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Simultaneous observability of networks of beams and strings

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ABSTRACT: In this paper we investigate a finite system of vibrating beams and strings. We obtain results on simultaneous observability by observing a common endpoint.

Contents

1. Introduction

Consider a system of N vibrating beams with fixed endpoints, one of which is common to all of them. Denoting by l_j the lengths of the beams, we study the following uncoupled system:

$$\begin{cases} u_{j,tt} + u_{j,xxxx} = 0 & \text{in} \quad (0, l_j) \times \mathbb{R}, \\ u_j(0,\cdot) = u_j(l_j,\cdot) = 0 & \text{in} \quad \mathbb{R}, \\ u_{j,xx}(0,\cdot) = u_{j,xx}(l_j,\cdot) = 0 & \text{in} \quad \mathbb{R}, \\ u_j(\cdot,0) = u_{j0}, u_{j,t}(\cdot,0) = u_{j1} & \text{in} \quad (0, l_j), \\ j = 1, \dots, N. \end{cases} \tag{1}$$

Assume that we can measure the *total* force

$$f := \sum_{j=1}^{N} u_{j,x}(0,\cdot)$$

exerced on the beams at the common endpoint during some time. Investigating the observability of the problem, our question is whether this information is sufficient

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in order to identify all initial data? This problem was first studied for vibrating strings in the case $N = 2$ in [7] and for arbitrary N in [2], and for beams in [2] in the case $N = 2$.

Recall (see e.g. [10]) that for every initial data

$$(u_{j0}, u_{j1})_{j=1}^{N} \in \prod_{j=1}^{N} \left(H_0^1 (0, l_j) \times H^{-1} (0, l_j)\right)$$

in the natural energy space, there exists a unique solution satisfying

$$u_j \in C\left(\mathbb{R}; H_0^1 (0, l_j)\right) \cap C^1 \left(\mathbb{R}; H^{-1} (0, l_j)\right),\ j = 1, \ldots, N,$$

(well-posedness) and that

$$u_{j,x} \in L^2_{loc} (\mathbb{R}),\ j = 1, \ldots, N,$$

(hidden reguralty). See, e.g., Lasiecka and Triggiani [8] and [9] for results of such type. Moreover, the linear maps

$$(u_{j0}, u_{j1}) \longmapsto u_{j,x} (0, \cdot)$$

are continuous with respect to these topologies. It follows that for every bounded interval I there exists a constant c such that

$$\int_I |f (t)|^2 \, dt \leq c \sum_{j=1}^{N} \left(\|u_{j0}\|^2_{H_0^1(0,l_j)} + \|u_{j1}\|^2_{H^{-1}(0,l_j)} \right) \tag{2}$$

for all initial data.

Now our question is whether the linear map

$$(u_{j0}, u_{j1})_{j=1}^{N} \longmapsto f \mid_I \tag{3}$$

is one-to-one? If yes, we can ask whether the inverse linear map is also bounded, that is, whether the inverse inequality to (2) holds true. It would mean that there exists another constant c' such that

$$\sum_{j=1}^{N} \left(\|u_{j0}\|^2_{H_0^1(0,l_j)} + \|u_{j1}\|^2_{H^{-1}(0,l_j)} \right) \leq c' \int_I |f (t)|^2 \, dt \tag{4}$$

for all initial data.

2. Statement of the theorem and starting idea of the proof

Let us begin with a simple but important observation: if there exist two beams with commeasurable lengths, then the map (3) is not one-to-one for any interval I. Indeed, if for example

$$\frac{l_1}{l_2} = \frac{p}{q}$$

with two positive integers p and q, then the formulae

$$\begin{cases} u_1(x,t) := \sin \frac{p\pi x}{l_1} \exp\left(ip^2\pi^2 t/l_1^2\right), \\ u_2(x,t) := -\sin \frac{q\pi x}{l_2} \exp\left(iq^2\pi^2 t/l_2^2\right), \\ u_j(x,t) := 0, \ j = 3, \dots, N, \end{cases}$$

define a nonzero solution of (1) with suitable initial data for which f vanishes identically on $\mathbb{R}$. Thus we cannot hope positive results unless

$$\frac{l_j}{l_k} \text{ is irrational for all } j \neq k. \tag{5}$$

Remark 1 *The set of excluded N-tuples $(l_1, ..., l_N)$, where at least one of the fractions $\frac{l_j}{l_k}$ is rational, has zero measure. Consequently, the complement set of admissible N-tuples is dense in $(0,\infty)^N$.*

The following result can be obtained.

