Maximal chain transitive sets for local groups

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ABSTRACT: Let $\mathcal{H}$ be a locally transitive local group. We characterize the maximal chain sets for a family $\mathcal{F}$ of subsets of $\mathcal{H}$ as intersections of control sets for certain shadowing semigroups.

Contents

1. Introduction

The concepts of control set and chain control set for control systems were introduced by Colonius and Kliemann [1]. Extending this notion for general classes of semigroups Braga Barros and San Martin [3] defined chain control sets for a family of subsets of a semigroup acting on a homogeneous space. In that paper chain control sets were characterized as intersection of control sets for the semigroups generated by the neighborhoods of the subsets in the family. For a metric space M we denote by $\mathrm{loc}\,(M)$ the set of local homeomorphisms of M. Let $\mathcal{H} \subset \mathrm{loc}\,(M)$ be a local group (see Definition 1) and $\mathcal{F}$ a family of subsets of $\mathcal{H}$. In this paper we define maximal chain transitive sets for $\mathcal{F}$ and characterize these sets as intersections of control sets for certain shadowing semigroups. In case $\mathcal{F}$ is contained in a local semigroup (see Definition 3) a maximal chain transitive set for $\mathcal{F}$ (with non empty interior in M) is a $\mathcal{F}$-chain control set as defined in [3]. Let A be a subset of a local group $\mathcal{H}$. The shadowing semigroups (see Definition 4) $S_{\varepsilon,A}$, $\varepsilon > 0$ and $A \in \mathcal{H}$ are semigroups obtained by successively composing the local homeomorphisms which are ε-close (in their domains) to some $\phi \in A$. The characterization of maximal chain transitive sets as intersections of control sets is possible since we relate chain attainability with the action of the shadowing semigroup (see Propositions 2 and 3). The approach here is different from that of [3]. In this new context, we are considering local semigroups contained in locally transitive (see Definition 5) local groups. We also consider local semigroups acting on metric spaces instead of homogeneous spaces.

The maximal chain transitive sets for flows on metric spaces were studied in [5]. Now, let ϕ_t be a flow on a metric space M. It is well known that the set

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$\mathcal{H} = \{\phi_t : t \in \mathbb{R}\}$ is a local group. For $T > 0$ we define $A_T = \{\phi_t : t \geq T\}$ and $\mathcal{F}_\phi = \{A_T : T \geq 0\}$. In this paper it is shown that the maximal chain transitive sets for the family $\mathcal{F}_\phi$ are the maximal chain transitive sets for the flow ϕ as defined by Conley in [2]. It follows that the Theorem 1 of this paper applied to the family $\mathcal{F}_\phi$ is the Theorem 4.7 in [5].

In the case of a flow on a metric space the domain of attraction of a chain transitive component of the flow was defined and studied in [5]. In this paper we also define and study the domain of attraction of a maximal chain transitive set for a family of subsets of a local group.

Apart from this general characterization of maximal chain transitive sets as intersection of control sets we also study the behavior of maximal chain transitive sets on fiber bundles. The action of semigroups in fiber bundles arises naturally in many contexts. For instance in nonlinear control systems the linearized flow evolves on a fiber bundle over the state space of the system (see [1]). The action of semigroups of diffeomorphisms on fiber bundles were studied by Barros and San Martin [4]. In [4] the control sets were described from their projections onto the base space and their intersections with the fibers.

In this paper it is shown the same kind of results of [4] for the maximal chain transitive sets. We show that a maximal chain transitive set in the total space of a fiber bundle projects inside a maximal chain transitive set in the base space. On the other hand, under certain conditions, we also prove that a maximal chain transitive set in the fiber is contained in a maximal chain transitive set in the total space.

2. Maximal chain transitive sets

In this section we start defining the shadowing semigroups. They are associated with a subset A contained in a local group $\mathcal{H}$ and a positive real number ε. We also relate chain attainability and the shadowing semigroups. As a consequence we characterize the chain control sets as intersections of control sets for shadowing semigroups. Finally we define and study the domain of attraction of a maximal chain transitive set for a family of subsets of a local group.

Let M be a metric space. We denote by $\mathrm{loc}\,(M)$ the set of *local homeomorphisms* of M, that is, homeomorphisms $\xi : U \to V$ between open subsets of M.

Definition 1 *A subset $\mathcal{H} \subset \mathrm{loc}\,(M)$ is a local group if it is closed under the operations of inverses and compositions (when they are allowed).*

Given $A \subset \mathcal{H}$ and a real $\varepsilon > 0$, we define a (ε, A)-chain.

Definition 2 *Take $x, y \in M$, a real $\varepsilon > 0$ and $A \subset \mathcal{H}$. A (ε, A)-chain from x to y consists of points $x_0 = x$, $x_1, \ldots, x_{n-1}$, $x_n = y$ in M and $\phi_0, \ldots, \phi_{n-1} \in A$ such that $d(\phi_j(x_j), x_{j+1}) < \varepsilon$ for $j = 0, \ldots, n-1$.*

We denote by

$$\mathcal{C}_{\varepsilon,A}(x) = \{y \in M : \text{ there is a } (\varepsilon, A)\text{-chain from } x \text{ to } y\}$$

We use $\mathrm{dom}\,(\cdot)$ for the domain of local homeomorphisms and for $\xi, \eta \in \mathrm{loc}\,(M)$ whose domains overlap put

$$d'(\xi, \eta) = \sup d(\xi(x), \eta(x))$$

where the supremum is taken over $\mathrm{dom}\,(\xi) \cap \mathrm{dom}\,(\eta)$. Note that for $\xi, \tau, \eta \in \mathrm{loc}\,(M)$ it holds

$$d'\,(\xi\eta, \tau\eta) \leq d'\,(\xi, \tau)\,, \tag{1}$$

since the supremum in the left hand side is taken over a smaller set than in the right hand side.

