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Global Solutions for a System of Klein-Gordon Equations with Memory

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ABSTRACT: In this paper we study the existence and uniqueness of solutions of a system of Klein-Gordon equations with memory.

Contents

1. Introduction

In this paper we study the global existence and uniqueness of solutions $(u, v) = (u(t, x), v(t, x))$ of the following nonlinear system

$$u_{tt} - \Delta u + f(u, v) + k * \Delta u = 0 \quad \text{in } [0, T] \times \Omega, \tag{1.1}$$

$$v_{tt} - \Delta v + g(u, v) + l * \Delta v = 0 \quad \text{in } [0, T] \times \Omega, \tag{1.2}$$

with boundary conditions $u = v = 0$ in $[0, T] \times \partial\Omega$ and initial conditions $u(0) = u_0$, $v(0) = v_0$, $u_t(0) = u_1$ and $v_t(0) = v_1$ in Ω. Here Ω is a bounded domain in $\mathbb{R}^n$, with smooth boundary, $T > 0$, and

$$(\eta * w)(t) = \int_0^t \eta(t - s) w(s) ds.$$

This system is a generalization of the following coupled system of Klein-Gordon equations

$$u_{tt} - \Delta u + m_1 u + k_1 u v^2 = 0,$$
$$v_{tt} - \Delta v + m_2 u + k_2 u^2 v = 0,$$

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where m_1, m_2, k_1, k_2 are nonnegative constants, which is considered in the study of the quantum field theory. We refer the reader to Schiff [8], Segal [7] and Struwe [9] for some classical results in Klein-Gordon equations.

The generalized system (1.1)-(1.2), without memory terms, were early considered by several authors. For instance, Medeiros & Milla Miranda [2], proved the existence and uniqueness of global weak solutions. Later, da Silva Ferreira [1] proved that the first order energy decays exponentially in the presence of frictional local damping. Quite recently, Cavalcanti et al in [3] considered the asymptotic behaviour for an analogous hyperbolic-parabolic system, with boundary damping, using arguments from Komornik and Zuazua [4].

Our objective is to study the system (1.1)-(1.2) when the memory terms $k * \Delta u$ and $l * \Delta v$ have dissipative properties. More precisely, if the kernels k and l are nonnegative C^2 functions satisfying

$$1 - \int_0^\infty k(s)ds > 0 \quad \text{and} \quad 1 - \int_0^\infty l(s)ds > 0, \tag{1.3}$$

$$k'', l'' \in L^1(0, \infty), \tag{1.4}$$

then the system has a unique strong global solution. We also use these conditions, there exist $\alpha, \beta > 0$ such that

$$-\alpha k(t) \leq k'(t) \leq -\beta k(t) \quad \text{and} \quad -\alpha\, l(t) \leq l'(t) \leq -\beta\, l(t). \tag{1.5}$$

We think that the strong solution decays uniformly as time goes to infinity. This is done by using multipliers techniques as in Muñoz Rivera [5]. But because of the coupled nonlinearities $f(u, v)$ and $g(u, v)$, the analysis of the dissipative effect of the memory terms requires new arguments.

To simplify our analysis, we assume that

$$f(u, v) = |u|^{\rho-2}u|v|^\rho \qquad g(u, v) = |v|^{\rho-2}v|u|^\rho,$$

with

$$\rho > 1 \ \text{ if } \ n = 1, 2 \qquad \text{and} \qquad 1 < \rho \leq \frac{n-1}{n-2} \ \text{ if } \ n \geq 3. \tag{1.6}$$

Note that (1.6) holds for the classical power $\rho = 2$ provided that $n \leq 3$.

2. Existence of Global Solutions

We begin with some notations that will be used throughout the paper. For the Sobolev space $H_0^1(\Omega)$ we consider the norm $\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2$, where $\|\cdot\|_p$ denotes the standard norm in $L^p(\Omega)$. The inner product in L^2 is denoted by $\langle\cdot,\cdot\rangle$. Now, if $w = w(t,x)$ is a function in $L^2(0,T;H_0^1(\Omega))$ and k is continuous, we put

$$(k\Box w)(t) = \int_0^t k(t-s)\|\nabla w(t) - \nabla w(s)\|_2^2\,ds.$$

Then, by differentiation, the following Lemma holds for $w \in C^1([0,T);H_0^1(\Omega))$ and $k \in C^1(0,\infty)$:

