Bol. Soc. Paran. Mat.
(3s.) **v. 21** 1/2 (2003): 1–10.
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Multiple Positive Solutions for a Fourth-order Boundary Value Problem

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ABSTRACT: In this paper, we discuss the existence of multiple positive solutions for the fourth-order boundary value problem

$$u^{(4)}(t) = f(t, u(t)),\ 0 < t < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. Existence theorems are established via the theory of fixed point index in cones.

1. Introduction

The deformations of an elastic beam in equilibrium state, whose two ends are simply supported, can be described by the fourth-order boundary value problem

$$u^{(4)}(t) = g(t, u(t), u''(t)),\ 0 < t < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $g : [0, 1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous[3,4]. Owing to its importance in physics, the existence of solutions to this problem has been studied under various kinds of restrictions or conditions by many authors, see for example [1-15]. However in pratice only its solutions are significant. In this paper, we discuss the existence of multiple positive solutions for the fourth-order boundary value problem (abbrev. as BVP)

$$u^{(4)}(t) = f(t, u(t)),\ 0 < t < 1, \tag{1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{2}$$

We assume the following conditions throughout this paper:

(P_1) $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous.

The existence of positive solutions of the BVP(1)-(2) has been studied by Ma and Wang [14]. They show the existence of one positive solution when $f(t, u)$ is either superlinear or sublinear in u by employing a cone extension or compression theorem. The purpose of this paper is to extend this result. Our argument is based on fixed point index theory in cones [16].

For convenience, we introduce the following notations

$$f_0 = \liminf_{v\to 0+} \min_{x\in[0,1]} \frac{f(x,v)}{v},\ f^0 = \limsup_{v\to 0+} \max_{x\in[0,1]} \frac{f(x,v)}{v},$$

2000 *Mathematics Subject Classification:* 34B15

* Partially supported by the Natural Science Foundation of Guangdong Province (011471)

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$$f_\infty = \liminf_{v\to+\infty} \min_{x\in[0,1]} \frac{f(x,v)}{v},\ f^\infty = \limsup_{v\to+\infty} \max_{x\in[0,1]} \frac{f(x,v)}{v}.$$

Let λ_1 be the first eigenvalue of the problem

$$u^{(4)} = \lambda u,\ u(0) = u(1) = u''(0) = u''(1) = 0.$$

We know from [6,7] that $\lambda_1 = \pi^4$, and $\phi_1(t) = \sin \pi t$ is the first eigenfunction.

In this paper, some of the following hypotheses are satisfied:

(H_1) $f_0 > \lambda_1,\ f_\infty > \lambda_1$;

(H_2) $f^0 < \lambda_1,\ f^\infty < \lambda_1$;

(H_3) There is a $p > 0$ such that $0 \le v \le p$ and $0 \le t \le 1$ implies

$$f(t, v) < \eta p,$$

where $\eta = [\int_0^1 \int_0^1 G(\tau,\tau)G(\tau,s)dsd\tau]^{-1}$, and $G(t,s)$ is the Green's function (see Section 2) of

$$-u'' = 0, \quad u(0) = u(1) = 0;$$

(H_4) There is a $p > 0$ such that $\frac{p}{4} \le v \le p$ implies

$$f(t, v) > \lambda p,$$

where $\lambda = [\int_0^1 \int_{\frac14}^{\frac34} G(\sigma,\tau)G(\tau,s)dsd\tau]^{-1}$, and $\sigma \in [0,1]$ is such that

$$\int_0^1 \int_{\frac14}^{\frac34} G(\sigma,\tau)G(\tau,s)dsd\tau = \max_{t\in[0,1]} \int_0^1 \int_{\frac14}^{\frac34} G(t,\tau)G(\tau,s)dsd\tau.$$

Remark 1. In fact $\eta = 60, \sigma = \frac12, \lambda = \frac{6144}{57}$, see the appendix in section 4.

The following theorems are our main results.

Theorem 1. *Assume that* $(P_1), (H_1)$ *and* (H_3) *are satisfied. Then the BVP(1)-(2) has at least two positive solutions* u_1 *and* u_2 *with*

$$0 < ||u_1|| < p < ||u_2||,$$

here $||u|| = \sup_{t\in[0,1]} |u(t)|$.

Corollary 1. *The conclusion of Theorem 1 is valid if* (H_1) *is replaced by:*
(H_1^*) $f_0 = \infty,\ f_\infty = \infty$.

Theorem 2. *Assume that* $(P_1), (H_2)$ *and* (H_4) *are satisfied. Then the BVP (1)-(2) has at least two positive solutions* u_1 *and* u_2 *with*

$$0 < ||u_1|| < p < ||u_2||.$$

Corollary 2. *The conclusion of Theorem 2 is valid if* (H_2) *is replaced by:*
(H_2^*) $f^0 = 0,\ f^\infty = 0$.

Theorem 3. *Assume that* (P_1) *is satisfied. Also suppose the following condition is satisfied:*

$$f_0 > \lambda_1, \ f^\infty < \lambda_1.$$

Then the BVP (1)-(2) has at least one positive solution.

Corollary 3. *Assume that* (P_1) *is satisfied. Also suppose the following condition is satisfied:*

$$f_0 = \infty, \ f^\infty = 0 \quad (sublinear).$$

Then the BVP (1)-(2) has at least one positive solution.