Given a local group $\mathcal{H}$ and $A \subset \mathcal{H}$ we define the neighborhood

$$B_\varepsilon\,(A, \mathcal{H}) = \{\eta \in \mathcal{H} : \text{there is } \xi \in A \text{ such that } d'(\xi, \eta) < \varepsilon\}$$

(or simply $B_\varepsilon\,(A)$ if $\mathcal{H}$ is understood).

Definition 3 *We say that $\mathcal{S} \subset \mathrm{loc}\,(M)$ is a local semigroup in case $\mathcal{S}$ is closed under the allowed compositions.*

Definition 4 *Let $\mathcal{H}$ be a local group and $A \subset \mathcal{H}$. Given a positive real number ε we define the* shadowing semigroup $S_{\varepsilon,A}\,(\mathcal{H})$ *(or simply $S_{\varepsilon,A}$) as the local subsemigroup of $\mathcal{H}$ generated by the set $B_\varepsilon(A, \mathcal{H})$.*

For a set $\mathcal{S} \subset \mathrm{loc}\,(M)$ and $x \in M$, we use the notation

$$\mathcal{S}x = \{\phi\,(x) : \phi \in \mathcal{S}\}.$$

Using the standard notation of control theory we say that a local semigroup $\mathcal{S}$ satisfies the accessibility property if $\mathrm{int}(\mathcal{S}x) \neq \emptyset$ for every $x \in M$.

In the following we show that the shadowing semigroups $S_{\varepsilon,A}$ satisfy the accessibility property. In order to discuss this result we require that the set of local homeomorphisms $\mathcal{H}$ is locally transitive. This locally transitive property means intuitively that we can map any $x \in M$ to neighboring points using "small" local homeomorphisms of M. More specifically, we introduce the following class of local groups.

Definition 5 *We say that a local group $\mathcal{H} \subset \mathrm{loc}\,(M)$ is* locally transitive *with parameters $c, \rho > 0$ if for every $x \in M$ and y in the ball $B_\rho\,(x)$ there exists $\xi \in \mathcal{H}$ such that $\xi\,(x) = y$ and $d\,(\xi\,(x)\,, x) \geq cd'\,(\xi, \mathrm{id})$.*

We observe that this condition was extensively discussed in [5]. Although restrictive (see examples in section 3 of [5]) this condition is weak enough so that many classes of reasonable metric spaces are allowed, like e.g. compact Riemannian manifolds or open sets in Frechet spaces.

Note that by the very definition $S_{\varepsilon,A} \subset S_{\varepsilon_1,A_1}$ if $\varepsilon \leq \varepsilon_1$ and $A \subset A_1$. Actually, the next lemma, which is a reformulation of [5] Lemma 4.2, shows that in a certain sense $S_{\varepsilon_1,A}$ is contained in the interior of $S_{\varepsilon_2,A}$ if $\varepsilon_1 < \varepsilon_2$.

Lemma 1 *Suppose that $\xi \in \mathcal{H}$ satisfies $d'(\xi, \mathrm{id}) < \delta$. Then for $\psi \in S_{\varepsilon,A}$, the composition $\xi\psi \in S_{\varepsilon+\delta,A}$.*

Proof: Write $\psi = \psi_1 \cdots \psi_k$ with $\psi_i \in B_\varepsilon(A, \mathcal{H})$, $i = 1, \ldots, k$. To prove the lemma it is enough to check that $\xi\psi_1 \in S_{\varepsilon+\delta,A}$, because $\psi_2 \cdots \psi_k \in S_{\varepsilon,A} \subset S_{\varepsilon+\delta,A}$. By inequality (1), $d'(\xi\psi_1, \psi_1) \leq d'(\xi, \mathrm{id})$, so that $d'(\xi\psi_1, \psi_1) < \delta$. However, $\psi_1 \in B_\varepsilon(A, \mathcal{H})$ and there is $\phi \in A$ such that $d'(\psi_1, \phi) < \varepsilon$. Hence for any z in $\mathrm{dom}(\xi\psi_1) \cap \mathrm{dom}(\psi_1) = \mathrm{dom}(\psi_1)$ it holds,

$$
\begin{aligned}
d(\xi\psi_1(z), \phi(z)) &\leq d(\xi\psi_1(z), \psi_1(z)) + d(\psi_1(z), \phi(z)) \\
&< \delta + \varepsilon,
\end{aligned}
$$

showing that $\xi\psi_1 \in B_{\varepsilon+\delta}(A, \mathcal{H})$, concluding the proof.   □

With the aid of the Lemma 1 we show under the hypothesis of locally transitivity that the shadowing semigroups satisfy the accessibility property.

Proposition 1 *Suppose that A is contained in a locally transitive local group $\mathcal{H}$ and ε is a positive real number. Then the shadowing semigroup $S_{\varepsilon,A}$ satisfy the accessibility property.*

Proof: Take $x \in M$. We show that $S_{\varepsilon,A}x \subset \mathrm{int}(S_{\varepsilon_1,A}x)$ if $\varepsilon < \varepsilon_1$. Given $\eta \in S_{\varepsilon,A}$ let us show that $\eta x \in \mathrm{int}(S_{\varepsilon_1,A}x)$. Write $\eta = \eta_1 \cdots \eta_k$ with $\eta_i \in B_\varepsilon(A, \mathcal{H})$, $i = 1, \ldots, k$. Now, let $c, \rho > 0$ be the parameters of local transitivity of $\mathcal{H}$, and choose $\rho' \leq \min\{\rho, c(\varepsilon_1 - \varepsilon)\}$. Then for any $y \in B_{\rho'}(\eta x)$ there exists $\xi \in \mathcal{H}$ with $\xi\eta(x) = y$ and $d(\xi(\eta x), \eta x) \geq cd'(\xi, \mathrm{id})$. By Lemma 1, one has $\xi\eta \in S_{\varepsilon_1,A}$, because the choice of ρ' ensures that $d'(\xi, \mathrm{id}) \leq \varepsilon_1 - \varepsilon$. Therefore, every $y \in B_{\rho'}(\eta x)$ belongs to $S_{\varepsilon_1,A}x$, proving the lemma.   □

The basic facts relating chain attainability and the shadowing semigroups are the following two propositions which are essentially a reformulation of Propositions 3.1 and 3.2 of [3] (see also Proposition 4.5 of [5]).