Lemma 2.1

$$\int_0^t k(t-s)\langle\nabla w(s),\nabla w'(t)\rangle\,ds = -\frac{1}{2}\frac{d}{dt}(k\Box w)(t)$$
$$+\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)\,ds\right)\|\nabla w(t)\|_2^2 + (g'\Box w)(t) - g(t)\|\nabla w(t)\|_2^2 \quad (2.1)$$

Theorem 2.1 *Assume that f and g satisfy condition* (1.6) *and k,l satisfy* (1.3). *Then if $u_0,v_0 \in H_0^1(\Omega)$ and $u_1,v_1 \in L^2(\Omega)$, problem* (1.1)-(1.2) *has weak solution (u,v) such that*

$$u,v \in L^\infty(0,T;H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)), \quad (2.2)$$

$$u'',v'' \in L^2(0,T;H^{-1}(\Omega)).$$

Assume in addition that $\rho \geq 2$ and (1.4) *holds. Then if*

$$u_0,v_0 \in H_0^1(\Omega)\cap H^2(\Omega) \quad \text{and} \quad u_1,v_1 \in H_0^1(\Omega), \quad (2.3)$$

problem (1.1)-(1.2) *has a unique solution such that*

$$u,v \in C^0([0,T];H_0^1(\Omega)\cap H^2(\Omega)) \cap C^1([0,T];H_0^1(\Omega)), \quad (2.4)$$

$$u'',v'' \in L^\infty(0,T;L^2(\Omega)).$$

The proof of Theorem 2.1 is based on a standard Galerkin approximation. Let $\{w_j\}$ be a basis for both $H_0^1(\Omega)$ and $L^2(\Omega)$, given by the eigenfunctions of $-\Delta$ in Ω, with Dirichlet condition. For each positive integer m we put

$$V_m = \text{Span}\{w_1, w_2, \cdots, w_m\}.$$

We search for functions

$$u^m(t) = \sum_{i=1}^{m} \alpha_{im}(t) w_i \qquad \text{and} \qquad v^m(t) = \sum_{i=1}^{m} \beta_{im}(t) w_i$$

satisfying the approximate problem

$$\int_\Omega \{u_{tt}^m - \Delta u^m + f(u^m, v^m)\}\, w_j\, dx - \int_0^t k(t-s)\langle \nabla u^m(s), \nabla w_j\rangle\, ds = 0, \tag{2.5}$$

$$\int_\Omega \{v_{tt}^m - \Delta v^m + g\,(u^m, v^m)\}\, w_j\, dx - \int_0^t l(t-s)\langle \nabla v^m(s), \nabla w_j\rangle\, ds = 0, \tag{2.6}$$

with initial conditions

$$u^m(0) = u_0^m, \;\; v^m(0) = v_0^m, \;\; u_t^m(0) = u_1^m, \;\; v_t^m(0) = v_1^m,$$

satisfying

$$u_0^m \to u_0 \text{ and } v_0^m \to v_0 \quad \text{strongly in} \quad H_0^1(\Omega),$$

$$u_1^m \to u_1 \text{ and } v_1^m \to v_1 \quad \text{strongly in} \quad L^2(\Omega).$$

The above system of o.d.e. has a local solution $(u^m(t), v^m(t))$ defined in some interval $[0, T_m)$.

Existence of Weak Solutions: Let us put

$$\begin{aligned} 2E_1^m(t) &= \|u_t^m(t)\|_2^2 + \|v_t^m(t)\|_2^2 \\ &\quad + \left(1 - \int_0^t k(s)ds\right)\|\nabla u^m(t)\|_2^2 + \left(1 - \int_0^t l(s)ds\right)\|\nabla v^m(t)\|_2^2 \\ &\quad + (k\Box u^m)(t) + (l\Box v^m)(t) + \frac{2}{\rho}\|u^m(t)v^m(t)\|_\rho^\rho. \end{aligned} \tag{2.7}$$

Then, multiplying (2.5) by $u_t^m(t)$, (2.6) by $v_t^m(t)$ and using identity (2.1) we get

$$\frac{d}{dt}E_1^m(t) = \frac{1}{2}\{(k'\Box u^m)(t)+(l'\Box v^m)(t)-k(t)\|\nabla u^m(t)\|_2^2-l(t)\|\nabla v^m(t)\|_2^2\} \leq 0.$$

