} + h_{n-t+k-l-m} x_{n-t-l-m} = 0.$$

In view of (29) and the above equality, we obtain

$$\Delta\left(z_n - z_{n-m}\right) + h_{n-t+k-l} z_{n-t-l} \le \left\{h_{n-t+k-l-m} - r_{n-t-l} h_{n-t+k-l}\right\} x_{n-t-l-m} \le 0,$$

i.e. $\{z_n\}$ is an eventually positive solution of the recurrence relation

$$\Delta\left(z_n - z_{n-m}\right) + h_{n-t+k-l} z_{n-t-l} \le 0,$$

which yields a contradiction by Lemma 2.1 and 2.2. The proof is complete.

If the assumption (28) is not satisfied, the following result is available.

Theorem 3 *Suppose there exists an integer $t \in \{0, 1, \cdots, k-l\}$ such that (3) holds eventually. Further assume that there exists a constant $c \in [0, 1)$ such that*

$$r_{n-k}\left(p_n - q_{n-k+l}\right) \ge c\left(p_{n-m} - q_{n-k+l-m}\right). \tag{30}$$

Then every solution of Eq.(1) is oscillatory provided that there exists a constant $\bar{c} \in [0, c)$ such that the following recurrence relation

$$\Delta u_n + \frac{\bar{c}}{1-c} h_{n-t+k-l} u_{n-t-l-m} \le 0, \quad n = 0, 1, 2, ... \tag{31}$$

does not have an eventually positive solution.

Proof. Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1). Then, from Lemma 2.1, we have $z_n > 0$ and $\Delta z_n \le 0$ for all large n. As in the proof of Theorem 2, we get

$$\Delta\left(z_n - c z_{n-m}\right) + h_{n-t+k-l} z_{n-t-l} \le 0,$$

for all large n. Let $u_n = z_n - cz_{n-m}$. Similar to the proof of Lemma 2.1, we have $u_n > 0$ and $\Delta u_n \leq 0$ for all large n. Hence there exists an integer $N > 0$ such that $z_n > 0$ and $\Delta z_n \leq 0$, and $u_n > 0$ and $\Delta u_n \leq 0$ for $n \geq N$. Thus

$$\begin{aligned} z_n &= u_n + cz_{n-m} = u_n + c\left(u_{n-m} + cz_{n-2m}\right) = \cdots \\ &= u_n + cu_{n-m} + \cdots + c^i u_{n-im} + cz_{n-(i+1)m} \\ &\geq \left(c + c^2 + \cdots + c^i\right) u_{n-m} = \frac{c\left(1 - c^{i+1}\right)}{1-c} u_{n-m} \end{aligned}$$

for $n \geq (i+1)\,m + N$. Hence (31) holds for all large n, which is contrary to the hypothesis that (31) has no eventually positive solution. The proof is complete.

3. Examples

In this section, we give two examples to illustrate our results.

Example 3.1. Consider the neutral difference equation

$$\Delta\left(x_n - r_n x_{n-m}\right) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n = 1, 2, \cdots, \tag{32}$$

where $k = l + 1$ and $m > 0$, $l \geq 0$, and $r_n = 1 - c$, $p_n = c$ and $q_n = c - (n+1)^{-d}$ for $n \in \{0, 1, 2, \cdots\}$, where $c \in (0, 1)$ and d is a real constant.

Let $t_1 = 1$ and $t = 0$. Then it is easy to see that

$$R_n\left(t_1\right) = R_n\left(1\right) = r_n + q_{n-1} = 1 - n^{-d} < 1$$

for any positive integer n, and

$$R_n\left(t\right) = R_n\left(0\right) = r_n + p_n = 1 \geq 1.$$

Since $h_n = p_n - q_{n-1} = n^{-d}$ satisfies

$$\liminf_{n\to\infty}\left(n\sum_{s=n}^{\infty} h_{s+k-l}\right) = \liminf_{n\to\infty}\left(n\sum_{s=n+1}^{\infty} s^{-d}\right) = \begin{cases} \infty, & d \in (-\infty, 2) \\ 1, & d = 2 \end{cases} .$$

Hence by Theorem 1, every solution of (32) oscillates when $d \in (-\infty, 2)$ or $d = 2$ and $\max\{m, k\} < 4$. But it seems difficult to obtain the same conclusion from the results in [5-7].

Example 3.2. Consider the neutral difference equation (32), where $m = l = 1$, $k = 3$ and $r = 0.5$, and

$$p_n = 0.25 + (n+1)^{-2}, \ n = 0, 1, 2, ...,$$

$$q_n = 0.25 - (n+3)^{-2}, \ n = 0, 1, 2, ...$$

Take $t_1 = 2$ and $t = 0$, then

$$R_n\left(t_1\right) = R_n\left(2\right) = r_n + q_{n-2} + q_{n-1} = 1 - (n+1)^{-2} - (n+2)^{-2} < 1$$

and

$$R_n(t) = R_n(0) = r_n + p_n + p_{n+1} = 1 + (n+1)^{-2} + (n+2)^{-2} > 1$$

for all large n, and $h_n = p_n - q_{n-2} = 2(n+1)^{-2}$. Thus

$$\liminf_{n\to\infty}\left(n\sum_{s=n}^{\infty} h_{s+k-l}\right) = 2\times\liminf_{n\to\infty}\left(n\sum_{s=n+3}^{\infty} s^{-2}\right) = 2 > \frac{3}{4} = \frac{\max(m,k)}{4}.$$

Hence by Theorem 1, every solution of (32) oscillates. But it seems difficult to obtain the same conclusion from the results in [5-7].

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