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On Nonlinear Coupled System with Nonlocal Boundary Conditions

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ABSTRACT: We discuss the existence, uniqueness and stability exponential and polynomial of global solutions for a nonlinear coupled system with nonlocal boundary conditions given by

$$\begin{array}{rcl} u_{tt}+\Delta^2 u+f(u-v) &=& 0 \quad \text{in} \quad \Omega\times(0,\infty),\\ v_{tt}-\Delta v-f(u-v) &=& 0 \quad \text{in} \quad \Omega\times(0,\infty),\\ u=0 \quad \text{on} \quad \Gamma_0\times(0,\infty), \quad -u+\displaystyle\int_0^t g_1(t-s)\mathcal{B}_2 u(s)ds &=& 0 \quad \text{on} \quad \Gamma_1\times(0,\infty),\\ \dfrac{\partial u}{\partial\nu}=0, \quad \text{on} \quad \Gamma_0\times(0,\infty), \quad \dfrac{\partial u}{\partial\nu}+\displaystyle\int_0^t g_2(t-s)\mathcal{B}_1 u(s)ds &=& 0 \quad \text{on} \quad \Gamma_1\times(0,\infty),\\ v=0 \quad \text{on} \quad \Gamma_0\times(0,\infty), \quad v+\displaystyle\int_0^t g_3(t-s)\frac{\partial v}{\partial\nu}(s)ds &=& 0 \quad \text{on} \quad \Gamma_1\times(0,\infty),\\ (u(0,x),v(0,x)=(u_0(x),v_0(x)), \quad (u_t(0,x),v_t(0,x)) &=& (u_1(x),v_1(x)) \quad \text{in} \quad \Omega \end{array}$$

where Ω is a bounded region in $\Re^2$ whose boundary is partitioned into disjoint sets Γ_0, Γ_1. We show that such dissipation is strong enough to produce uniform rate of decay. Besides, the coupled is nonlinear which brings up some additional difficulties, which makes the problem interesting.

Contents

1. Introduction

The main purpose of this work is study the asymptotic behavior of the solutions to a nonlinear coupled system with a boundary conditions of memory type. For this, let Ω be a open bounded set of $\Re^2$ with regular boundary Γ. We divide the boundary into two parts:

$$\Gamma=\Gamma_0\cup\Gamma_1 \quad \text{with} \quad \bar{\Gamma}_0\cap\bar{\Gamma}_1=\emptyset; \quad \text{and} \quad \Gamma_0\neq\emptyset. \tag{1.1}$$

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Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal to Γ, and let us denotes by $\eta = (-\nu_2, \nu_1)$ the unit tangent positively oriented on Γ. We consider the following initial boundary value problem:

$$u_{tt} + \Delta^2 u + f(u - v) = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.2}$$

$$v_{tt} - \Delta v - f(u - v) = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.3}$$

$$u = v = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty), \tag{1.4}$$

$$-u + \int_0^t g_1(t-s)\mathcal{B}_2 u(s)ds = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{1.5}$$

$$\frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\mathcal{B}_1 u(s)ds = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{1.6}$$

$$v + \int_0^t g_3(t-s)\frac{\partial v}{\partial \nu}(s)ds = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{1.7}$$

$$(u(0,x), v(0,x) = (u_0(x), v_0(x)), \quad (u_t(0,x), v_t(0,x)) = (u_1(x), v_1(x)) \quad \text{in} \; \Omega, \tag{1.8}$$

where

$$\begin{aligned} \mathcal{B}_1 u &= \Delta u + (1-\mu) B_1 u, \\ \mathcal{B}_2 u &= \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta} \end{aligned}$$

and

$$\begin{aligned} B_1 u &= 2\nu_1\nu_2 \frac{\partial^2 u}{\partial x \partial y} - \nu_1^2 \frac{\partial^2 u}{\partial y^2} - \nu_2^2 \frac{\partial^2 u}{\partial x^2}, \\ B_2 u &= (\nu_1^2 - \nu_2^2)\frac{\partial^2 u}{\partial x \partial y} + \nu_1\nu_2(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}). \end{aligned}$$

Considering the history condition, we must add to conditions (1.4)-(1.7) the one given by

$$u = \frac{\partial u}{\partial \nu} = v = 0 \quad \text{on} \quad \Gamma_0 \times]-\infty, 0].$$

Here, u and v are the transverse displacements. The relaxation functions g_i are positive and non decreasing, μ is the Poisson coeficient, with $\mu \in]0, \frac{1}{2}[$, and the function $f \in C^1(\Re)$ satisfy

$$f(s)s \geq 0 \quad \forall s \in \Re.$$

Additionally, we suppose that f is superlinear, that is

$$f(s)s \geq (2+\delta)F(s), \quad F(z) := \int_0^z f(s)ds, \quad \forall s \in \Re,$$

for some $\delta > 0$ with the following growth condition

$$|f(x) - f(y)| \le C(1 + |x|^{\rho-1} + |y|^{\rho-1})|x - y|, \quad \forall x, y \in \Re,$$

for some $C > 0$ and $\rho \ge 1$ such that $(n-2)\rho \le n$. We assume that there exists $x_0 \in \Re^2$ such that

$$\Gamma_0 = \{x \in \Gamma; \nu(x) \cdot (x - x_0) \le 0\}, \tag{1.9}$$

$$\Gamma_1 = \{x \in \Gamma; \nu(x) \cdot (x - x_0) > 0\}. \tag{1.10}$$

Denoting by $m(x) = x - x_0$, since Γ_1 is a compact set, there exists $\delta_0 \in \Re^+$ such that

$$0 < \delta_0 \le m(x) \cdot \nu(x), \quad \forall x \in \Gamma_1. \tag{1.11}$$

Dissipative linear coupled systems of the wave equations with boundary feedback was investigated by several authors, see for example [1,2,3,6,16] among others. There is not much in literature regarding the existence and asymptotic behavior of solutions of systems with memory acting on the domain or on the boundary. It is worth mentioning some papers in connection with viscoelastic effects on the domain or on the boundary. In this direction we can cite the work by Santos [14] who consider the linear coupled system with memory and proved uniform (exponential and polynomial) decay rates. Also, we can cite the article of Cavalcanti et al. [5] where it was considered a linear coupled degenerate system with boundary memory effect. In this work the authors showed that the dissipation occasioned by the memory terms was strong enough to guarantee global estimates and, consequently, to prove existence of global smooth solution and obtain exponential (or polynomial) decay provided the kernels decays exponentially (or polynomially). Concerning with subject this paper we can mention the work of Bae [4] that studied the existence of global solutions and uniform exponential decay of the following system:

$$\begin{aligned}
u_{tt} + \Delta^2 u + av + g_1(u_t) &= 0 \quad \text{on} \quad \Omega \times (0,\infty),\\
v_{tt} - \Delta v + au + g_2(v_t) &= 0 \quad \text{on} \quad \Omega \times (0,\infty),\\
u = \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \Gamma \times (0,\infty),\\
v &= 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty),\\
\frac{\partial v}{\partial \nu} + v + v_t + g(t)|v_t|^\rho v_t &= g * |v|^\rho v \quad \text{on} \quad \Gamma_1 \times (0,\infty),\\
(u(0), v(0)) = (u_0, v_0), \quad (u_t(0), v_t(0)) &= (u_1, v_1) \quad \text{on} \quad \Omega.
\end{aligned}$$

The main goal of the present paper is to complement the above mentioned works. The results are obtained for linear coupled while our paper deals with nonlinear coupled which brings up some additional difficulties. Notice that the coupled used in our work is more general than used by Bae [4]. Besides we despised the frictional damping and we just used memory effect.

As we have said before we study the asymptotic of the solutions of system (1.2)-(1.8). We show that the energy decays to zero with the same rate of decay as g_i. That is, when the relaxation functions g_i decays exponentially then the energy decays exponentially. But if g_i decays polynomially then the energy decay polynomially with the same rate. This means that the memory effect produces strong dissipation capable of making a uniform rate of decay for the energy. To see the dissipative properties of the system we have to construct a suitable functional whose derivative is negative and is equivalent to the first order energy. This functional is obtained using the multiplicative technique following Komornik [7] or Rivera [12].