It follows that $E_1^m(t)$ is a decreasing function and hence there exists a positive constant M_1, independent of m and t such that

$$\|u_t^m(t)\|_2^2 + \|v_t^m(t)\|_2^2 + \|\nabla u^m(t)\|_2^2 + \|\nabla v^m(t)\|_2^2 + \|u^m v^m(t)\|_\rho^\rho \leq M_1. \tag{2.8}$$

From this estimate we can extend the approximate solutions $(u^m(t), v^m(t))$ to the whole interval $[0, T]$. In addition, we get

$$u^m,\ v^m \quad \text{is bounded in } L^\infty(0,T;H_0^1(\Omega)), \tag{2.9}$$

$$u_t^m,\ v_t^m \quad \text{is bounded in } L^\infty(0,T;L^2(\Omega)). \tag{2.10}$$

Therefore, going to a subsequence if necessary, there exists u, v such that

$$u^m \to u,\ v^m \to v \quad \text{weakly star in } \ L^\infty(0,T;H_0^1(\Omega)), \tag{2.11}$$

$$u_t^m \to u_t,\ v_t^m \to v_t \quad \text{weakly star in } \ L^\infty(0,T;L^2(\Omega)). \tag{2.12}$$

Besides, from Lions-Aubin Lemma we also have

$$u^m \to u,\ v^m \to v \quad \text{strongly in } L^\infty(0,T;L^2(\Omega)). \tag{2.13}$$

These convergence allow us easily to pass to the limit the linear terms. For the nonlinear terms, we get for any $\theta \in (0, \rho/(\rho-1))$,

$$f(u^m, v^m) \to f(u,v) \quad \text{and} \quad g(u^m, v^m) \to g(u,v)$$

weakly in $L^\infty(0,T;L^\theta(\Omega))$. Therefore the existence of weak solutions is proved. $\Box$

To prove the existence of strong solutions we need the following two Lemmas.

Lemma 2.2 *Suppose that $\rho \geq 2$. Then there exists a constant $C > 0$ independent of m and t such that*

$$\int_\Omega |(f(u^m, v^m))_t u^m_{tt} + (g(u^m, v^m))_t v^m_{tt}|\, dx \leq C \left\{ \|u^m_{tt}\|_2^2 + \|v^m_{tt}\|_2^2 + \|\nabla u^m_t\|_2^2 + \|\nabla v^m_t\|_2^2 \right\}. \tag{2.14}$$

Proof. To simplify drop the upper index m and the time-variable t. First we note that

$$\int_\Omega (f(u,v))_t u_{tt}\, dx = \int_\Omega f_u(u,v) u_t u_{tt}\, dx + \int_\Omega f_v(u,v) v_t u_{tt}\, dx.$$

Now since

$$\int_\Omega |f_u(u,v)|\, |u_t|\, |u_{tt}|\, dx \leq \frac{\rho - 1}{2} \left\{ \int_\Omega |u|^{2(\rho-2)} |v|^{2\rho} |u_t|^2\, dx + \|u_{tt}\|_2^2 \right\} \tag{2.15}$$

we must assume $\rho \geq 2$. But then from (1.6), we have that $\rho = 2$ and $n \leq 3$ or $\rho > 2$ and $n = 1, 2$. Suppose $\rho = 2$. Then

$$\int_\Omega |u|^{2(\rho-2)} |v|^{2\rho} |u_t|^2\, dx = \int_\Omega |v|^4 |u_t|^2\, dx \leq \frac{1}{2} \|v\|_6^4\, \|u_t\|_6^2.$$

From the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and (2.8), there exists $C > 0$ such that

$$\int_\Omega |f_u(u,v)|\, |u_t|\, |u_{tt}|\, dx \leq C \left\{ \|\nabla u_t\|_2^2 + \|u_{tt}\|_2^2 \right\}, \tag{2.16}$$

If $\rho > 2$ and $n = 1, 2$, we take

$$\alpha = \frac{\rho - 1}{\rho - 2}, \quad \beta = \gamma = 2(\rho - 1)$$

so that $\alpha^{-1} + \beta^{-1} + \gamma^{-1} = 1$. Then we have

$$\begin{aligned} \int_\Omega |u|^{2(\rho-2)} |v|^{2\rho} |u_t|^2\, dx &\leq \left(\int_\Omega |u|^{2(\rho-2)\alpha} dx \right)^{\frac{1}{\alpha}} \left(\int_\Omega |v|^{2\rho\beta} dx \right)^{\frac{1}{\beta}} \left(\int_\Omega |u_t|^{2\gamma} dx \right)^{\frac{1}{\gamma}} \\ &= \|u\|_{2(\rho-1)}^{2(\rho-2)} \|v\|_{4\rho(\rho-1)}^{2\rho} \|u_t\|_{4(\rho-1)}^2 \\ &\leq C \|\nabla u_t\|_2^2, \end{aligned}$$

since in this case $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for all $p > 1$. Therefore in any case we have that (2.16) holds. Working similarly with $\int_\Omega f_v(u,v) v_t u_{tt} dx$ we conclude that

$$\int_\Omega |(f(u,v))_t|\, |u_{tt}|\, dx \leq C \left\{ \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|u_{tt}\|_2^2 \right\}.$$