Proposition 2 *Keep the above notations and take $x, y \in M$. Then $y \in \mathcal{C}_{\varepsilon,A}(x)$ if $y \in S_{\varepsilon,A}x$. Also $y \in \mathcal{C}_{\varepsilon',A}(x)$ for every $\varepsilon' > \varepsilon$ if $y \in \mathrm{cl}(S_{\varepsilon,A}x)$.*

Proof: Take $y \in S_{\varepsilon,A}x$ and let $\psi \in S_{\varepsilon,A}$ be such that $y = \psi(x)$. By definition of $S_{\varepsilon,A}$ it follows that $\psi = \psi_{k-1} \cdots \psi_0$ with $\psi_i \in B_\varepsilon(A)$, $i = 0, \ldots, k-1$. Thus there are $\phi_i \in A$ such that $d(\phi_i(z), \psi_i(z)) < \varepsilon$ for every $z \in \mathrm{dom}(\psi_i)$. The sequences $x_0 = x$, $x_1 = \psi_0(x_0), \ldots, x_k = \psi_{k-1}(x_{k-1}) = y$ and $A \in \mathcal{F}$ determine an (ε, A)-chain from x to y, since

$$
d(\phi_{i-1}(x_{i-1}), x_i) = d(\phi_{i-1}(x_{i-1}), \psi_{i-1}(x_{i-1})) < \varepsilon
$$

for every i, showing the existence of a (ε, A)-chain from x to y.

Now, take $y \in \mathrm{cl}(S_{\varepsilon,A}x)$. Then there exists a sequence $\psi_n \in S_{\varepsilon,A}$ with $\psi_n(x) \to y$. Take $\varepsilon' > \varepsilon$ and let n_0 be such that $d(\psi_{n_0}(x), y) < \varepsilon' - \varepsilon$. As before, there

exists a (ε, A)-chain from x to $\psi_{n_0}(x)$. Let this chain be given by $y_0 = x,....,$ $y_n = \psi_{n_0}(x_0)$, and $\phi_0, \ldots, \phi_{n-1} \in A$. Thus $d(\phi_i(y_i), y_{i+1}) < \varepsilon$ for $i = 0, \ldots, n-1$. Therefore, the chain $z_0 = x, z_1 = y_1, ..., z_{n-1} = y_{n-1}, z_n = y$ and $\phi_0, \ldots, \phi_{n-1} \in A$ determine a (ε', A)-chain from x to y, since

$$d(\phi_{n-1}(y_{n-1}), y) \le d(\phi_{n-1}(y_{n-1}), \psi_{n_0}(x)) + d(\psi_{n_0}(x), y) < \varepsilon',$$

so that $d(\phi_{i-1}(y_{i-1}), y_i) < \varepsilon < \varepsilon'$ for every i. □

As a converse one has.

Proposition 3 *Suppose that $A \subset \mathcal{H}$ and $\mathcal{H}$ is a locally transitive local group with parameters c, ρ. Take ε with $0 < \varepsilon < \rho$ and put $\varepsilon' = \varepsilon/c$. Let $x_0, \ldots, x_n \in M$ and $\phi_0, \ldots, \phi_{n-1}$ determine a (ε, A)-chain from x_0 to x_n. Then $x_n \in \mathrm{int}(S_{\varepsilon', A} x_0)$.*

Proof: Since $d(\phi_i(x_i), x_{i+1}) < \varepsilon < \rho$, the locally transitivity property of $\mathcal{H}$ implies that there exists $\xi_i \in \mathcal{H}$ such that

$$d(x_{i+1}, \phi_i(x_i)) = d(\xi_i(\phi_i(x_i)), \phi_i(x_i)) \ge cd'(\xi_i, \mathrm{id})$$

for $i = 0, \ldots, n-1$. Hence $d'(\xi_i, \mathrm{id}) < \varepsilon/c = \varepsilon'$. Define $\eta_i = \xi_i \phi_i$. Then

$$d'(\eta_i, \phi_i) = d'(\xi_i \phi_i, \phi_i) \le d'(\xi_i, \mathrm{id}) < \varepsilon'$$

because multiplication on the right diminishes d'. Therefore, $\eta_i \in B_{\varepsilon'}(A)$. On the other hand, $\eta_i(x_i) = \xi_i \phi_i(x_i) = x_{i+1}$, and $x_n = \eta_{n-1} \cdots \eta_0(x_0)$, concluding the proof since $\psi = \eta_{n-1} \cdots \eta_0 \in S_{\varepsilon', A}$. □

This proposition ensures that we can replace an (ε, A)-chain by the action of an element in $S_{\varepsilon, A}$.

Now, we recall the definition of a control set. For a more detailed study of the control sets we refer [10]. From now on, and in the whole paper we assume that S is a local semigroup satisfying the accessibility property.

Definition 6 *A control set for S on M is a subset $D \subset M$ which satisfies*

1. $\mathrm{int}(D) \ne \emptyset$

2. For every $x \in D$, $D \subset \mathrm{cl}(Sx)$ and

3. D is maximal with these properties.

The control sets are the subsets where the semigroup is approximately transitive. This approximate transitivity can be improved to exact transitivity inside a dense subset of D. We define

$$D_0 = \{x \in D : x \in \mathrm{int}(Sx) \cap \mathrm{int}(S^{-1}x)\}.$$

In general, D_0 may be empty. However, in case it is not empty the set D_0 is called the *set of transitivity (or core)* of the control set D. The control set D is an *effective* control set in case $D_0 \neq \emptyset$. We also recall that a control set is called an *invariant* control set if it is invariant under the action of the semigroup $\mathcal{S}$.