Because of condition (1.4) the solution of the system (1.2)-(1.8) must belong to the following spaces:

$$W := \{w \in H^2(\Omega) : w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0\}, \quad V := \{v \in H^1(\Omega) : v = 0 \quad \text{on} \quad \Gamma_0\}.$$

The notations we use in this paper are standard and can be found in Lion's book [11]. In the sequel by C (sometime $C_1, C_2, \dots$) we denote various positive constants independent of t and on the initial data. The organization of this paper is as follows. In section 2 we establish the existence and regularity result. In section 3 prove the uniform rate of exponential decay. Finally in section 4 we prove the uniform rate of polynomial decay.

2. Notations and Main Results

In this section we present some notation and shall study the existence of regular solutions for the system (1.2)-(1.8). For this, let us define the bilinear form $a(.,.)$ as follows:

$$\begin{aligned} a(u,v) \quad = \quad & \int_\Omega \left\{ \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + \mu(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}) \right. \\ & \left. + 2(1-\mu)\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} \right\} dxdy. \end{aligned}$$

The following Lemma will be useful in that follows.

Lemma 2.1 *Let u and v be functions in $H^4(\Omega)\cap W$. Then we have*

$$\int_\Omega (\Delta^2 u) v dx = a(u,v) + \int_{\Gamma_1} \left\{ (\mathcal{B}_2 u) v - (\mathcal{B}_1 u)\frac{\partial v}{\partial \nu} \right\} d\Gamma_1. \tag{2.1}$$

Proof. ¿From Green's formula we get

$$\begin{aligned}
\int_\Omega (\Delta^2 u) v dx &= \int_{\Gamma_1} (\frac{\partial \Delta u}{\partial \nu}) v d\Gamma_1 - \int_{\Gamma_1} \Delta u \frac{\partial v}{\partial \nu} d\Gamma_1 + \int_\Omega \Delta u \Delta v dx dy \\
&= \int_{\Gamma_1} (\frac{\partial \Delta u}{\partial \nu}) v d\Gamma_1 - \int_{\Gamma_1} \Delta u \frac{\partial v}{\partial \nu} d\Gamma_1 + a(u,v) \\
&\quad +(1-\mu)\int_\Omega \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} dx dy \\
&\quad -2(1-\mu)\int_\Omega \frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} dx dy.
\end{aligned}$$

By recalling the definition of B_1 and B_2 and using

$$\begin{aligned}
&\int_\Omega \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} dx dy - 2\int_\Omega \frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} dx dy \\
&= \int_{\Gamma_1} \{ (\frac{\partial B_2 u}{\partial \eta}) v - (B_1 u)\frac{\partial v}{\partial \nu} \} d\Gamma_1,
\end{aligned}$$

our result follows. ∎

We will assume that the relaxation functions g_i are positive and we shall use equations (1.5)-(1.7) to estimate the values of $\mathcal{B}_1, \mathcal{B}_2, \frac{\partial v}{\partial \nu}$ on Γ_1. Denoting by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s) ds,$$

the convolution product operator and differentiating the equations (1.5)-(1.7) we arrive at the following Volterra equations

$$\begin{aligned}
\mathcal{B}_2 u + \frac{1}{g_1(0)} g_1' * \mathcal{B}_2 u &= \frac{1}{g_1(0)} u_t, \\
\mathcal{B}_1 u + \frac{1}{g_2(0)} g_2' * \mathcal{B}_1 u &= -\frac{1}{g_2(0)} \frac{\partial u_t}{\partial \nu}, \\
\frac{\partial v}{\partial \nu} + \frac{1}{g_3(0)} g_3' * \frac{\partial v}{\partial \nu} &= -\frac{1}{g_3(0)} v_t.
\end{aligned}$$

Applying the Volterra's inverse operator, we get

$$\begin{aligned}\mathcal{B}_2 u &= \frac{1}{g_1(0)}\{u_t + k_1 * u_t\},\\ \mathcal{B}_1 u &= -\frac{1}{g_2(0)}\left\{\frac{\partial u_t}{\partial \nu} + k_2 * \frac{\partial u_t}{\partial \nu}\right\},\\ \frac{\partial v}{\partial \nu} &= -\frac{1}{g_3(0)}\{v_t + k_3 * v_t\},\end{aligned}$$

where the resolvent kernels satisfies

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = \frac{1}{g_i(0)} g_i', \quad \forall i = 1, 2, 3.$$

Denoting by $\tau_i = \frac{1}{g_i(0)}$, $i = 1, 2, 3$, the last iqualities can be written as

$$\mathcal{B}_2 u = \tau_1\{u_t + k_1(0)u - k_1(t)u_0 + k_1' * u\}, \tag{2.2}$$

$$\mathcal{B}_1 u = \tau_2\left\{-\frac{\partial u_t}{\partial \nu} - k_2(0)\frac{\partial u}{\partial \nu} + k_2(t)\frac{\partial u_0}{\partial \nu} - k_2' * \frac{\partial u}{\partial \nu}\right\} \tag{2.3}$$

$$\frac{\partial v}{\partial \nu} = -\tau_3\{v_t + k_3(0)v - k_3(t)v_0 + k_3' * v\}. \tag{2.4}$$

Reciprocally, taking initial data such that $u_0 = v_0 = \frac{\partial u_0}{\partial \nu} = 0$ on Γ_1, the identities (2.2)-(2.4) imply (1.5)-(1.7). Since we are interested in relaxation functions of exponential or polynomial type and the identities (2.2)-(2.4) involve the resolvent kernels k_i, we want to know whether k_i has the same properties. The following Lemma answers this question.

Lemma 2.2 *If h is a positive continuous function, then k also is a positive continuous function. Moreover,*

1. *If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that*

$$h(t) \leq c_0 e^{-\gamma t},$$

then, the function k satisfies

$$k(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},$$

for all $0 < \epsilon < \gamma - c_0$.

2. *Given $p > 1$, let us denote by $c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p}\, ds$. If there exists a positive constant c_0 with $c_0 c_p < 1$ such that*

$$h(t) \leq c_0 (1+t)^{-p},$$

then, the function k satisfies

$$k(t) \leq \frac{c_0}{1 - c_0 c_p}(1+t)^{-p}.$$

Proof. See e. g. [15, Lemma 2.1]

Remark: The finiteness of the constant c_p can be found in [13, Lemma 7.4].

Due to this Lemma, in the remainder of this paper, we shall use (2.2)-(2.4) instead of (1.5)- (1.7). Let us denote by

$$(g\Box\varphi)(t) \quad = \quad \int_0^t g(t-s)|\varphi(t)-\varphi(s)|^2 ds.$$

The next Lemma gives a identity for the convolution.

Lemma 2.3 *For $g, \varphi \in C^1([0,\infty[: \Re)$ we have*

$$\begin{aligned}\int_0^t g(t-s)\varphi(s)ds\varphi_t \quad &= \quad -\frac{1}{2}g(t)|\varphi(t)|^2 + \frac{1}{2}g'\Box\varphi \\ &\quad -\frac{1}{2}\frac{d}{dt}\left[g\Box\varphi - (\int_0^t g(s)ds)|\varphi|^2\right].\end{aligned}$$

The proof of this lemma follows by differentiating the term $g\Box\varphi$.

To show the regularity result we will use the following Lemma,

Lemma 2.4 *Suppose that $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\Gamma_1)$ and $h \in H^{\frac{3}{2}}(\Gamma_1)$ then, any solution of*

$$a(u,w) = \int_\Omega fwdx + \int_{\Gamma_1} gwd\Gamma_1 + \int_{\Gamma_1} h\frac{\partial w}{\partial\nu}d\Gamma_1, \quad \forall w \in W$$

satisfies

$$u \in H^4(\Omega)$$

and also:

$$\begin{aligned}&\Delta^2 u = f,\\ &u = \frac{\partial u}{\partial \nu} = 0 \quad on \quad \Gamma_0\\ &\mathcal{B}_1 u = h, \quad \mathcal{B}_2 u = g \quad on \quad \Gamma_1.\end{aligned}$$

Proof. See, e.g., [10]

Since $\Gamma_0 \neq \emptyset$, Korn's Lemma implies that $\sqrt{a(.,.)}$ is a norm equivalent to the usual

Sobolev norm $||\cdot||_{H^2}$ on W. Let us introduce the energy function

$$\begin{aligned}E(t) := E(t,u,v) &= \frac{1}{2}\{\int_\Omega (|u_t|^2 + |v_t|^2 + |\nabla v|^2)dx + a(u,u) + 2\int_\Omega F(u-v)dx \\ &+\tau_1 \int_{\Gamma_1} (k_1(t)|u|^2 - k_1'\Box u)d\Gamma_1 \\ &+\tau_2 \int_{\Gamma_1} (k_2(t)|\frac{\partial u}{\partial \nu}|^2 - k_2'\Box \frac{\partial u}{\partial \nu})d\Gamma_1 \\ &+\tau_3 \int_{\Gamma_1} (k_3(t)|v|^2 - k_3'\Box v)d\Gamma_1\};\end{aligned}$$

and let us denote by $\{w_i \in W : i \in \mathbb{N}\}$ an orthonormal basis of W. In these conditions we are able to prove the existence of strong solutions.