The same argument shows that

$$\int_\Omega |(g(u,v))_t|\, |v_{tt}|\, dx \leq C\{ \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|v_{tt}\|_2^2 \},$$

and the Lemma follows. $\Box$.

Lemma 2.3 *There exists $C > 0$, depending only on the data, such that*

$$\begin{aligned} &-\int_0^t \int_\Omega \{(k * \Delta u^m)_t u_{tt}^m + (l * \Delta v^m)_t v_{tt}^m\}\, dx ds \\ &\leq C E_2(0) + C\left\{ \|\nabla u_t^m\|_2^2 + \|\nabla v_t^m\|_2^2 \right\} + \frac{1}{2}\int_0^t \{ \|\nabla u_t^m\|_2^2 + \|\nabla v_t^m\|_2^2 \}\, ds \end{aligned} \tag{2.17}$$

Proof. Here we also drop the upper index m. We note that

$$\begin{aligned} &-\int_\Omega (k * \Delta u)_t u_{tt}\, dx = k(0) \int_\Omega \nabla u(t) \nabla u_{tt}(t)\, dx \\ &\quad + \int_0^t k'(t-s) \langle \nabla u(s), \nabla u_{tt}(t) \rangle\, ds \\ &= \frac{d}{dt}\{k(0)\langle \nabla u(s), \nabla u_t(t)\rangle\} - k(0)\|\nabla u_t\|_2^2 \\ &\quad + \frac{d}{dt}\left\{ \int_0^t k'(t-s)\langle \nabla u(s), \nabla u_t(t)\rangle\, ds - \frac{k'(0)}{2}\|\nabla u_t(t)\|_2^2 \right\} \\ &\quad - \int_0^t k''(t-s)\langle \nabla u(s), \nabla u_t(t)\rangle\, ds. \end{aligned} \tag{2.18}$$

From assumption (1.4) and estimate (2.8), applying (2.1)

$$\int_0^t -k''(t-s)\langle \nabla u(s), \nabla u_t(t)\rangle\, ds \leq C\{\|\nabla u_t\|_2^2 + \|\nabla u\|_2^2\| + k' \Box \nabla u\}. \tag{2.19}$$

Then combining (2.18) and (2.19) and since $-k(0)\|\nabla u_t(t)\|_2^2 \leq 0$, we have

$$\begin{aligned}
-\int_0^t \int_\Omega (k * \Delta u)_t u_{tt}\, dxds &\leq \int_0^t k'(t-s)\langle \nabla u(s), \nabla u_t(t)\rangle\, ds - \frac{k(0)}{2}\|\nabla u_t(t)\|_2^2 \\
&+ \frac{1}{2}\int_0^t \|\nabla u_t(s)\|_2^2\, ds + \frac{T}{2}M_1\|k''\|^2_{L^1(0,\infty)}
\end{aligned}$$

As in (2.19) we infer that

$$\int_0^t k'(t-s)\langle \nabla u(s), \nabla u_t(t)\rangle\, ds \leq C\|\nabla u_t(t)\|_2^2 + CM_1\|k'\|^2_{L^1(0,\infty)}.$$

Then there exists a constant $\hat{C} = C(k,T) > 0$ such that

$$-\int_0^t \int_\Omega (k * \Delta u)_t u_{tt}\, dxds \leq 2\|\nabla u_t(t)\|_2^2 + \frac{1}{2}\int_0^t \|\nabla u_t(s)\|_2^2\, ds.$$

A similar argument proves that

$$-\int_0^t \int_\Omega (l * \Delta v)_t v_{tt}\, dxds \leq 2\|\nabla v_t(t)\|_2^2 + \frac{1}{2}\int_0^t \|\nabla v_t(s)\|_2^2\, ds.$$

This ends the proof. □.