These control sets have the following properties, proved in [4], Proposition 2.2 (see also [10], Proposition 2.2).

Proposition 4 *Suppose $D_0 \neq \emptyset$, that is, D is an effective control set. Then*

1. $D \subset \mathrm{int}\left(\mathcal{S}^{-1}x\right)$ for every $x \in D_0$.

2. $D_0 = \mathrm{int}\left(\mathcal{S}^{-1}x\right) \cap \mathrm{int}\left(\mathcal{S}x\right)$ for every $x \in D_0$.

3. For every $x, y \in D_0$ there exist $g \in \mathcal{S}$ with $gx = y$.

4. D_0 is dense in D.

5. D_0 is $\mathcal{S}$-invariant inside D, i.e., $\xi(x) \in D_0$ if $\xi \in \mathcal{S}$, $x \in D_0$ and $\xi(x) \in D$.

As a complement to the above proposition we have the following statement which ensures the existence of effective control sets.

Proposition 5 *Let $x \in M$ be such that*

$$x \in \mathrm{int}\left(\mathcal{S}x\right) \cap \mathrm{int}\left(\mathcal{S}^{-1}x\right).$$

Then there exists a unique effective control set D such that $x \in D_0$.

Proof: See [4], Proposition 2.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

On the sets of the transitivity of the control sets for the shadowing semigroups we have.

Lemma 2 *With the same assumptions as the previous proposition, take $\varepsilon_1 < \varepsilon_2$ and suppose that $D_{\varepsilon_1,A}$ and $D_{\varepsilon_2,A}$ are effective control sets for $S_{\varepsilon_1,A}$ and $S_{\varepsilon_2,A}$, respectively, such that $(D_{\varepsilon_1,A})_0 \cap (D_{\varepsilon_2,A})_0 \neq \emptyset$. Then $D_{\varepsilon_1,A} \subset (D_{\varepsilon_2,A})_0$.*

Proof: Take $x \in (D_{\varepsilon_1,A})_0 \cap (D_{\varepsilon_2,A})_0$. Then for any $y \in (D_{\varepsilon_1,A})_0$, $y \in S_{\varepsilon_1,A}x$ and $x \in S_{\varepsilon_1,A}y$. Since $S_{\varepsilon_1,A} \subset S_{\varepsilon_2,A}$, the maximality property in the definition of control sets ensures that $y \in D_{\varepsilon_2,A}$, and a fortiori, by Proposition 4, $y \in (D_{\varepsilon_2,A})_0$. Hence, $(D_{\varepsilon_1,A})_0 \subset (D_{\varepsilon_2,A})_0$. To conclude the proof we show that $z \in S_{\varepsilon_2,A}x$ and $x \in S_{\varepsilon_2,A}z$. By Proposition 4, $x \in S_{\varepsilon_1,A}z \subset S_{\varepsilon_2,A}z$. On the other hand, $D_{\varepsilon_1,A} \subset \mathrm{cl}\left(D_{\varepsilon_2,A}\right)_0$, so that any $z \in D_{\varepsilon_1,A}$ belongs to $\mathrm{cl}\left(S_{\varepsilon_1,A}x\right)$. Hence by Proposition 3, it follows that $z \in \mathrm{int}\left(S_{\varepsilon_2,A}x\right) \subset S_{\varepsilon_2,A}x$, as we desired to show. □

We define maximal chain transitive sets for a family $\mathcal{F}$ of subsets of a local group $\mathcal{H}$. We use the notation

$$\mathcal{C}(x) = \bigcap_{\varepsilon>0, A\in\mathcal{F}} \mathcal{C}_{\varepsilon,A}(x)$$

Definition 7 *Let $\mathcal{F}$ be a family of subsets of a local group $\mathcal{H}$. A subset $\mathcal{E} \subset M$ is* chain transitive *for the family $\mathcal{F}$ if for all $x \in \mathcal{E}$, $\mathcal{E} \subset \mathcal{C}(x)$. A chain transitive subset $\mathcal{E}$ is a* maximal chain transitive *for $\mathcal{F}$ if $\mathcal{E}$ is maximal with respect to set inclusion.*

Definition 8 *Let $\mathcal{S}$ be a local semigroup and assume that $\mathcal{F}$ is contained in $\mathcal{S}$. A maximal transitive set for $\mathcal{F}$ is called a $\mathcal{F}$-chain control set if $\operatorname{int}_M(\mathcal{E}) \neq \emptyset$.*

It follows quickly from the maximallity condition that two maximal chain transitive sets for $\mathcal{F}$ are either disjoint or coincident. On the other hand, a simple application of Zorn's Lemma shows that any chain transitive set for a family $\mathcal{F}$ is contained in a maximal chain transitive set for $\mathcal{F}$.

Finally we can give a characterization of the maximal chain transitive sets in terms of the control sets of the shadowing semigroups.