Theorem 1 *Let I be an arbitrarily short bounded interval and $s < 1$. Then for almost all N-tuples $(l_1, \dots, l_N)$ of positive real numbers statisfying (5) there exists a constant $c = c(|I|, s)$ such that*

$$\sum_{j=1}^{N} \left(\|u_{j0}\|^2_{H^s(0,l_j)} + \|u_{j1}\|^2_{H^{s-2}(0,l_j)} \right) \leq c \int_I |f(t)|^2 \, dt \tag{6}$$

for all initial data.

Corollary 1A *The map (3) is one-to-one for almost all set of N-tuples $(l_1, ..., l_N)$ satisfying (5).*

The starting idea of the proof is the following. The solution of (1) is given by the formulas

$$u_j(x,t) = \sum_{0 \neq k \in \mathbb{Z}} b_{jk} \sin \frac{k\pi x}{l_j} \exp \frac{ik|k|\pi^2 t}{l_j^2}, \ j = 1, \dots, N,$$

with suitable complex coefficients depending on the initial data, and

$$f(t) = \sum_{j=1}^{N} \sum_{0 \neq k \in \mathbb{Z}} \frac{k\pi b_{jk}}{l_j} \exp \frac{ik|k|\pi^2 t}{l_j^2} =: \sum_{n=-\infty}^{\infty} b_n e^{i\lambda_n t} \tag{7}$$

by rearranging the exponents $k|k|\pi^2 l_j^{-2}$ into an increasing sequence (λ_n) and denoting the corresponding coefficients $k\pi b_{jk} l_j^{-1}$ by b_n. It follows from (5) that

$$\lambda_n \neq \lambda_m \text{ if } n \neq m.$$

A straightforward computation shows that the estimate (6) is equivalent to the inequality

$$\sum_n |\lambda_n|^{s-1} |b_n|^2 \leq c \int_I |f(t)|^2 \, dt. \tag{8}$$

In the following section we present some results concerning this type of estimates.

3. Preliminary results

Let (λ_n) be a strictly increasing sequence of real numbers satisfying the following *uniform gap condition*:

$$\exists \gamma > 0 \ \forall n : \lambda_{n+1} - \lambda_n \geq \gamma. \tag{9}$$

We have the following theorem due to Ingham [6].

Theorem 2 *Assume* (9). *If I is a bounded interval of length $|I| > 2\pi/\gamma$, then there exist two positive constants c_1 and c_2 such that*

$$c_1 \sum |b_n|^2 \leq \int_I \left| \sum b_n e^{i\lambda_n t} \right|^2 dt \leq c_2 \sum |b_n|^2 \tag{10}$$

for all square summable sequences of complex numbers b_n.

An optimal condition for the length of I satisfying the above inequalities was given by Beurling. This is expressed by the so-called *upper density* of the sequence (λ_n), a notion due to Pólya (see [11]).

Definition 3.1 *Let us denote by $n^+(r)$ the maximal possible number of elements of (λ_n) contained in an interval of length $r > 0$. Then the limit*

$$\lim_{r \to \infty} \frac{n^+(r)}{r}$$

exists and is equal to

$$D^+ = D^+((\lambda_n)) := \inf_{r > 0} \frac{n^+(r)}{r}.$$

We call D^+ the upper density *of the sequence (λ_n).*

The result of Beurling [4] is as follows.

Theorem 3 *Assume* (9) *again.*
(a) If I is a bounded interval of length $|I| > 2\pi D^+$, then the inequalities (10) *hold true.*
(b) If $|I| < 2\pi D^+$, then the first inequality of (10) *does not hold true (with a constant independent of the choice of (b_n)).*