Theorem 2.1 *Let $k_i \in C^2(\Re^+)$ be such that*

$$k_i, -k_i', k_i'' \geq 0, \quad \forall i = 1,2,3.$$

If $(u_0, v_0) \in (H^4(\Omega)\cap W) \times (H^2(\Omega)\cap V)$ and $(u_1, v_1) \in W \times V$ satisfy the compatibility conditions

$$\mathcal{B}_2 u_0 - \tau_1 u_1 = 0, \quad on \quad \Gamma_1 \tag{2.5}$$

$$\mathcal{B}_1 u_0 + \tau_2 \frac{\partial u}{\partial \nu} = 0, \quad on \quad \Gamma_1 \tag{2.6}$$

$$\frac{\partial v_0}{\partial \nu} + \tau_3 v_1 = 0, \quad on \quad \Gamma_1 \tag{2.7}$$

then there exists only one strong solution (u,v) for equations (1.2)-(1.8) satisfying:

$$u \in L^\infty(0,T; H^4(\Omega)\cap W) \cap W^{1,\infty}(0,T;W) \cap W^{2,\infty}(0,T;L^2(\Omega))$$

$$v \in L^\infty(0,T; H^2(\Omega)\cap V) \cap W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;L^2(\Omega)).$$

In addition, considering the assumption (1.11) and assuming that there exist positive constants b_1, b_2 such that

$$k_i(0) > 0, \quad k_i'(t) \leq -b_1 k_i(t), \quad k_i''(t) \geq -b_2 k_i'(t), \quad or \tag{2.8}$$

$$k_i(0) > 0, \quad k_i'(t) \leq -b_1 k_i^{1+\frac{1}{p}}, \quad k_i''(t) \geq b_2[-k_i(t)]^{1+\frac{1}{p+1}}, \quad \forall i = 1,2,3, \quad p > 1, \tag{2.9}$$

then, the energy $E(t)$ associated to problem (1.2)-(1.8) decays, respectively, with the following rates of decay

$$E(t) \leq \alpha_1 e^{-\alpha_2 t} E(0), \tag{2.10}$$

$$E(t) \leq \frac{C}{(1+t)^{p+1}} E(0), \tag{2.11}$$

where α_1, α_2 and C are positive constants.

Proof. Our starting point is to construct the Galerkin aproximation (u^m, v^m) of the solution

$$(u^m(\cdot,t), v^m(\cdot,t)) = \sum_{j=1}^{m}(h_{j,m}(t), \psi_{j,m}(t))w_j(\cdot),$$

which is given by the solution of the aproximated equations

$$\begin{aligned}
&\int_\Omega u^m_{tt} w_j dx + a(u^m, w_j) + \int_\Omega f(u-v) w_j dx \\
&= -\tau_1 \int_{\Gamma_1} \{u^m_t + k_1(0)u^m - k_1(t)u_{0,m} + k_1' * u^m\} w_j d\Gamma_1 \\
&+\tau_2 \int_{\Gamma_1} \{-\frac{\partial u^m_t}{\partial \nu} - k_2(0)\frac{\partial u^m}{\partial \nu} + k_2(t)\frac{\partial u_{0,m}}{\partial \nu} - k_2' * \frac{\partial u^m}{\partial \nu}\}\frac{\partial w_j}{\partial \nu} d\Gamma_1 \qquad (2.12)\\
&\int_\Omega v^m_{tt} w_j dx + \int_\Omega \nabla v^m \cdot \nabla w_j dx - \int_\Omega f(u^m - v^m) w_j dx \\
&= -\tau_3 \int_{\Gamma_1} \{v^m_t + k_3(0)v^m - k_3(t)v^m(0) + k_3' * v^m\} w_j d\Gamma_1 \qquad (2.13)\\
&(u^m(\cdot,0), v^m(\cdot,0)) = (u_{0,m}, v_{0,m}); \quad (u^m_t(\cdot,0), v^m_t(\cdot,0)) = (u_{1,m}, v_{1,m}) \qquad (2.14)
\end{aligned}$$

where

$$(u_{0,m}, v_{0,m}) = (u_0, v_0), \quad (u_{1,m}, v_{1,m} = (u_1, v_1).$$

Note that we can be solve the system (2.12)-(2.14) by Picard's method. In fact, the system (2.12)-(2.14) have unique solution on some interval $[0, T_m)$. The extension of the solution to the whole interval $[0, \infty)$ is a consequence of the first estimate which we are going to prove below.

A priori estimate I

To this end, let us multiply equation (2.12) by $h'_{j,m}$ and equation (2.13) by $\psi'_{j,m}$, summing up the product results in j and using Lemma 2.3 we conclude that

$$\frac{d}{dt}E(t, u^m, v^m) \le CE(0, u^m, v^m).$$

Integrating the last inequality we obtain

$$E(t, u^m, v^m) \le CE(0, u^m, v^m).$$

¿From our choice of $(u_{0,m}, v_{0,m})$ and $(u_{1,m}, v_{1m})$ it follows that

$$E(t, u^m, v^m) \le C, \quad \forall t \in [0,T], \quad \forall m \in \mathbb{N}. \qquad (2.15)$$

A priori estimate II

Next, we shall find a estimate for the second order energy. First, let us estimate the initial data $u_{tt}^m(0)$ and $v_{tt}^m(0)$ in the L^2-norm. Letting $t \to 0^+$ in the equations (2.12)-(2.13), multiplying the result by $h''_{j,m}(0)$ and $\psi''_{j,m}(0)$, respectively, and using the compatibility conditions (2.5)-(2.7) we get

$$\int_\Omega |u_{tt}^m(0)|^2 \, dx = -\int_\Omega \Delta^2 u_0 u_{tt}^m(0) dx - \int_\Omega f(u_0 - v_0) u_{tt}^m(0) dx,$$
$$\int_\Omega |v_{tt}^m(0)|^2 \, dx = \int_\Omega \Delta v_0 v_{tt}^m(0) dx + \int_\Omega f(u_0 - v_0) v_{tt}^m(0) dx.$$

Since $(u_0, v_0) \in [H^2(\Omega)]^2$, the growth hypothesis for the function f together with the Sobolev's imbedding imply that $f(u_0 - v_0) \in L^2(\Omega)$. Hence

$$\int_\Omega \left(|u_{tt}^m(0)|^2 + |v_{tt}^m(0)|^2\right) dx \le M_1, \quad \forall m \in \mathbb{N}. \tag{2.16}$$

Differentiating the equations (2.12)-(2.13) with respect to the time, we get

$$\begin{aligned}
&\int_\Omega u_{ttt}^m w_j dx + a(u_t^m, w_j) + \int_\Omega f'(u^m - v^m)(u_t^m - v_t^m) dx \\
&= -\tau_1 \int_{\Gamma_1} \{u_{tt}^m + k_1(0) u_t^m + k_1' * u_t^m\} w_j d\Gamma_1 \\
&+\tau_2 \int_{\Gamma_1} \{-\frac{\partial u_{tt}^m}{\partial \nu} - k_2(0) \frac{\partial u_t^m}{\partial \nu} - k_2' * \frac{\partial u_t^m}{\partial \nu}\} \frac{\partial w_j}{\partial \nu} d\Gamma_1,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&\int_\Omega (v_{ttt}^m w_j + \nabla v_t^m \cdot \nabla w_j - f'(u^m - v^m)(u_t^m - v_t^m)) dx \\
&= -\tau_3 \int_{\Gamma_1} \{v_{tt}^m + k_3(0) v_t^m + k_3' * v_t^m\} w_j d\Gamma_1.
\end{aligned} \tag{2.18}$$