Theorem 1 *Suppose that $\mathcal{F}$ is a family of subsets contained in a locally transitive group $\mathcal{H}$. Assume that for each $\varepsilon > 0$ and $A \in \mathcal{F}$ there exists a control set $D_{\varepsilon,A}$ of $S_{\varepsilon,A}(\mathcal{H})$ such that $\mathcal{E}' = \bigcap_{\varepsilon>0 A \in \mathcal{F}} D_{\varepsilon,A} \neq \emptyset$. Then $\mathcal{E}'$ is a maximal chain transitive set for $\mathcal{F}$.*

Conversely let $\mathcal{E}$ be a maximal chain transitive set for $\mathcal{F}$. Then for every $\varepsilon > 0$ and $A \in \mathcal{F}$, there exists an effective control set $D_{\varepsilon,A}(\mathcal{E})$ of $S_{\varepsilon,A}(\mathcal{H})$ such that $\mathcal{E}$ is contained in the set of transitivity $D_{\varepsilon,A}(\mathcal{E})_0$. Furthermore,

$$\mathcal{E} = \bigcap_{\varepsilon>0, A \in \mathcal{F}} D_{\varepsilon,A}(\mathcal{E}) = \bigcap_{\varepsilon>0, A \in \mathcal{F}} D_{\varepsilon,A}(\mathcal{E})_0. \tag{2}$$

Proof: If $x, y \in \mathcal{E}'$ then for all $\varepsilon > 0$ and $A \in \mathcal{F}$ one has $x, y \in D_{\varepsilon,A}$, so that $y \in \operatorname{cl}(S_{\varepsilon,A}x)$. Hence by Proposition 2 there exists an (ε, A)-chain from x to y. This shows that $\mathcal{E}'$ is chain transitive. The maximality follows by Proposition 3. In fact, if $x \in \mathcal{E}'$ and for every $\varepsilon > 0$, $A \in \mathcal{F}$ there is a (ε, A)-chain from x to z and from z to x then $z \in D_{\varepsilon,A}$, so that $z \in \mathcal{E}'$.

For the second part take $x \in \mathcal{E}$. Since $\mathcal{E}$ is chain recurrent, $x \in \mathcal{C}_{\varepsilon,A}(x)$ for all $\varepsilon > 0$, $A \in \mathcal{F}$. By Proposition 3 and Corollary 1, it follows that $x \in \operatorname{int}(S_{\varepsilon,A}x)$ for every $\varepsilon > 0$, $A \in \mathcal{F}$. Now applying Proposition 5 we conclude that there exists a control set $D_{\varepsilon,A}(\mathcal{E}, x)$ of $S_{\varepsilon,A}$ such that $x \in D_{\varepsilon,A}(\mathcal{E}, x)_0$. We claim that $D_{\varepsilon,A}(\mathcal{E}, x) = D_{\varepsilon,A}(\mathcal{E}, y)$ for all $x, y \in \mathcal{E}$. In fact, since $\mathcal{E}$ is chain transitive, $y \in \mathcal{C}_{\varepsilon,A}(x)$ for all $\varepsilon > 0$, $A \in \mathcal{F}$. Hence, by Proposition 3, $y \in S_{\varepsilon,A}x$. The same way $x \in S_{\varepsilon,A}y$, showing that x and y belong to the same control set.

As to the equalities in (2), note that the second one is a consequence of Corollary 2. Hence it remains to prove that $\bigcap_{\varepsilon,A} D_{\varepsilon,A}(\mathcal{E}) \subset \mathcal{E}$. Using Proposition 2, we see that any two points $x, y \in \bigcap_{\varepsilon,A} D_{\varepsilon,A}(\mathcal{E})$ are attainable to each other by (ε, A)-chains, so that this intersection is indeed contained in a maximal chain transitive set for $\mathcal{F}$, which must be $\mathcal{E}$. $\qquad\square$

Now, we present some applications of the last theorem.

Example 1 *Let G be a Lie group and G/H a homogeneous space of G. Let $X_0, X_1, \ldots, X_m$ be right invariant vector fields in G, and consider the control system*

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t)X_i(x(t))$$

where $u = (u_1, \ldots, u_m) \in \mathcal{U}$ for some class of admissible controls $\mathcal{U}$. Denote by $\varphi(g, u, t)$ the solution of the system at time t given by the control u and starting at $g \in G$. It is given by $\varphi(g, u, t) = \varphi(1, u, t)g$ where 1 stands for the identity in G. The attainable set from the identity at time t, $A(t)$ is given by $A(t) = \{\varphi(1, u, t) : u \in \mathcal{U}\}$. Their union

$$S = \bigcup_{t \geq 0} A(t)$$

is a subsemigroup of G known as the system's semigroup (see [7]).

Let $\mathcal{F}_{control}$ be the family of subsets of S defined by

$$\mathcal{F}_{control} = \{\bigcup_{t > T} A(t) : T \geq 0\}.$$

Then the $\mathcal{F}_{control}$-chain control sets on G/H are, in general, the chain control sets for control systems as defined by Colonius and Kliemann in [1](see [3] pg 260).

Now, if we apply Theorem 1 to the case where $M = G/H$ and $\mathcal{F}$ is a family of subsets of a semigroup contained in a locally transitive local group of $\mathrm{loc}(G/H))$ we obtain Theorem 3.7 in [3] for $\mathcal{F}$-chain control sets.

Example 2 *Regarding flows on metric spaces we refer to the books Colonius-Kliemann [1] (Appendix B) and Conley [2]. Let (M, d) be a metric space. Given a continuous-time flow $\phi : \mathbb{R} \times M \to M$ we write the corresponding homeomorphisms by $\phi_t(\cdot) = \phi(t, \cdot)$ or simply by $\phi_t(x) = t \cdot x$.*

For $x, y \in M$ and $\varepsilon, T > 0$ an ε, T-chain from x to y is given by points $x_0 = x, x_1, \ldots, x_n = y \in M$ and $t_0, \ldots, t_{n-1} \geq T$, for some $n \in \mathbb{N}$, such that

$$d(t_i \cdot x_i, x_{i+1}) < \varepsilon, \qquad i = 0, 1, \ldots, n - 1.$$

We denote by $\mathcal{C}_{\varepsilon,T}(x)$ the set of those $y \in Y$ such that there exists an ε, T-chain from x to y, and put $\mathcal{C}(x) = \bigcup_{\varepsilon,T} \mathcal{C}_{\varepsilon,T}(x)$. A subset $A \subset M$ is chain transitive for ϕ if for all $x \in A$, $A \subset \mathcal{C}(x)$. A chain transitive subset A is maximal transitive for ϕ (with respect to set inclusion) if and only if for all $x \in A$, $A = \mathcal{C}(x)$. The set $\mathcal{H} = \{\phi_t : t \in \mathbb{R}\}$ is a local group. Now, define $A_T = \{\phi_t : t \geq T\}$ and $\mathcal{F}_\phi = \{A_T : T \geq 0\}$. It follows that the maximal chain transitive sets for $\mathcal{F}_\phi$ are the maximal chain transitive sets for the flow ϕ. If we apply the Theorem 1 using the family $\mathcal{F}_\phi$ we obtain Theorem 4.7 in [5].