In our original problem on the observability of beams we have

$$\{\lambda_n\} = \left\{ \frac{k\,|k|\,\pi^2}{l_j^2} : 0 \neq k \in \mathbb{Z}, j = 1, \ldots, N \right\},$$

whence

$$D^+ = 0.$$

Indeed, for each fixed j, an interval of length r contains at least $\left(\sqrt{r} l_j^2/\pi^2\right) - 1$ and at most $\left(\sqrt{r} l_j^2/\pi^2\right) + 1$ elements of the sequence $\left(k\,|k|\,\pi^2/l_j^2\right)_{k\in\mathbb{Z}}$. Hence

$$\frac{n^+(r)}{r} \to 0 \text{ as } r \to \infty.$$

It is thus tempting to apply Beurling's theorem which would also yield the condition

$$|I| > 2\pi D^+ = 0.$$

But there is a serious obstacle in our case: the uniform gap condition (9) is not satisfied if $N \geq 2$. Therefore we have to generalize our condition.

Let $(\lambda_n)_{n=-\infty}^{+\infty}$ again be a strictly increasing sequence of real numbers. The following result in [3] establishes a connection between the assumptions of Ingham and Beurling.

Lemma 3.1 *Let x be a positive number satisfying $x > 2\pi D^+$. Then there exists a real number $\gamma' > 0$ and an integer $M \geq 1$ such that*

$$x > \frac{2\pi}{\gamma'}$$

and

$$\lambda_{n+M} - \lambda_n \geq M\gamma' \textit{ for all } n. \tag{11}$$

Fixing such a γ' and M, we introduce the divided differences of the close exponential functions.

Definition 3.2 *Fix a number $0 < \gamma'' \leq \gamma'$. We say that $\lambda_m, \ldots, \lambda_{m+k-1}$ is a* chain of close exponents belonging to γ'' *if*

$$\lambda_{n+1} - \lambda_n < \gamma'', \; n = m, \ldots, m+k-2,$$

but

$$\lambda_m - \lambda_{m-1} \geq \gamma'' \textit{ and } \lambda_{m+k} - \lambda_{m+k-1} \geq \gamma''.$$

It follows from the property (11) and from the choice of γ'' that $k \leq M$, and that every λ_n belongs to a unique chain.

For each chain $\lambda_m, \dots, \lambda_{m+k-1}$ let us denote by $e_m(t), \dots, e_{m+k-1}(t)$ the divided differences of the exponential functions $\exp(i\lambda_m t), \dots, \exp(i\lambda_{m+k-1} t)$, defined by the formula

$$e_n(t) := (it)^{n-m} \int_0^1 \int_0^{s_1} \dots \int_0^{s_{n-m-1}} \exp\left(i\left(s_{n-m}\cdot[\lambda_n-\lambda_{n-1}]+\dots+s_1\cdot[\lambda_{m+1}-\lambda_m]+\lambda_m\right)t\right) ds_{n-m}\dots ds_1 \tag{12}$$

for $n = m, \dots, m+k-1$. In particular we have $e_m(t) = \exp(i\lambda_m t)$. If $\lambda_m, \dots, \lambda_n$ are pairwise distinct, then we have the more familiar expressions

$$e_n(t) = \sum_{p=m}^{n} \left[\prod_{\substack{q=m\\q\neq p}}^{n} (\lambda_p - \lambda_q)\right]^{-1} \exp(i\lambda_p t). \tag{13}$$

We recall the following result of [3].

Theorem 4 *If $|I| > 2\pi D^+$ and $(\lambda_n)_{n=-\infty}^{+\infty}$ satisfies (11), then there exist two constants c_1 and c_2 such that, using the above notation, we have*

$$c_1 \sum |a_n|^2 \le \int_I \left|\sum a_n e_n(t)\right|^2 dt \le c_2 \sum |a_n|^2$$

for every sequence $(a_n)_{n=-\infty}^{+\infty}$ of complex numbers.

4. Proof of the theorem

Let us consider the sequence (λ_n) in our problem, defined in (7). We know from the previous section that $D^+ = 0$. By Lemma 3.1, for every bounded interval there exist an integer $M \ge 1$ and a positive number γ' such that the sequence satisfies the "generalized uniform gap condition" (11). Let us now fix a number $0 < \gamma'' \le \gamma'$ and a chain of close exponents, $\lambda_m, \dots, \lambda_{m+k-1}$ belonging to γ''. Define

$$d_m = \dots = d_{m+k-1} := \min\{|\lambda_p - \lambda_q| : m \le p < q \le m+k-1\}. \tag{14}$$