Multiplying (2.17) by $h''_{j,m}$ and (2.18) by $\psi''_{j,m}$, summing up the product result in j we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \{|u_{tt}|^2+|v_{tt}|^2+|\nabla v_t|^2\}dx+\frac{1}{2}\frac{d}{dt}a(u_t^m,u_t^m)\\
&=-\int_\Omega f'(u^m-v^m)(u_t^m-v_t^m)u_{tt}^m dx\\
&+\int_\Omega f'(u^m-v^m)(u_t^m-v_t^m)v_{tt}^m dx\\
&-\tau_1\int_{\Gamma_1}\{|u_{tt}^m|^2+k_1(0)u_t^m u_{tt}^m+k_1'*u_t^m u_{tt}^m\}d\Gamma_1\\
&+\tau_2\int_{\Gamma_1}\{-|\frac{\partial u_{tt}^m}{\partial\nu}|^2-k_2(0)\frac{\partial u_t^m}{\partial\nu}\frac{\partial u_{tt}^m}{\partial\nu}-k_2'*\frac{\partial u_t^m}{\partial\nu}\frac{\partial u_{tt}^m}{\partial\nu}\}d\Gamma_1\\
&-\tau_3\int_{\Gamma_1}\{|v_{tt}^m|^2+k_3(0)v_t^m v_{tt}^m+k_3'*v_t^m v_{tt}^m\}d\Gamma_1. \qquad (2.19)
\end{aligned}
$$

Using the Lemma 2.3 and denoting by

$$
\begin{aligned}
E_0(t,u,v) \quad = \quad & \frac{1}{2}\int_\Omega(|u_t|^2+|v_t|^2+|\nabla v|^2)dx+\frac{1}{2}a(u,u)\\
&+\frac{\tau_1}{2}\int_{\Gamma_1}(k_1(t)|u|^2-k_1'\Box u)d\Gamma_1\\
&+\frac{\tau_2}{2}\int_{\Gamma_1}(k_2(t)|\frac{\partial u}{\partial\nu}|^2-k_2'\Box\frac{\partial u}{\partial\nu})d\Gamma_1\\
&+\frac{\tau_3}{2}\int_{\Gamma_1}(k_3(t)|v|^2-k_3'\Box v)d\Gamma_1
\end{aligned}
$$

we find that

$$
\begin{aligned}
\frac{d}{dt}E_0(t,u_t^m,v_t^m) \quad = \quad & -\tau_1\int_{\Gamma_1}|u_{tt}^m|^2 d\Gamma_1+\frac{\tau_1}{2}\int_{\Gamma_1}(k_1'(t)|u_t^m|^2-k_1''\Box u_t^m)d\Gamma_1\\
&-\tau_2\int_{\Gamma_1}|\frac{\partial u_{tt}^m}{\partial\nu}|^2 d\Gamma_1+\frac{\tau_2}{2}\int_{\Gamma_1}(k_2'(t)|\frac{\partial u_t^m}{\partial\nu}|^2-k_2''\Box\frac{\partial u_t^m}{\partial\nu})d\Gamma_1\\
&-\tau_3\int_{\Gamma_1}|v_{tt}^m|^2 d\Gamma_1+\frac{\tau_3}{2}\int_{\Gamma_1}(k_3'(t)|v_t^m|^2-k_3''\Box v_t^m)d\Gamma_1\\
&-\int_\Omega f'(u^m-v^m)(u_t^m-v_t^m)u_{tt}^m dx\\
&+\int_\Omega f'(u^m-v^m)(u_t^m-v_t^m)v_{tt}^m dx. \qquad (2.20)
\end{aligned}
$$

Let us take $p_n = \frac{2n}{n-2}$. From the growth condition of the function f and the Sobolev imbedding we have

$$\begin{aligned}
&\int_\Omega f'(u^m - v^m) u_t^m u_{tt}^m dx \le c\int_\Omega (1+2|u^m - v^m|^{\rho-1})|u_t^m||u_{tt}^m| dx \\
&\le c\left[\int_\Omega (1+2|u^m - v^m|^{\rho-1})^n dx\right]^{\frac{1}{n}}\left[\int_\Omega |u_t^m|^{p_n} dx\right]^{\frac{1}{p_n}}\left[\int_\Omega |u_{tt}^m|^2 dx\right]^{\frac{1}{2}} \\
&\le c\left[\int_\Omega (1+|\nabla u^m - \nabla v^m|^2) dx\right]^{\frac{\rho-1}{2}}\left[\int_\Omega |\nabla u_t^m|^2 dx\right]^{\frac{1}{2}}\left[\int_\Omega |u_{tt}^m|^2 dx\right]^{\frac{1}{2}}.
\end{aligned}$$

Taking into account the first estimate (2.15) we conclude that

$$\begin{aligned}
\int_\Omega f_1'(u^m) u_t^m u_{tt}^m dx \quad &\le \quad c\left[\int_\Omega |\nabla u_t^m|^2 dx\right]^{\frac{1}{2}}\left[\int_\Omega |u_{tt}^m|^2 dx\right]^{\frac{1}{2}} \\
&\le c\left\{\int_\Omega |\nabla u_t^m|^2 dx + \int_\Omega |u_{tt}^m|^2 dx\right\}. \qquad (2.21)
\end{aligned}$$

Similarly we get

$$-\int_\Omega f'(u^m - v^m) v_t^m u_{tt}^m dx \le c\left\{\int_\Omega |\nabla v_t^m|^2 dx + \int_\Omega |u_{tt}^m|^2 dx\right\}, \qquad (2.22)$$

$$\int_\Omega f'(u^m - v^m) u_t^m v_{tt}^m dx \le c\left\{\int_\Omega |\nabla u_t^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx\right\}, \qquad (2.23)$$

$$-\int_\Omega f'(u^m - v^m) v_t^m v_{tt}^m dx \le c\left\{\int_\Omega |\nabla v_t^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx\right\}. \qquad (2.24)$$

Substitution of the inequalities (2.21)-(2.24) into (2.20) we get

$$\frac{d}{dt} E_0(t, u_t^m, v_t^m) \quad \le \quad c_2\left\{\int_\Omega |u_{tt}^m|^2 + |\nabla u_t^m|^2 dx + \int_\Omega |v_{tt}^m|^2 + |\nabla v_t^m|^2 dx\right\}.$$

Integrating with respect to the time and applying Gronwall's inequality we conclude that

$$E_0(t, u_t^m, v_t^m) \le c, \quad \forall t \in [0,T], \quad \forall m \in \mathbb{N}. \qquad (2.25)$$

¿From (2.15) and (2.25) we obtain

$$u^m \quad \text{is bounded in} \quad L^\infty(0,T; H^2(\Omega)), \qquad (2.26)$$

$$u_t^m \quad \text{is bounded in} \quad L^\infty(0,T; H^2(\Omega)), \qquad (2.27)$$

$$u_{tt}^m \quad \text{is bounded in} \quad L^\infty(0,T; L^2(\Omega)), \qquad (2.28)$$

$$v^m \quad \text{is bounded in} \quad L^\infty(0,T; V), \qquad (2.29)$$

$$v_t^m \quad \text{is bounded in} \quad L^\infty(0,T; V), \qquad (2.30)$$

$$v_{tt}^m \quad \text{is bounded in} \quad L^\infty(0,T; L^2(\Omega)). \qquad (2.31)$$

Letting $m \to \infty$ in the eq. (2.12) we get

$$\begin{aligned} a(u,w) &= -\int_\Omega (u_{tt} + f(u-v))w dx - \tau_1 \int_{\Gamma_1} \{u_t + k_1(0)u - k_1(t)u_0 + k_1' * u\} w d\Gamma_1 \\ &\quad + \tau_2 \int_{\Gamma_1} \left\{ -\frac{\partial u_t}{\partial \nu} - k_2(0)\frac{\partial u}{\partial \nu} + k_2(t)\frac{\partial u_0}{\partial \nu} - k_2' * \frac{\partial u}{\partial \nu} \right\} \frac{\partial w}{\partial \nu} d\Gamma_1, \end{aligned}$$

for any $w \in W$. From Lemma 2.4 we get that

$$u \in L^\infty(0,T; H^4(\Omega)).$$

Moreover we have that u, v verify the system (1.2)-(1.8) in the strong sense. The uniqueness follows by standard method for hyperbolic equations. The present proof is now complete. ■

3. Uniform Exponential Decay

In this section we shall study the asymptotic behavior of the solutions of system (1.2)-(1.8). For this, we will use the following Lemmas.