Existence of Strong Solutions: Our starting is to get second order estimates of the solutions of (1.1)-(1.2). Let us put

$$E_2^m(t) = \frac{1}{2}\left\{\|u_{tt}^m(t)\|_2^2 + \|v_{tt}^m(t)\|_2^2 + \|\nabla u_t^m(t)\|_2^2 + \|\nabla v_t^m(t)\|_2^2\right\}. \tag{2.20}$$

Then we differentiate equation (2.5) and multiply by $u_{tt}^m(t)$ and differentiate equation (2.6) and multiply by $v_{tt}^m(t)$. Summing up the result, we have

$$\begin{aligned}
\frac{d}{dt}E_2^m(t) &= -\int_\Omega \{(f(u^m, v^m))_t u_{tt}^m + (g(u^m, v^m))_t v_{tt}^m\}\, dx \\
&\quad - \int_\Omega \{(k * \Delta u^m)_t u_{tt}^m + (l * \Delta v^m)_t v_{tt}^m\}\, dx. \qquad (2.21)
\end{aligned}$$

From (2.21) and Lemma 2.2, there exists a constant $C_1 > 0$ such that

$$\frac{d}{dt}E_2^m(t) \leq C_1 E_2^m(t) - \int_\Omega \{(k * \Delta u^m)_t u_{tt}^m + (l * \Delta v^m)_t v_{tt}^m\}\, dx.$$

Now we integrate the above relation from 0 to t and taking into account Lemma 2.3 and since $u_{tt}^m(0)$ and $v_{tt}^m(0)$ are bounded, there exists a positive constant C_2, not depending on m, such that

$$\begin{aligned} E_2^m(t) &\leq E_2(0) + C_2 \int_0^t E_2^m(s)\, ds \\ &+ C_2 \int_0^t \{\|\nabla u_t(s)\|_2^2 + \|\nabla v_t(s)\|_2^2\} ds. \end{aligned}$$

Then there exists a constant $C_3 > 0$, independently of m, such that

$$E_2^m(t) \leq E_2(0) + C_3 \int_0^t E_2^m(s)\, ds.$$

Then from the Gronwall's Lemma we finally get a positive constant M_2, depending on T but not on m, such that

$$\|u_{tt}^m(t)\|_2^2 + \|v_{tt}^m(t)\|_2^2 + \|\nabla u_t^m(t)\|_2^2 + \|\nabla v_t^m(t)\|_2^2 \leq M_2 \tag{2.22}$$

From this estimate we have

$$u_t^m,\ v_t^m \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)) \tag{2.23}$$
$$u_{tt}^m,\ v_{tt}^m \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{2.24}$$

and therefore

$$u_t^m \to u_t,\ v_t^m \to v_t \quad \text{wealky star in} \quad L^\infty(0,T;H_0^1(\Omega)), \tag{2.25}$$
$$u_{tt}^m \to u_{tt},\ v_{tt}^m \to v_{tt} \quad \text{weakly star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{2.26}$$

Besides, from Lions-Aubin Lemma we also have

$$u_t^m \to u_t,\ v_t^m \to v_t \quad \text{strongly in } L^\infty(0,T;L^2(\Omega)). \tag{2.27}$$

Now it is a matter of routine to verify that (u,v) satisfies (2.4) and the initial conditions of the problem (1.1)-(1.2). This conclude the proof of the existence part of Theorem 2.1.

Finally to prove that $u \in L^2(0,T;H^2(\Omega))$ for $n=3$, (for $n=1$ and $n=2$ follows immediately from the equation). In this case $\rho \leq 2$,

$$\int_\Omega |f(u,v)|^2 dx \leq \int_\Omega |u|^4|v|^4 dx \leq \|u\|_{L^2}^2 \int_\Omega |u|^2|v|^4 dx.$$

Using Gagliardo-Nirenberg inequalities,

$$\|u\|_{L^\infty} \leq C\|u\|^{\frac{1}{2}}|u\|^{\frac{1}{2}}_{H^2},$$

we get

$$\int_\Omega |f(u,v)|^2 dx \leq \|u\|^2_{L^\infty}\|u\|^2_{L^6}\|v\|^4_{L^6}.$$

So we have

$$\int_\Omega |f(u,v)|^2 dx \leq C\|u\|_{H^2}\|u\|^2_{H^1}\|v\|^4_{H^1} \leq CE(0)^{\frac{7}{2}}\|\Delta u\|_{L^2}.$$