Using the definition of the divided differences, since the elements of the sequence (λ_n) are pairwise distinct, one can show by (13) that

$$\sum_{n=m}^{m+k-1} b_n e^{i\lambda_n t} = \sum_{n=m}^{m+k-1} a_n e_n(t)$$

for suitable coefficients $a_n, n = m, \dots, m+k-1$. Moreover, by (14) there exists a constant c such that

$$|b_n| \, d_n^{N-1} \le c\,|a_n|$$

for all $m \le n \le m+k-1$. Therefore, applying Theorem 4, we obtain the inequality

$$\sum |b_n|^2 d_n^{2N-2} \le \frac{c^2}{c_1} \int_I |f(t)|^2 dt.$$

In order to obtain the estimate (8), it suffices to show the existence of a constant c' such that

$$c' \left|\lambda_n\right|^{s-1} \leq d_n^{2N-2} \tag{15}$$

for all $n \in \mathbb{Z}$.

For this we need some classical results on Diophantine approximation (see [5], Chapter VII., Theorem 1).

Theorem 5 *Let $\phi(q)$ be a decreasing function of the integer variable $q > 0$ with $0 \leq \phi(q) \leq 1/2$. Then the set of inequalities*

$$\|q\theta_k\| < \phi(q), \quad 1 \leq k \leq n,$$

(the „norm" is the distance from the set $\mathbb{Z}$) has infinitely many integer solutions $q > 0$ for almost no or for almost all n-tuples $(\theta_1, \dots, \theta_n)$ of real numbers according to whether

$$\sum (\phi(q))^n$$

converges or diverges.

Let $\varepsilon > 0$, and $\phi(q) := q^{-1-\varepsilon}$. Then by Theorem 5 for almost all choices of the lengths l_j and for all $\varepsilon > 0$ we can find $c_\varepsilon > 0$ such that

$$\left|k - r\frac{l_j}{l_i}\right| \geq c_\varepsilon r^{-1-\varepsilon} \text{ for all } r = 1, 2, \dots \tag{16}$$

Let $p, q \in \{m, \dots, m+k-1\}$ arbitrary, $\lambda_p := \frac{k^2\pi^2}{l_j^2}$, $\lambda_q := \frac{r^2\pi^2}{l_i^2}$ (for simplicity we assume that $k, r \geq 0$). Since $(N-1)\gamma'' > \left|\frac{k^2\pi^2}{l_j^2} - \frac{r^2\pi^2}{l_i^2}\right|$, we have $k \asymp r$. (We write $k \asymp r$ if k can be estimated by r multiplied by a constant and vice versa). Hence it follows that

$$\lambda_p \asymp \lambda_q \asymp k^2 \asymp r^2.$$

Thus for all $\varepsilon > 0$ there exists c_ε such that

$$\begin{aligned}
|\lambda_p - \lambda_q| &= \left|\frac{k^2\pi^2}{l_j^2} - \frac{r^2\pi^2}{l_i^2}\right| = \left|\frac{k\pi}{l_j} - \frac{r\pi}{l_i}\right| \left|\frac{k\pi}{l_j} + \frac{r\pi}{l_i}\right| \\
&\asymp \left|k - r\frac{l_j}{l_i}\right| \cdot r \\
&\geq c_\varepsilon r^{-1-\varepsilon} \cdot r = c_\varepsilon r^{-\varepsilon}.
\end{aligned} \tag{17}$$

Since all the integers belonging to the lambda's of the chain are equivalent (see the above meaning of $\asymp$), we have

$$d_n^{2N-2} \geq (c_\varepsilon r^{-\varepsilon})^{2N-2} \geq c'_\varepsilon \left|\lambda_n\right|^{-\varepsilon(N-1)}$$

for all $n \in \mathbb{Z}$ and for every positive ε. Taking $\varepsilon = (1-s)/(N-1)$, inequality (15) and hence Theorem 1 follows.