Lemma 3.1 *For every $\varphi \in H^4(\Omega)$ and for every $\mu \in \mathbb{R}$, we have*

$$\begin{aligned} \int_\Omega (m \cdot \nabla \varphi)\Delta^2 \varphi dx &= a(\varphi,\varphi) + \frac{1}{2}\int_\Gamma m \cdot \nu \left[\left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2 + \left(\frac{\partial^2 \varphi}{\partial y^2}\right)^2 \right. \\ &\quad \left. + 2\mu \frac{\partial^2 \varphi}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 \varphi}{\partial x \partial y}\right)^2 \right] d\Gamma \\ &\quad + \int_\Gamma [(\mathcal{B}_2 \varphi) m \cdot \nabla \varphi - (\mathcal{B}_1 \varphi)\frac{\partial}{\partial \nu}(m \cdot \nabla \varphi)] d\Gamma. \end{aligned}$$

Proof. See, e.g., [8].

Lemma 3.2 *For any strong solution* (u, v) *of system (1.2)-(1.8) we have*

$$\begin{aligned}\frac{d}{dt}E(t) \quad \leq \quad & -\frac{\tau_1}{2}\int_{\Gamma_1}|u_t|^2 d\Gamma_1 + \frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma_1 + \frac{\tau_1}{2}k_1'(t)\int_{\Gamma_1}|u|^2 d\Gamma_1 \\ & -\frac{\tau_1}{2}\int_{\Gamma_1}k_1''(t)\Box u d\Gamma_1 - \frac{\tau_2}{2}\int_{\Gamma_1}|\frac{\partial u_t}{\partial \nu}|^2 d\Gamma_1 + \frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_1}|\frac{\partial u_0}{\partial \nu}|^2 d\Gamma_1 \\ & +\frac{\tau_2}{2}k_2'(t)\int_{\Gamma_1}|\frac{\partial u}{\partial \nu}|^2 d\Gamma_1 - \frac{\tau_2}{2}\int_{\Gamma_1}k_2''\Box\frac{\partial u}{\partial \nu}d\Gamma_1 \\ & -\frac{\tau_3}{2}\int_{\Gamma_1}|v_t|^2 d\Gamma_1 + \frac{\tau_3}{2}k_3^2(t)\int_{\Gamma_1}|v_0|^2 d\Gamma_1 + \frac{\tau_3}{2}k_3'(t)\int_{\Gamma_1}|v|^2 d\Gamma_1 \\ & -\frac{\tau_3}{2}\int_{\Gamma_1}k_3''(t)\Box v d\Gamma_1.\end{aligned}$$

Proof. Multiplying the equations (1.2) by u_t and (1.3) by v_t, integrating over Ω, summing up the product result and using Lemma 2.1 we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\int_\Omega(|u_t|^2 + |v_t|^2 + |\nabla v|^2 + 2F(u-v))dx + a(u,u)\} \\ &= -\int_{\Gamma_1}(\mathcal{B}_2 u)u_t d\Gamma_1 + \int_{\Gamma_1}(\mathcal{B}_1 u)\frac{\partial u_t}{\partial \nu}d\Gamma_1 + \int_{\Gamma_1}v_t\frac{\partial v}{\partial \nu}d\Gamma_1.\end{aligned}$$

Substituting the boundary conditions (2.2)-(2.4) and using Lemma 2.3 our conclusion follows. ■

Let us consider the following binary operator

$$(k \diamond \varphi)(t) := \int_0^t k(t-s)(\varphi(t) - \varphi(s))ds.$$

Then applying the Schwarz inequality for $0 \leq \mu \leq 1$ we have

$$|(k \diamond \varphi)(t)|^2 \leq \left[\int_0^t |k(s)|^{2(1-\mu)}ds\right](|k|^{2\mu}\Box\varphi)(t). \tag{3.1}$$

Let us introduce the following functionals

$$\mathcal{N}(t) := \int_\Omega(|u_t|^2 + |v_t|^2 + |\nabla v|^2 + F(u-v))dx + a(u,u),$$

$$\psi(t) := \int_\Omega\{(m\cdot\nabla u) + (1-\theta)u\}u_t dx + \int_\Omega\{(m\cdot\nabla v) + (1-\theta)v\}v_t dx$$

where θ is a small positive constant. The following Lemma plays an important role for the construction of the Lyapunov functional.

Lemma 3.3 *For any strong solution (u,v) of system (1.2)-(1.8) we get*

$$\begin{aligned}
\frac{d}{dt}\psi(t) \quad \leq \quad & -\frac{\theta}{2}\mathcal{N}(t)+\frac{1}{2}\int_{\Gamma_1} m\cdot\nu(|u_t|^2+|v_t|^2)d\Gamma_1 \\
& -\frac{1}{2}\int_{\Gamma_1} m\cdot\nu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\left(\frac{\partial^2 u}{\partial y^2}\right)^2+2\mu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right]d\Gamma_1 \\
& -\int_{\Gamma_1}(\mathcal{B}_2 u)m\cdot\nabla u d\Gamma_1+\int_{\Gamma_1}(\mathcal{B}_1 u)\frac{\partial}{\partial\nu}(m\cdot\nabla u)d\Gamma_1 \\
& +C\int_{\Gamma_1}(|u_t|^2+|k_1(t)u|^2+|k_1'\diamond u|^2+|k_1(t)u_0|^2)d\Gamma_1 \\
& +C\int_{\Gamma_1}(|\frac{\partial v_t}{\partial\nu}|^2+|k_2(t)\frac{\partial v}{\partial\nu}|^2+|k_2'\diamond\frac{\partial v}{\partial\nu}|^2+|k_2(t)\frac{\partial v_0}{\partial\nu}|^2)d\Gamma_1 \\
& +C\int_{\Gamma_1}(|v_t|^2+|k_3(t)v|^2+|k_3'\diamond v|^2+|k_3(t)v_0|^2)d\Gamma_1
\end{aligned}$$

for some positive constant C.

Proof. Using the equation (1.2) and Lemma 3.1 we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\{(m\cdot\nabla u)+(1-\theta)u\}u_t dx=\frac{1}{2}\int_{\Gamma_1} m\cdot\nu|u_t|^2 d\Gamma_1-\theta\int_\Omega|u_t|^2dx-(2-\theta)a(u,u) \\
&-\frac{1}{2}\int_{\Gamma} m\cdot\nu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\left(\frac{\partial^2 u}{\partial y^2}\right)^2+2\mu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right]d\Gamma \\
&-\int_\Gamma[(\mathcal{B}_2 u)m\cdot\nabla u-(\mathcal{B}_1 u)\frac{\partial}{\partial\nu}(m\cdot\nabla u)]d\Gamma-(1-\theta)\int_{\Gamma_1}[(\mathcal{B}_2 u)u-(\mathcal{B}_1 u)\frac{\partial u}{\partial\nu}]d\Gamma_1 \\
&-\int_\Omega(m\cdot\nabla u)f(u-v)dx-(1-\theta)\int_\Omega f(u-v)udx. \qquad (3.2)
\end{aligned}$$

Let us next examine the integrals over Γ_0 in (3.2). Since $u=\frac{\partial u}{\partial\nu}=0$ on Γ_0 we have $B_1u=B_2u=0$ there and

$$\frac{\partial}{\partial\nu}(m\cdot\nabla u)=(m\cdot\nu)\frac{\partial^2 u}{\partial\nu^2}=(m\cdot\nu)\Delta u, \qquad (3.3)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\left(\frac{\partial^2 u}{\partial y^2}\right)^2+2\mu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2=(\Delta u)^2 \qquad (3.4)$$

on Γ_0

since $\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 0$ on Γ_0. Observing (3.3)-(3.4) and noting that $m \cdot \nu \leq 0 \quad \text{on} \quad \Gamma_0$ we obtain

$$\frac{d}{dt}\int_\Omega \{(m\cdot\nabla u) + (1-\theta)u\}u_t dx \leq \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|u_t|^2 d\Gamma_1 - \theta\int_\Omega |u_t|^2 dx - (2-\theta)a(u,u)$$
$$-\frac{1}{2}\int_{\Gamma_1} m\cdot\nu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right] d\Gamma_1$$
$$-\int_{\Gamma_1}[(\mathcal{B}_2 u)m\cdot\nabla u - (\mathcal{B}_1 u)\frac{\partial}{\partial\nu}(m\cdot\nabla u)]d\Gamma_1 - (1-\theta)\int_{\Gamma_1}[(\mathcal{B}_2 u)u - (\mathcal{B}_1 u)\frac{\partial u}{\partial\nu}]d\Gamma_1$$
$$-\int_\Omega (m\cdot\nabla u)f(u-v)dx - (1-\theta)\int_\Omega f(u-v)u dx. \tag{3.5}$$