Using the equation

$$-\Delta u + \int_0^t g(t-\tau)\Delta u(t)d\tau == u_{tt} - f(u,v)$$

and the resolvent operator we conclude that

$$\begin{aligned}\|\Delta u(\cdot)\|_{L^2(0,T;L^2)} &\leq \|u_{tt}\|_{L^2} + \|f(u,v)\|_{L^2(0,T;L^2)}\\ &\leq CE_2(0) + CE(0)^{\frac{7}{2}}\|\Delta u\|_{L^2(0,T;L^2)}.\end{aligned}$$

Then, $\|\Delta u\|_{L^2(0,T;L^2)} \leq CE_2(0) + E_1(0)^{\frac{7}{2}}$. Similar results holds to $g(u,v)$. From where we conclusion follows. □

Uniqueness: Let us suppose that (u,v) and $(\hat{u},\hat{v})$ are two solutions of (1.1)-(1.2). Then $U = u - \hat{u}$ and $V = v - \hat{v}$ satisfy

$$U_{tt} - \Delta U + k * \Delta U + f(u,v) - f(\hat{u},\hat{v}) = 0 \quad \text{in } [0,T]\times\Omega, \tag{2.28}$$

$$V_{tt} - \Delta V + l * \Delta V + g(u,v) - g(\hat{u},\hat{v}) = 0 \quad \text{in } [0,T]\times\Omega, \tag{2.29}$$

with $U(0) = V(0) = 0$ and $U_t(0) = V_t(0) = 0$. Let us put

$$\begin{aligned}2E_3(t) &= \|U_t(t)\|_2^2 + \|V_t(t)\|_2^2 + (k\square U)(t) + (l\square V)(t)\\ &\quad + \left(1 - \int_0^t k(s)ds\right)\|\nabla U_t(t)\|_2^2 + \left(1 - \int_0^t l(s)ds\right)\|\nabla V_t(t)\|_2^2.\end{aligned}$$

Multiplying (2.28) by $U_t(t)$, (2.29) by $V_t(t)$ and summing up the product result we have

$$\frac{d}{dt}E_3(t) \leq \int_\Omega |f(u,v) - f(\hat{u},\hat{v})|\,\|U_t\|_2 dx + \int_\Omega |g(u,v) - g(\hat{u},\hat{v})|\,\|V_t\|_2\, dx. \tag{2.30}$$

After some rearrangement

$$\int_\Omega (f(u,v) - f(\hat{u},\hat{v}))\,\|U_t\|_2\,dx = \int_\Omega (|v|^\rho - |\hat{v}|^\rho)|u|^{\rho-1}\,\|U_t\|_2\,dx + \int_\Omega (|u|^{\rho-2}u - |\hat{u}|^{\rho-2}\hat{u})\,|\hat{v}|^2\,\|U_t\|_2\,dx.$$

Then by the Mean Value Theorem we have

$$\int_\Omega |f(u,v) - f(\hat{u},\hat{v})|\,\|U_t\|_2\,dx \le \rho \int_\Omega (|v|^{\rho-1} + |\hat{v}|^{\rho-1})|u|^{\rho-1}\,\|V\|_2\,\|U_t\|_2\,dx + (\rho-1)\int_\Omega (|u|^{\rho-2} + |\hat{u}|^{\rho-2})\,|\hat{v}|^2\,\|U\|_2\,\|U_t\|_2\,dx.$$

Working as in the proof of Lemma 2.2, there exists $C > 0$ such that

$$\int_\Omega |f(u,v) - f(\hat{u},\hat{v})|\,\|U_t\|_2\,dx \le C\left\{\|\nabla U\|_2^2 + \|\nabla V\|_2^2 + \|U_t\|_2^2\right\}.$$

Similarly we see that

$$\int_\Omega |g(u,v) - g(\hat{u},\hat{v})|\,\|V_t\|_2\,dx \le C\left\{\|\nabla V\|_2^2 + \|\nabla U\|_2^2 + \|V_t\|_2^2\right\},$$

and hence from (2.30)

$$\begin{aligned}\frac{d}{dt}E_3(t) &\le C\left\{\|\nabla U\|_2^2 + \|\nabla V\|_2^2 + \|U_t\|_2^2 + \|V_t\|_2^2\right\}\\ &\le CE_3(t).\end{aligned}$$

Then from the Gronwall's Lemma we get

$$\|\nabla U\|_2 = \|\nabla V\|_2 = \|U_t\|_2 = \|V_t\|_2 = 0.$$

This proves the uniqueness statement. □

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