5. Network of strings and beams

Consider a vibrating system containing strings as well as beams with fixed endpoints, one common to all of them. That is, we have the following system:

$$\begin{cases} u_{j,tt} - u_{j,xx} = 0 & \text{in} \quad (0, l_j) \times \mathbb{R}, \\ u_j(0, \cdot) = u_j(l_j, \cdot) = 0 & \text{in} \quad \mathbb{R}, \\ u_j(\cdot, 0) = u_{j0}, u_{j,t}(\cdot, 0) = u_{j1} & \text{in} \quad (0, l_j), \\ j = 1, \dots, N_1, & \\ u_{j,tt} + u_{j,xxxx} = 0 & \text{in} \quad (0, l_j) \times \mathbb{R}, \\ u_j(0, \cdot) = u_j(l_j, \cdot) = 0 & \text{in} \quad \mathbb{R}, \\ u_{j,xx}(0, \cdot) = u_{j,xx}(l_j, \cdot) = 0 & \text{in} \quad \mathbb{R}, \\ u_j(\cdot, 0) = u_{j0}, u_{j,t}(\cdot, 0) = u_{j1} & \text{in} \quad (0, l_j), \\ j = N_1 + 1, \dots, N. & \end{cases} \tag{18}$$

In order to have an observability estimate, we need again the hypothesis as in (5) for the system of strings and for the system of beams.

$$\frac{l_j}{l_k} \text{ is irrational for } j \neq k, \quad j, k \in \{1, \dots, N_1\} \text{ or } j, k \in \{N_1 + 1, \dots, N\}. \tag{19}$$

Moreover, we also need the hypothesis

$$\frac{l_j \pi}{l_k^2} \text{ is irrational for all } j = 1, \dots, N_1, \quad k = N_1 + 1, \dots, N. \tag{20}$$

Indeed, if e.g.

$$\frac{l_j \pi}{l_k^2} = \frac{p}{q}, \quad j \in \{1, \dots, N_1\}, \quad k \in \{N_1 + 1, \dots, N\}$$

with two positive integers p and q, then putting

$$\begin{cases} u_j(x,t) := \sin \frac{qp\pi x}{l_j} \exp(iqp\pi t / l_j), \\ u_k(x,t) := -\frac{p l_k}{l_j} \sin \frac{q\pi x}{l_k} \exp\left(iq^2 \pi^2 t / l_k^2\right), \\ u_l(x,t) := 0, \ l \neq j, k, \end{cases}$$

we obtain a nonzero solution of (18) with suitable initial data such that f vanishes identically on $\mathbb{R}$.

Concerning the observability of this system, we can prove the following.

Theorem 6 *Let I be a bounded interval with $|I| > 2(l_1 + \dots + l_{N_1})$ and $\alpha < 2 - N$, $\beta < 3 - 2N$. Then for almost all N-tuples $(l_1, \dots, l_N)$ of positive real numbers* *there exists a constant* $c = c(|I|, \alpha, \beta)$ *such that*

$$\sum_{j=1}^{N_1} \left(\|u_{j0}\|^2_{H^{\alpha}(0,l_j)} + \|u_{j1}\|^2_{H^{\alpha-1}(0,l_j)} \right) +$$
$$\sum_{j=N_1+1}^{N} \left(\|u_{j0}\|^2_{H^{\beta}(0,l_j)} + \|u_{j1}\|^2_{H^{\beta-2}(0,l_j)} \right) \le c \int_I |f(t)|^2 \, dt$$

for all initial data.

Remark 2 *Similar results were announced without proof in* [1], *writing that the proofs are more complicated and will be presented elsewhere. Our proof below, based on earlier results of Baiocchi et al., is short and elementary.*

Proof. We proceed as above. For the solution of the system of strings we have

$$u_j(x,t) = \sum_{0 \ne k \in \mathbb{Z}} b_{jk} \sin \frac{k\pi x}{l_j} \exp \frac{ik\pi t}{l_j}, \; j = 1, \ldots, N_1,$$

and for the solution for the system of beams

$$u_j(x,t) = \sum_{0 \ne k \in \mathbb{Z}} b_{jk} \sin \frac{k\pi x}{l_j} \exp \frac{ik|k|\pi^2 t}{l_j^2}, \; j = N_1+1, \ldots, N,$$

as above. For the total force measured at the common endpoint we have

$$\begin{aligned} f(t) &= \sum_{j=1}^{N_1} \sum_{0 \ne k \in \mathbb{Z}} \frac{k\pi b_{jk}}{l_j} \exp \frac{ik\pi t}{l_j} + \sum_{j=N_1+1}^{N} \sum_{0 \ne k \in \mathbb{Z}} \frac{k\pi b_{jk}}{l_j} \exp \frac{ik|k|\pi^2 t}{l_j^2} \\ &=: \sum_{n=-\infty}^{\infty} b_n e^{i\lambda_n t}. \end{aligned}$$

It follows from (19) and (20) that

$$\lambda_n \ne \lambda_m \text{ if } n \ne m,$$

and it is easy to see that we have excluded a set of measure 0 from the N-tuples $(l_1, \ldots, l_N)$.