Next, we define and study the domain of attraction of maximal chain transitive sets. In the case of a flow on a metric space the domain of attraction of a maximal chain transitive set was defined and studied in [5]. The definition of the domain of attraction of a maximal chain transitive set given bellow generalizes the definition given in [5].

Definition 9 *Let $\mathcal{F}$ be a family of subsets of a local group $\mathcal{H}$. Let $\mathcal{E}$ be a maximal chain transitive set for $\mathcal{F}$. We define the domain of attraction of $\mathcal{E}$ as*

$$\mathcal{A}(\mathcal{E}) = \{y \in M : \text{ there exist a } x \in \mathcal{E} \text{ and } y \in \mathcal{C}(x)\}$$

We define the following relation among the maximal chain transitive sets for $\mathcal{F}$ in M:

$$\mathcal{E}_1 \preceq \mathcal{E}_2 \text{ if and only if there are } x \in \mathcal{E}_1, y \in \mathcal{E}_2 \text{ and } y \in \mathcal{C}(x)$$

Proposition 6 *The relation $\preceq$ is an order among the maximal chain transitive sets for $\mathcal{F}$.*

Proof: It is immediate from the definition and properties of the maximal chain transitive sets for $\mathcal{F}$. □

Equivalently, $\mathcal{E}_1 \preceq \mathcal{E}_2$ if and only if $\mathcal{E}_1 \cap \mathcal{A}(\mathcal{E}_2) \neq \emptyset$.

Proposition 7 *Let $\mathcal{F}$ be a family contained in a locally transitive local group $\mathcal{H}$. Suppose that $\mathcal{E}$ is a maximal chain transitive set for $\mathcal{F}$ given by $\mathcal{E} = \bigcap_{\varepsilon,A} D_{\varepsilon,A}$ where $D_{\varepsilon,A}$ are invariant control sets for the shadowing semigroups $S_{\varepsilon,A}$ with $\varepsilon > 0$ and $A \in \mathcal{F}$. Then $\mathcal{E}$ is maximal with respect to the order $\preceq$, defined above.*

Proof: It is enough to show that if a point $z \in \mathcal{E}$ can be linked to a point x by a (ε, A)-chain for every $\varepsilon > 0$ and $A \in \mathcal{F}$ then $x \in \mathcal{E}$. By contradiction we assume that there exists a (ε, A)-chain from z to $x \notin \mathcal{E}$ for every $\varepsilon > 0$ and $A \in \mathcal{F}$. Since $x \notin \mathcal{E}$ we have $x \notin D_{\varepsilon_1,A_1}$ for some $\varepsilon_1 > 0$ and $A_1 \in \mathcal{F}$. Let c be the parameter given by the locally transitivity. There is a $(c\varepsilon_1, A_1)$-chain from z to x and by the Proposition 3 we conclude that $x \in S_{\varepsilon_1,A_1}z$. Since $z \in D_{\varepsilon_1,A_1}$ and D_{ε_1,A_1} is invariant by the action of S_{ε_1,A_1} one has that $x \in D_{\varepsilon_1,A_1}$, which is a contradiction. □

Now, we show that the domain of attraction of a maximal chain transitive set is the intersection of the domains of attraction of the control sets for the shadowing semigroups.

Proposition 8 *With the hypothesis of Theorem 1 one has*

$$\mathcal{A}(\mathcal{E}) = \bigcap_{\varepsilon,A} \mathcal{A}(D_{\varepsilon,A})$$

Proof: Take $z \in \mathcal{A}(\mathcal{E})$. Then, there is $x \in \mathcal{E}$ and a (ε, A)-chain from z to x for every $\varepsilon > 0$ and $A \in \mathcal{F}$. By the Proposition 3 there exist $\phi_{\varepsilon,A} \in S_{\varepsilon,A}$ such that $\phi_{\varepsilon,A}(z) = x$ for every $\varepsilon > 0$ and $A \in \mathcal{F}$. Therefore $z \in \mathcal{A}(D_{\varepsilon,A})$ for every $\varepsilon > 0$ and $A \in \mathcal{F}$, i.e., $z \in \bigcap_{\varepsilon,A} \mathcal{A}(D_{\varepsilon,A})$. For the converse, assume that $z \in \bigcap_{\varepsilon,A} \mathcal{A}(D_{\varepsilon,A})$. Then, $z \in \mathcal{A}(D_{\varepsilon,A})$ for every $\varepsilon > 0$ and $A \in \mathcal{F}$. For $\varepsilon > 0$ and $A \in \mathcal{F}$ there is $\phi_{\varepsilon,A} \in S_{\varepsilon,A}$ and $x_{\varepsilon,A} \in D_{\varepsilon,A}$ such that $\phi_{\varepsilon,A}(z) = x_{\varepsilon,A}$. Take $x \in \mathcal{E} \subset D_{\varepsilon,A}$. By the Proposition 2 there exist a (ε, A)-chain from z to $x_{\varepsilon,A}$ and therefore from z to x. □

As a corollary we have.