Using the equation (1.3) and performing a integration by parts we get

$$\frac{d}{dt}\int_\Omega \{m\cdot\nabla v + (1-\theta)v\}v_t dx \leq \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|v_t|^2 d\Gamma_1 - \theta\int_\Omega |v_t|^2 dx$$
$$+\int_{\Gamma_1}\frac{\partial v}{\partial\nu}\{m\cdot\nabla v + (1-\theta)v\}d\Gamma_1 - \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|\nabla v|^2 d\Gamma_1 - (1-\theta)\int_\Omega |\nabla v|^2 dx$$
$$+\int_\Omega f(u-v)m\cdot\nabla v dx + (1-\theta)\int_\Omega f(u-v)v dx. \tag{3.6}$$

Summing the inequalities (3.5), (3.6) and taking into account that f is superlinear we arrive at

$$\frac{d}{dt}\psi(t) \leq \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|u_t|^2 d\Gamma_1 - \theta\int_\Omega |u_t|^2 dx - (2-\theta)a(u,u)$$
$$-\frac{1}{2}\int_{\Gamma_1} m\cdot\nu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right] d\Gamma_1$$
$$-\int_{\Gamma_1}[(\mathcal{B}_2 u)m\cdot\nabla u - (\mathcal{B}_1 u)\frac{\partial}{\partial\nu}(m\cdot\nabla u)]d\Gamma_1 - (1-\theta)\int_{\Gamma_1}[(\mathcal{B}_2 u)u - (\mathcal{B}_1 u)\frac{\partial u}{\partial\nu}]d\Gamma_1$$
$$+\frac{1}{2}\int_{\Gamma_1} m\cdot\nu|v_t|^2 d\Gamma_1 - \theta\int_\Omega |v_t|^2 dx + \int_{\Gamma_1}\frac{\partial v}{\partial\nu}\{m\cdot\nabla v + (1-\theta)v\}d\Gamma_1$$
$$-\frac{1}{2}\int_{\Gamma_1} m\cdot\nu|\nabla v|^2 d\Gamma_1 - (1-\theta)\int_\Omega |\nabla v|^2 dx - (\delta - (2+\delta)\theta)\int_\Omega F(u-v)dx.$$

Taking θ small enough we obtain

$$\frac{d}{dt}\psi(t) \leq -\theta\mathcal{N}(t) + \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|u_t|^2 d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|v_t|^2 d\Gamma_1$$
$$-\frac{1}{2}\int_{\Gamma_1} m\cdot\nu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right] d\Gamma_1$$
$$-\int_{\Gamma_1}[(\mathcal{B}_2 u)m\cdot\nabla u - (\mathcal{B}_1 u)\frac{\partial}{\partial\nu}(m\cdot\nabla u)]d\Gamma_1 - (1-\theta)\int_{\Gamma_1}[(\mathcal{B}_2 u)u - (\mathcal{B}_1 u)\frac{\partial u}{\partial\nu}]d\Gamma_1$$
$$+\int_{\Gamma_1}\frac{\partial v}{\partial\nu}\{m\cdot\nabla v + (1-\theta)v\}d\Gamma_1 - \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|\nabla v|^2 d\Gamma_1. \tag{3.7}$$

Now, we analyze some boundary term of the above inequality. Applying Young and Poincaré's inequalities we have, for $\epsilon > 0$

$$\int_{\Gamma_1}\frac{\partial v}{\partial\nu}\{m\cdot\nabla v + (1-\theta)v\}d\Gamma_1$$
$$\leq \epsilon\int_{\Gamma_1}\{|m\cdot\nabla v|^2 + (1-\theta)^2|v|^2\}d\Gamma_1 + C_\epsilon\int_{\Gamma_1}\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma$$
$$\leq \epsilon C\{\int_{\Gamma_1} m\cdot\nu|\nabla v|^2 d\Gamma_1 + \mathcal{N}(t) + C_\epsilon\int_{\Gamma_1}\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma_1.$$

Similarly, we obtain

$$-(1-\theta)\int_{\Gamma_1}[(\mathcal{B}_2 u)u - (\mathcal{B}_1 u)\frac{\partial u}{\partial\nu}]d\Gamma_1 \leq \epsilon\mathcal{N}(t) + C_\epsilon\int_{\Gamma_1}[|\mathcal{B}_2 u|^2 + |\mathcal{B}_1 u|^2]d\Gamma_1.$$

By substitution of these last inequalities into (3.7) with ϵ small and taking into account that the boundary conditions (2.2)-(2.4) can be written as

$$\begin{aligned}
\mathcal{B}_2 u &= \tau_1\{u_t + k_1(t)u - k_1'\diamond u - k_1(t)u_0\},\\
\mathcal{B}_1 u &= -\tau_2\{\frac{\partial u_t}{\partial\nu} + k_2(t)\frac{\partial v}{\partial\nu} - k_2'(t)\diamond\frac{\partial v}{\partial\nu} - k_2(t)\frac{\partial v_0}{\partial\nu}\},\\
\frac{\partial v}{\partial\nu} &= -\tau_3\{v_t + k_3(t)v - k_3'\diamond v - k_3(t)v_0\}
\end{aligned}$$

our conclusion follows. ■

Let us introduce the functional

$$\mathcal{L}(t) = NE(t) + \psi(t), \tag{3.8}$$

with $N > 0$. Using Young's inequality and taking N large enough we find that

$$q_0 E(t) \leq \mathcal{L}(t) \leq q_1 E(t), \tag{3.9}$$

for some positive constants q_0 and q_1. We will show later that the functional $\mathcal{L}$ satisfies the inequality of the following Lemma

Lemma 3.4 *Let f be a real positive function of class C^1. If there exists positive constants γ_0, γ_1 and c_0 such that*

$$f'(t) \le -\gamma_0 f(t) + c_0 e^{-\gamma_1 t},$$

then there exist positive constants γ and c such that

$$f(t) \le (f(0)+c)e^{-\gamma t}.$$

Proof. See e. g. [14, Lemma 3.4].

Finally, we shall show the inequality (2.10). Using the Lemma 3.2 and Lemma 3.3 we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \le & -\frac{N\tau_1}{2}\int_{\Gamma_1}|u_t|^2 d\Gamma_1 + \frac{N\tau_1}{2}k_1^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma_1 + \frac{N\tau_1}{2}k_1'(t)\int_{\Gamma_1}|u|^2 d\Gamma_1 \\
& -\frac{N\tau_1}{2}\int_{\Gamma_1}k_1''(t)\Box u d\Gamma_1 - \frac{N\tau_2}{2}\int_{\Gamma_1}|\frac{\partial u_t}{\partial\nu}|^2 d\Gamma_1 + \frac{N\tau_2}{2}k_2^2(t)\int_{\Gamma_1}|\frac{\partial u_0}{\partial\nu}|^2 d\Gamma_1 \\
& +\frac{N\tau_2}{2}k_2'(t)\int_{\Gamma_1}|\frac{\partial u}{\partial\nu}|^2 d\Gamma_1 - \frac{N\tau_2}{2}\int_{\Gamma_1}k_2''\Box\frac{\partial u}{\partial\nu}d\Gamma_1 - \frac{N\tau_3}{2}\int_{\Gamma_1}|v_t|^2 d\Gamma_1 \\
& +\frac{N\tau_3}{2}k_3^2(t)\int_{\Gamma_1}|v_0|^2 d\Gamma_1 + \frac{N\tau_3}{2}k_3'(t)\int_{\Gamma_1}|v|^2 d\Gamma_1 - \frac{N\tau_3}{2}\int_{\Gamma_1}k_3''(t)\Box v d\Gamma_1 \\
& -\frac{\theta}{2}\mathcal{N}(t) + \frac{1}{2}\int_{\Gamma_1}m\cdot\nu(|u_t|^2+|v_t|^2)d\Gamma_1 + \frac{\epsilon_1}{2}\int_{\Gamma_1}(|m\cdot\nabla u|^2 + |\frac{\partial}{\partial\nu}(m\cdot\nabla u)|^2)d\Gamma_1 \\
& -\frac{1}{2}\int_{\Gamma_1}m\cdot\nu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right]d\Gamma_1 \\
& +C\int_{\Gamma_1}(|u_t|^2 + |k_1(t)u|^2 + |k_1'\diamond u|^2 + |k_1(t)u_0|^2)d\Gamma_1 \\
& +C\int_{\Gamma_1}(|\frac{\partial v_t}{\partial\nu}|^2 + |k_2(t)\frac{\partial v}{\partial\nu}|^2 + |k_2'\diamond\frac{\partial v}{\partial\nu}|^2 + |k_2(t)\frac{\partial v_0}{\partial\nu}|^2)d\Gamma_1 \\
& +C\int_{\Gamma_1}(|v_t|^2 + |k_3(t)v|^2 + |k_3'\diamond v|^2 + |k_3(t)v_0|^2)d\Gamma_1, \qquad (3.10)
\end{aligned}$$