Now, we have to prove the estimate

$$\sum_{\substack{n: \\ \lambda_n = k\pi/l_j}} |\lambda_n|^{2\alpha-2} |b_n|^2 + \sum_{\substack{n: \\ \lambda_n = r^2\pi^2/l_i^2}} |\lambda_n|^{\beta-1} |b_n|^2 \le c \int_I |f(t)|^2 \, dt.$$

We introduce the divided differences for the chains of close exponents, see (12) and (13), and define $d_n, \; n \in \mathbb{Z}$, as in (14). Applying Theorem 4, it leads again to the following inequalities to be verified for an appropriate constant C:

$$C\,|\lambda_n|^{2\alpha-2} \le d_n^{2N-2} \text{ when } \lambda_n = \frac{k\pi}{l_j} \text{ for some } 0\neq k\in\mathbb{Z},\quad j\in\{1,\dots,N_1\}\,,$$
$$C\,|\lambda_n|^{\beta-1} \le d_n^{2N-2} \text{ when } \lambda_n = \frac{r^2\pi^2}{l_i^2} \text{ for some } 0\neq r\in\mathbb{Z},\quad i\in\{N_1+1,\dots,N\}\,. \tag{21}$$

Computing the upper density of the sequence (λ_n), we obtain that

$$D^+ = (l_1+\cdots+l_{N_1})\,/\pi;$$

see [3] and apply the result on the system of beams. So the assumption for the length of I in Theorem 4 follows.

A chain of close exponents can contain exponents corresponding to both the strings and the beams. If two elements of the chain have the form $\lambda_p := \frac{k\pi}{l_j}$, $\lambda_q := \frac{r\pi}{l_i}$, then

$$\lambda_p \asymp \lambda_q \asymp k \asymp r.$$

Using Diophantine approximation from Theorem 5 as in (16), we obtain

$$|\lambda_p-\lambda_q| \ge c_\varepsilon r^{-1-\varepsilon} \tag{22}$$

for every $\varepsilon>0$. For $\lambda_p := \frac{k^2\pi^2}{l_j^2}$, $\lambda_q := \frac{r^2\pi^2}{l_i^2}$ we conclude also from (16) that

$$|\lambda_p-\lambda_q| \ge c'_\varepsilon (r^2)^{-1-\varepsilon} \tag{23}$$

for every $\varepsilon>0$. For $\lambda_p := \frac{k^2\pi^2}{l_j^2}$, $\lambda_q := \frac{r\pi}{l_i}$ we can apply Theorem 5 for $\theta_k = \frac{\pi l_i}{l_j^2}$ as follows:

$$\begin{aligned}|\lambda_p-\lambda_q| &= \left|\frac{k^2\pi^2}{l_j^2}-\frac{r\pi}{l_i}\right| = c\left|k^2\frac{\pi l_i}{l_j^2}-r\right| \\ &\ge c''_\varepsilon r^{-1-\varepsilon} \asymp c''_\varepsilon (k^2)^{-1-\varepsilon}\end{aligned} \tag{24}$$

for all $\varepsilon>0$.

Using the fact that the exponents belonging to one chain are all equivalent, we obtain from (22), (23) and (24) that

$$d_n^{2N-2} \ge C_\varepsilon\cdot|\lambda_n|^{(-1-\varepsilon)(2N-2)} \text{ for every } \varepsilon>0,$$

that is

$$d_n^{2N-2} \ge C_\gamma\cdot|\lambda_n|^{\gamma}$$

for every $\gamma<2-2N$. Since the condition $2\alpha-2<2-2N$ is equivalent to $\alpha<2-N$ and $\beta-1<2-2N$ is equivalent to $\beta<3-2N$, the proof of (21) thus Theorem 6 is complete.

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