Corollary 1 *Let $\mathcal{F}$ be a family contained in a locally transitive local group $\mathcal{H}$. Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are maximal chain transitive sets for $\mathcal{F}$. Then $\mathcal{E}_1 \preceq \mathcal{E}_2$ if and only if $\mathcal{E}_1 \subset \mathcal{A}(\mathcal{E}_2)$.*

Proof: Suppose $\mathcal{E}_1 \preceq \mathcal{E}_2$. By the definition of order among the $\mathcal{F}$-chain control sets we have $\mathcal{E}_1 \cap \mathcal{A}(\mathcal{E}_2) \neq \emptyset$. Applying Theorem 1 we obtain $\mathcal{E}_1 = \bigcap_{\varepsilon,A} D^1_{\varepsilon,A}$ and $\mathcal{E}_2 = \bigcap_{\varepsilon,A} D^2_{\varepsilon,A}$. Proposition 8 implies that $\mathcal{A}(\mathcal{E}_2) = \bigcap_{\varepsilon,A} \mathcal{A}(D^2_{\varepsilon,A})$. Therefore $D^1_{\varepsilon,A} \cap \mathcal{A}(D^2_{\varepsilon,A}) \neq \emptyset$ for every $\varepsilon > 0$ and $A \in \mathcal{F}$. Thus $D^1_{\varepsilon,A} \subset \mathcal{A}(D^2_{\varepsilon,A})$ (see, Proposition 2.1 in [9]) and $\mathcal{E}_1 \subset \mathcal{A}(\mathcal{E}_2)$. For the converse we observe that $\mathcal{E}_1 \subset \mathcal{A}(\mathcal{E}_2)$ implies immediately that $\mathcal{E}_1 \preceq \mathcal{E}_2$ $\square$

3. Fiber bundles

In this section we present some properties concerning the behavior of maximal chain transitive sets on principal bundles and their associated bundles. We refer to [6] and [8] for the theory of fiber bundles.

We start by settling some notation. Let G be a topological group. We start with a principal bundle $\pi : Q \to M$ with structural group G. Thus G acts freely on the right on the metric space Q and its orbits are the fibers $Q_x = \pi^{-1}\{x\}$, $x \in M$. Each fiber is homeomorphic to G. We assume always that $Q \to M$ is locally trivial.

Recall that if G acts on the left on a space F we can construct the associated bundle with typical fiber F by taking in $Q \times F$ the equivalence relation $(q_1, v_1) \sim (q_2, v_2)$ if and only if there exists $g \in G$ such that $q_2 = q_1 g$ and $v_2 = g^{-1} v_1$. Let E be the quotient space by this equivalence relation and denote by $[q, v]$ the class in E of $(q, v) \in Q \times F$. Then $[q, v] \mapsto \pi(q)$ defines a projection $E \to M$, also denoted by π or π_E if we wish to distinguish it from the projection $\pi_Q : Q \to M$ of Q. It is well known that the map $v \in F \mapsto [q, v] \in E$ establishes a bijection between F and the fiber above $x = \pi(q)$.

The associated bundle $E \to M$ is locally trivial when this happens to $Q \to M$. In locally trivial bundles over metric spaces we use the following metric.

Proposition 9 *Let $\pi : E \to M$ be a locally trivial bundle with (M, d) a metric space as well as the fiber (F, d_F). Fix a covering U_α of M with $\pi^{-1}(U_\alpha) \approx U_\alpha \times F$. Then there exists a metric d_E on E such that on each trivialization $U_\alpha \times F$ it holds*

$$d_E\left((x, v), (y, w)\right) = \max\{d(x, y), d_F(v, w)\}.$$

Also, $d(\pi e, \pi f) \leq d_E(e, f)$ for all $e, f \in E$.

Proof: See [1], [11]. $\square$

Let $\pi : Q \to M$ be a principal bundle with structure group G. An element $\phi \in \mathrm{loc}(Q)$ is called right invariant if $\phi(q.g) = \phi(q).g$, $g \in G$. We denote by $\mathrm{Aut}(Q)$ the local group of the right invariant local homeomorphisms ϕ of Q having domain $\mathrm{dom}(\phi) = \pi^{-1}(U)$ with U open in M.

Now, let $E \to M$ be a bundle associated to $Q \to M$ with typical fiber F where G acts on the left. Any $\phi \in \text{Aut}(Q)$ induces homeomorphisms on both M and E. In fact, if $y \in M$ and $y = \pi(q)$ we define elements $b(\phi) \in \text{loc}(M)$ and $e(\phi) \in \text{loc}(E)$ as

$$b(\phi)(y) = \pi(\phi(q)) \text{ and } e(\phi)([q, v]) = [\phi(q), v],$$

if $\phi \in \text{Aut}(Q)$. Usually the induced maps are also denoted by ϕ. Note that the domain of $e(\phi)$ also has the form $\pi^{-1}(U)$, $U \subset M$. The maps $e : \text{Aut}(Q) \to \text{loc}(E)$ and $b : \text{Aut}(Q) \to \text{loc}(M)$ define actions of $\text{Aut}(Q)$ on E and M, respectively. The images of e and b are local groups in the corresponding spaces. In general b is not onto $\text{loc}(M)$.

Now, let $\mathcal{S}$ be a local semigroup contained in $\text{Aut}(Q)$. The images of e and b are local semigroups in the corresponding spaces. These semigroups are denoted by $e(\mathcal{S})$ and $b(\mathcal{S})$.

Given $q \in Q$ we define the subset

$$\mathcal{S}_q = \mathcal{S}(q) \cap \pi^{-1}(x), x = \pi(q)$$

Through the identification of the fiber over x with G via $g \in G \longmapsto q.g \in \pi^{-1}(x)$, $\mathcal{S}_q$ can be viewed as a subset of G

$$\mathcal{S}_q = \{g \in G : \exists \phi \in \mathcal{S}, \phi(q) = q.g\}$$

It follows immediately that $\mathcal{S}_q$ is a subsemigroup of G if $\mathcal{S}_q \neq \emptyset$.