where ϵ_1 is a positive number to be fixed later. Next, let us calculate the expression $\int_{\Gamma_1}|m\cdot\nabla u|^2 d\Gamma_1 + \int_{\Gamma_1}|\frac{\partial}{\partial\nu}(m\cdot\nabla u)|^2 d\Gamma_1$ that appears in (3.10). Since $a(u,v)$ is strictly coercive on W, we have using the trace theory

$$\begin{aligned}
& \int_{\Gamma_1}|m\cdot\nabla u|^2 d\Gamma_1 + \int_{\Gamma_1}|\frac{\partial}{\partial\nu}(m\cdot\nabla u)|^2 d\Gamma_1 \\
& \le \lambda_0 a(u,u) + \frac{\lambda_0}{\delta_0}\int_{\Gamma_1}m\cdot\nu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right]d\Gamma_1
\end{aligned}$$

where λ_0 is a constant depending on Ω and μ. Substituting the above inequality into (3.10), choosing N large enough and $\epsilon_1 < min(\frac{\theta}{2\lambda_0}, \frac{\delta_0}{\lambda_0})$ we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -q_2 E(t) + CR^2(t)E(0),$$

where $q_2 > 0$ is a small constant and $R(t) = k_1(t) + k_2(t) + k_3(t)$. Here we have used the assumptions (2.8) in order to conclude the following estimates

$$-\frac{\tau_1}{2}\int_{\Gamma_1} k_1''\Box u d\Gamma_1 \leq C_1 \int_{\Gamma_1} k_1'\Box u d\Gamma_1,$$
$$-\frac{\tau_2}{2}\int_{\Gamma_1} k_2''\Box \frac{\partial u}{\partial \nu} d\Gamma_1 \leq C_1 \int_{\Gamma_1} k_2'\Box \frac{\partial u}{\partial \nu} d\Gamma_1,$$
$$\frac{\tau_1}{2}\int_{\Gamma_1} k_1'|u|^2 d\Gamma_1 \leq -C_1 \int_{\Gamma_1} k_1|u|^2 d\Gamma_1,$$
$$\frac{\tau_2}{2}\int_{\Gamma_1} k_2'|\frac{\partial u}{\partial \nu}|^2 d\Gamma_1 \leq -C_1 \int_{\Gamma_1} k_2|\frac{\partial u}{\partial \nu}|^2 d\Gamma_1,$$
$$-\frac{\tau_3}{2}\int_{\Gamma_1} k_3''\Box v d\Gamma_1 \leq C_3 \int_{\Gamma_1} k_1'\Box v d\Gamma_1,$$
$$\frac{\tau_3}{2}\int_{\Gamma_1} k_3'|v|^2 d\Gamma_1 \leq -C_1 \int_{\Gamma_1} k_3|v|^2 d\Gamma_1$$

for the corresponding six terms appearing in the Lemma 3.2. Thus we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{q_2}{q_1}\mathcal{L}(t) + CR^2(t)E(0).$$

Using the exponential decay of k_1, k_2, k_3 and Lemma 3.4 we conclude that

$$\mathcal{L}(t) \leq \{\mathcal{L}(0) + C\}e^{-\gamma_1 t}$$

for all $t \geq 0$. From the inequality (3.9) our conclusion follows. ■

4. Uniform Polynomial decay

Here our attention will be focused on the uniform rate of decay when the resolvents k_i decays polynomially as $(1+t)^{-p}$ for some $p > 1$. For this, we will use the following lemmas.

Lemma 4.1 *Let us denote by* $(\phi_1, \phi_2, \phi_3) = (u, \frac{\partial u}{\partial \nu}, v)$. *Let* $p > 1$, $0 < r < 1$ *and* $t \geq 0$. *Then we have*

$$(\int_{\Gamma_1} |k_i'|\Box \phi_i d\Gamma_1)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}}$$
$$\leq 2(\int_0^t |k_i'(s)|^r ds||\phi_i||^2_{L^\infty((0,T),L^2(\Gamma_1))})^{\frac{1}{(1-r)(p+1)}}(\int_{\Gamma_1} |k_i'|^{1+\frac{1}{p+1}}\Box \phi_i d\Gamma_1)$$

while for $r = 0$ *we get*

$$
\begin{aligned}
&\left(\int_{\Gamma 1} |k_i'| \Box \phi_i d\Gamma_1\right)^{\frac{p+2}{p+1}} \\
&\leq 2\left(\int_0^t ||\phi_i(s,.)||^2_{L^2(\Gamma_1)} ds + t||\phi_i(s,.)||^2_{L^2(\Gamma_1)}\right)^{p+1}\left(\int_{\Gamma_1} |k_i'|^{1+\frac{1}{p+1}} \Box \phi_i d\Gamma_1\right),
\end{aligned}
$$

for all $i = 1, 2, 3$.

Proof. See e. g. [15, Lemma 4.1].

Lemma 4.2 *Let* $f \geq 0$ *be a differentiable function satisfying*

$$
f'(t) \leq \frac{-\bar{c}_1}{f(0)^{\frac{1}{\alpha}}} f(t)^{1+\frac{1}{\alpha}} + \frac{\bar{c}_2}{(1+t)^{\beta}} f(0), \quad \textit{for} \quad t \geq 0,
$$

for some positive constants $\bar{c}_1, \bar{c}_2$, α *and* β *such that*

$$
\beta \geq \alpha + 1.
$$

Then there exists a constant $\bar{c}_3 > 0$ *such that*

$$
f(t) \leq \frac{\bar{c}_3}{(1+t)^{\alpha}} f(0), \quad \textit{for} \quad t \geq 0.
$$

Proof. See e. g. [15, Lemma 4.2].

Finally, we shall show the inequality (2.11). We define the functional $\mathcal{L}$ as in (3.8) and we have the equivalence to the energy term E as given in (3.9) again. The negative terms

$$
-Ck_1(t)\int_{\Gamma_1} |u|^2 d\Gamma_1, \quad -Ck_2(t)\int_{\Gamma_1} |\frac{\partial u}{\partial \nu}|^2 d\Gamma_1, \quad -Ck_3(t)\int_{\Gamma_1} |v|^2 d\Gamma_1
$$

can be obtained from Lemma 3.3 and estimates

$$
\begin{aligned}
k_1(t)\int_{\Gamma_1} |u|^2 d\Gamma_1 &\leq C\int_{\Omega} |\nabla u|^2 dx, \\
k_2(t)\int_{\Gamma_1} |\frac{\partial u}{\partial \nu}|^2 d\Gamma_1 &\leq C||u||^2_{H^2(\Omega)}, \\
k_3(t)\int_{\Gamma_1} |v|^2 d\Gamma_1 &\leq C\int_{\Omega} |\nabla v|^2 dx.
\end{aligned}
$$

¿From the Lemmas 3.2 and 3.3 we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad &\leq \quad -C_1(\int_\Omega |u_t|^2dx + \int_\Omega |v_t|^2dx + a(u,u) + \int_\Omega |\nabla v|^2dx \\ &+k_1(t)\int_{\Gamma_1}|u|^2d\Gamma_1 + k_2(t)\int_{\Gamma_1}|\frac{\partial u}{\partial \nu}|^2d\Gamma_1 + N\int_{\Gamma_1}k_1''\Box ud\Gamma_1 \\ &+N\int_{\Gamma_1}k_2''\Box\frac{\partial u}{\partial\nu}d\Gamma_1 + k_3(t)\int_{\Gamma_1}|v|^2d\Gamma_1 + N\int_{\Gamma_1}k_3''\Box vd\Gamma_1 \\ &+\int_\Omega F(u-v)dx) + C_2R^2(t)E(0). \end{aligned} \tag{4.1}$$

¿From hypothesis (2.9) we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad &\leq \quad -C_1(\int_\Omega |u_t|^2dx + \int_\Omega |v_t|^2dx + a(u,u) + \int_\Omega |\nabla v|^2dx \\ &+k_1(t)\int_{\Gamma_1}|u|^2d\Gamma_1 + k_2(t)\int_{\Gamma_1}|\frac{\partial u}{\partial \nu}|^2d\Gamma_1 + k_3(t)\int_{\Gamma_1}|v|^2d\Gamma_1 \\ &+N\int_{\Gamma_1}(-k_1')^{1+\frac{1}{p+1}}\Box ud\Gamma_1 + N\int_{\Gamma_1}(-k_2')^{1+\frac{1}{p+1}}\Box\frac{\partial u}{\partial\nu}d\Gamma_1 \\ &+N\int_{\Gamma_1}(-k_3')^{1+\frac{1}{p+1}}\Box vd\Gamma_1 + \int_\Omega F(u-v)dx) + C_3R^2(t)E(0). \end{aligned} \tag{4.2}$$

Let us fix $0 < r < 1$ such that

$$\frac{1}{p+1} < r < \frac{p}{p+1}.$$

In this condition

$$\int_0^\infty |k_i'|^r \leq C\int_0^\infty \frac{1}{(1+t)^{r(p+1)}} < \infty, \quad \forall i = 1,2,3.$$

¿From Lemma 4.1 we get

$$\begin{aligned}&\int_{\Gamma_1}(-k_1')^{1+\frac{1}{p+1}}\Box ud\Gamma_1 \\ &\geq \frac{C}{E(0)^{\frac{1}{(1-r)(p+1)}}}(\int_{\Gamma_1}(-k_1')\Box ud\Gamma_1)^{1+\frac{1}{(1-r)(p+1)}}, \end{aligned} \tag{4.3}$$

$$\begin{aligned}&\int_{\Gamma_1}(-k_2')^{1+\frac{1}{p+1}}\Box\frac{\partial u}{\partial\nu}d\Gamma_1 \\ &\geq \frac{C}{E(0)^{\frac{1}{(1-r)(p+1)}}}(\int_{\Gamma_1}(-k_2')\Box\frac{\partial u}{\partial\nu}d\Gamma_1)^{1+\frac{1}{(1-r)(p+1)}} \end{aligned} \tag{4.4}$$

$$\begin{aligned}&\int_{\Gamma_1}(-k_3')^{1+\frac{1}{p+1}}\Box vd\Gamma_1 \\ &\geq \frac{C}{E(0)^{\frac{1}{(1-r)(p+1)}}}(\int_{\Gamma_1}(-k_3')\Box vd\Gamma_1)^{1+\frac{1}{(1-r)(p+1)}}. \end{aligned} \tag{4.5}$$

On the other hand since the energy is bounded we have

$$\Big(k_1(t)\int_{\Gamma_1}|u|^2d\Gamma_1+k_2(t)\int_{\Gamma_1}|\frac{\partial u}{\partial\nu}|^2d\Gamma_1+k_3(t)\int_{\Gamma_1}|v|^2d\Gamma_1$$
$$+\int_\Omega|u_t|^2dx+\int_\Omega|v_t|^2dx+a(u,u)+\int_\Omega|\nabla v|^2dx+\int_\Omega F(u-v)dx\Big)^{1+\frac{1}{(1-r)(p+1)}}$$
$$\leq CE(0)^{\frac{1}{(1-r)(p+1)}}\Big(k_1(t)\int_{\Gamma_1}|u|^2d\Gamma_1+k_2(t)\int_{\Gamma_1}|\frac{\partial u}{\partial\nu}|^2d\Gamma_1$$
$$+k_3(t)\int_{\Gamma_1}|v|^2d\Gamma_1+\int_\Omega|u_t|^2dx+\int_\Omega|v_t|^2dx+a(u,u)$$
$$+\int_\Omega|\nabla v|^2dx+\int_\Omega F(u-v)dx\Big). \tag{4.6}$$

Substituting (4.3)-(4.6) into (4.2) we arrive at

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad \leq \quad & -\frac{C}{E(0)^{\frac{1}{(1-r)(p+1)}}}[(k_1(t)\int_{\Gamma_1}|u|^2d\Gamma_1+k_2(t)\int_{\Gamma_1}|\frac{\partial u}{\partial\nu}|^2d\Gamma_1\\ &+k_3(t)\int_{\Gamma_1}|v|^2d\Gamma_1+\int_\Omega|u_t|^2dx+\int_\Omega|v_t|^2dx+a(u,u)\\ &+\int_\Omega|\nabla v|^2dx)^{1+\frac{1}{(1-r)(p+1)}}+(\int_{\Gamma_1}|k_1'|\square u d\Gamma_1)^{1+\frac{1}{(1-r)(p+1)}}\\ &+(\int_{\Gamma_1}|k_2'|\square\frac{\partial u}{\partial\nu}d\Gamma_1)^{1+\frac{1}{(1-r)(p+1)}}]+(\int_{\Gamma_1}|k_3'|\square v d\Gamma_1+\int_\Omega F(u-v)dx)^{1+\frac{1}{(1-r)(p+1)}}\\ &+CR^2(t)E(0).\end{aligned}$$

Taking into account (3.9) we conclude that

$$\frac{d}{dt}\mathcal{L}(t)\leq-\frac{C}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}}\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}}+CR^2(t)E(0). \tag{4.7}$$

Applying the Lemma 4.2 with $f=\mathcal{L}$ and $\beta=2p$ we have:

$$\mathcal{L}(t)\leq\frac{C}{(1+t)^{(1-r)(p+1)}}\mathcal{L}(0). \tag{4.8}$$

Since $(1-r)(p+1)>1$

$$\int_0^\infty E(s)ds\leq C\int_0^\infty\mathcal{L}(s)ds\leq c\mathcal{L}(0)<\infty, \tag{4.9}$$

$$t||u(t,.)||^2_{L^2(\Gamma_1)}+t||\frac{\partial u(t,.)}{\partial\nu}||^2_{L^2(\Gamma_1)}+t||v(t,.)||^2_{L^2(\Gamma_1)}\leq Ct\mathcal{L}(t)<\infty, \tag{4.10}$$

$$\int_0^t||u(s,.)||^2_{L^2(\Gamma_1)}+\int_0^t||\frac{\partial u(s,.)}{\partial\nu}||^2_{L^2(\Gamma_1)}$$
$$+\int_0^t||v(s,.)||^2_{L^2(\Gamma_1)}\leq C\int_0^\infty\mathcal{L}(t)dt<\infty,\forall t\geq0. \tag{4.11}$$

In this conditions applying Lemma 4.1 for $r = 0$ we get

$$\int_{\Gamma_1} (-k_1')^{1+\frac{1}{p+1}} \Box u d\Gamma_1 \geq \frac{C}{E(0)^{\frac{1}{p+1}}} (\int_{\Gamma_1} (-k_1')\Box u d\Gamma_1)^{1+\frac{1}{p+1}},$$

$$\int_{\Gamma_1} (-k_2')^{1+\frac{1}{p+1}} \Box \frac{\partial u}{\partial \nu} d\Gamma_1 \geq \frac{C}{E(0)^{\frac{1}{p+1}}} (\int_{\Gamma_1} (-k_2')\Box \frac{\partial u}{\partial \nu} d\Gamma_1)^{1+\frac{1}{p+1}},$$

$$\int_{\Gamma_1} (-k_3')^{1+\frac{1}{p+1}} \Box v d\Gamma_1 \geq \frac{C}{E(0)^{\frac{1}{p+1}}} (\int_{\Gamma_1} (-k_3')\Box v d\Gamma_1)^{1+\frac{1}{p+1}}.$$

Using these inequalities intead of (4.3)-(4.5) and reasoning in the same way we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{C}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}} + CR^2(t)E(0).$$

Applying the Lemma 4.2, we obtain

$$\mathcal{L}(t) \leq \frac{C}{(1+t)^{p+1}}\mathcal{L}(0).$$

Finally, from (3.9) we obtain

$$E(t) \leq \frac{C}{(1+t)^{p+1}}E(0),$$

which completes the present proof. ■

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