Suppose that $\mathcal{F}$ is a family of subsets of a local group $\mathcal{H}$. Using the maps b and e defined above the family $\mathcal{F}$ induces a family $\mathcal{F}_M$ in the local group $b(\mathcal{H})$ and a family $\mathcal{F}_E$ in $e(\mathcal{H})$. If $\mathcal{F}$ is contained in a local semigroup $\mathcal{S}$ for each $A \in \mathcal{F}$ we define $A_q = \{a \in G : \exists \phi \in A \text{ and } \phi(q) = q.a\}$. Thus we define the family $\mathcal{F}_q = \{A_q : A \in \mathcal{F}\}$ of $\mathcal{S}_q$.

The following theorem shows that maximal chain transitive sets in the total space of a fiber bundle project into maximal chain transitive sets in the base of the bundle.

Theorem 2 *Let E be a fiber bundle with projection $\pi : E \to M$. Let $\mathcal{F}$ be a family of subsets in a local group $\mathcal{H}$. Suppose that E is compact, and let $\mathcal{T} \subset E$ be a maximal chain transitive set for $\mathcal{F}_E$. Then there exists a maximal chain transitive $\mathcal{B} \subset M$ for $\mathcal{F}_M$ such that $\pi(\mathcal{T}) \subset \mathcal{B}$. For $\mathcal{F}$ contained in a local semigroup $\mathcal{S}$ one also has that $\mathcal{F}_E$-chain control sets project into $\mathcal{F}_M$-chain control sets.*

Proof: Take $\varepsilon > 0$, $A \in \mathcal{F}$ and $x', y' \in \pi(\mathcal{T})$. Pick $x, y \in \mathcal{T}$ such that $\pi(x) = x'$ and $\pi(y) = y'$. Let's show that there exists an $(\varepsilon, b(A))$-chain from x' to y'. Since E is compact, π is uniformly continuous so that there is $\delta > 0$ such that $d(\pi(z), \pi(z')) < \varepsilon$ if $d(z, z') < \delta$, $z, z' \in E$. Let $x_0 = x, x_1, ..., x_{n-1}, x_n = y$ in E together with $e(\phi_0), e(\phi_1), ..., e(\phi_{n-1})$ in $e(A)$ form a $(\delta, e(A))$-chain from x to y. Since $d(x_j, e(\phi_j)(x_j)) < \delta$ we have that $d(\pi(x_j), \pi(e(\phi_j)(x_j))) = d(\pi(x_j), b(\phi_j)(\pi(x_j))) < \varepsilon$ which shows that $\pi(x_i), b(\phi_i)$ determine a $(\varepsilon, b(A))$-chain from x' to y' and the result follows. For the chain control sets assume that $\text{int}(\mathcal{T}) \neq \emptyset$. Since π is an

open map, $\pi(\mathcal{T})$ has nonempty interior and therefore it is contained in a $\mathcal{F}_M$-chain control set. $\square$

The next theorem shows that a maximal chain transitive set for $\mathcal{F}_q$ in a fiber of a bundle is contained in a maximal chain transitive set in the total space.

Theorem 3 *Let $E \to M$ be a bundle associated to the locally trivial bundle $Q \to M$. Assume that $\mathcal{S}$ is a local semigroup contained in $\mathrm{Aut}\,(Q)$. Suppose that $\mathcal{F}$ is a family of subsets of $\mathcal{S}$ and take $q \in \pi^{-1}(x)$, $x \in M$. Assume that $\mathcal{T}$ is a maximal chain transitive set for $\mathcal{F}_q$ in the fiber F of the principal bundle. Then*

1. *Any maximal chain transitive set for $\mathcal{F}_q$ in F is contained in a maximal chain transitive set for $\mathcal{F}$ in Q.*

2. *$[q, \mathcal{T}]$ is contained in a maximal chain transitive set for $\mathcal{F}_E$ in E.*

Proof:

1. Let $\mathcal{T}$ be a maximal chain transitive set for $\mathcal{F}_q$ in F. Pick $z, z' \in \mathcal{T}$. Then for every $\varepsilon > 0$ and $A_q \in \mathcal{F}_q$ there exists $x_0 = z, x_1, \ldots, x_{n-1}, x_n = z'$ in $\pi^{-1}(x)$ and $a_0, a_1, \ldots, a_{n-1} \in A_q$ such that $d(x_j a_j, x_{j+1}) < \varepsilon$ for $j = 0, \ldots, n-1$. Let $\phi_j \in A$ be defined as $\phi_j(q) = q a_j$. Then $x_0, \ldots, x_n$ and ϕ_j, $j = 0, \ldots, n-1$ determine a $\mathcal{F}$-chain from z to z'.

2. Take $[q, v]$ and $[q, v']$ in $[q, \mathcal{T}]$. Since $\mathcal{T}$ is a $\mathcal{F}_q$-chain control set, for every $\varepsilon > 0$ and $A_q \in \mathcal{F}_q$ there exist $v_0 = v, \ldots, v_n = v'$ in F and $a_0, \ldots, a_{n-1} \in A_q$ such that $d_F(a_j v_j, v_{j+1}) < \varepsilon$ for $j = 0, \ldots, n-1$. Let $\phi_j \in A$ be defined as $\phi_j(q) = q a_j$. Then $[q, v_0], \ldots, [q, v_n]$ and $e(\phi_j)$, $j = 0, \ldots, n-1$ determine a $(\varepsilon, e(A))$-chain from $[q, v]$ to $[q, v']$. In fact, using Proposition 9, one has

$$
\begin{aligned}
d_E(E\phi_j([q, v_j]), [q, v_{j+1}]) &= d_E([q a_j, v_j], [q, v_{j+1}]) \\
&= d_E([q, a_j v_j], [q, v_{j+1}]) \\
&= d_F(a_j v_j, , v_{j+1}) \\
&< \varepsilon.
\end{aligned}
$$

and we conclude the proof.

$\square$

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