Theorem 4. *Assume that* (P_1) *is satisfied. Also suppose the following condition is satisfied:*

$$f^0 < \lambda_1, \ f_\infty > \lambda_1.$$

Then the BVP (1)-(2) has at least one positive solution.

Corollary 4. *Assume that* (P_1) *is satisfied. Also suppose the following condition is satisfied:*

$$f^0 = 0, \ f_\infty = \infty \quad (superlinear).$$

Then the BVP(1)-(2) has at least one positive solution.

Obviously, Theorems 3 and 4 extend the results in [14].

Remark 2. *Since* λ_1 *is an eigenvalue of the linear boundary value problem corresponding to the BVP(1)-(2), the conditions in Theorems 3 and 4 are optimal.*

2. Preliminaries

Suppose that u is a solution of the BVP(1)-(2). Then

$$u(t) = \int_0^1 \int_0^1 G(t,\tau)G(\tau,s)f(s,u(s))dsd\tau, \ 0 \le t \le 1, \tag{3}$$

where $G(t,s)$ is the Green's function to $-u'' = 0$, $u(0) = u(1) = 0$. In particular

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases}$$

and one can show that

$$\min\{t, 1-t\}G(s,s) \le G(t,s) \le G(s,s) = s(1-s), \ (t,s) \in [0,1] \times [0,1]. \tag{4}$$

By using (3) and (4), we see that for every solution u of the BVP (1)-(2), one has

$$\begin{array}{ll} ||u|| & \le \int_0^1 \int_0^1 G(\tau,\tau)G(\tau,s)f(s,u(s))dsd\tau, \\ u(t) & \ge \min\{t,1-t\} \int_0^1 \int_0^1 G(\tau,\tau)G(\tau,s)f(s,u(s))dsd\tau \\ & \ge \min\{t,1-t\}||u||, \end{array} \tag{5}$$

where $||u|| = \sup\{|u(t)|; 0 \le t \le 1\}$.

Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and let $A : K \cap \bar{\Omega} \to K$ be a continuous and completely continuous mapping. If $Au \neq u$ for every $u \in K \cap \partial\Omega$, then the fixed point index $i(A, K \cap \Omega, K)$ is defined. If $i(A, K \cap \Omega, K) \neq 0$, then A has a fixed point in $K \cap \Omega$.

For $r > 0$, let $K_r = \{u \in K : ||u|| < r\}$ and $\partial K_r = \{u \in K : ||u|| = r\}$, which is the relative boundary of K_r in K. The following three Lemmas are needed in our argument.

Lemma 1.[16] *Let $A : K \to K$ be a continuous and completely continuous mapping and $Au \neq u$ for $u \in \partial K_r$. Thus one has the following conclusions:*

(i) If $||u|| \leq \|Au\|$ for $u \in \partial K_r$, then $i(A, K_r, K) = 0$;

(ii) If $\|u\| \geq \|Au\|$ for $u \in \partial K_r$, then $i(A, K_r, K) = 1$.

Lemma 2.[16] *Let $A : K \to K$ be a continuous and completely continuous mapping with $\mu Au \neq u$ for every $u \in \partial K_r$ and $0 < \mu \leq 1$. Then $i(A, K_r, K) = 1$.*

Lemma 3.[16] *Let $A : K \to K$ be a continuous and completely continuous mapping. Suppose that the following two conditions are satisfied:*

(i) $\inf_{u \in \partial K_r} ||Au|| > 0$;

(ii) $\mu Au \neq u$ for every $u \in \partial K_r$ and $\mu \geq 1$.

Then, $i(A, K_r, K) = 0$.

3. Proof of Main Results

Let K be a cone in $E = C[0, 1]$ defined by

$$K = \{u \in E;\ \ u(t) \geq \min\{t, 1 - t\}\|u\|,\ t \in [0, 1]\}.$$

Define an operator $A : K \to K$ as follows

$$(Au)(t) = \int_0^1 \int_0^1 G(t, \tau)G(\tau, s)f(s, u(s))dsd\tau. \tag{6}$$

It is clear that $A : K \to K$ is continuous and completely continuous.

Then we have the following lemmas.

Lemma 4. *Assume that (P_1) holds. Then $A(K) \subset K$.*
Proof. We have from (4) and (6) that

$$\begin{aligned} (Au)(t) &\geq \min\{t, 1 - t\} \textstyle\int_0^1 \int_0^1 G(\tau, \tau)G(\tau, s)f(s, u(s))dsd\tau \\ &\geq \min\{t, 1 - t\}||Au||,\ t \in [0, 1]. \end{aligned}$$

Thus we have $A(K) \subset K$.

Lemma 5. *If (P_1) and (H_3) are satisfied, then $i(A, K_p, K) = 1$.*
Proof. For any $u \in \partial K_p$, we have

$$f(t, u(t)) < \eta p,\ \forall t \in [0, 1],$$

so we have

$$\begin{aligned} ||Au|| &\le \int_0^1 \int_0^1 G(\tau,\tau)G(\tau,s)f(s,u(s))dsd\tau \\ &< \eta p \int_0^1 \int_0^1 G(\tau,\tau)G(\tau,s)dsd\tau \\ &= p = ||u||. \end{aligned}$$

Therefore, from the second part of Lemma 1, we conclude that $i(A, K_p, K) = 1$.

Lemma 6. *If* (P_1) *and* (H_4) *are satisfied, then* $i(A, K_p, K) = 0$.
Proof. Let $u \in \partial K_p$. Then we have from (5) that

$$u(t) \ge \min\{t, 1-t\}||u|| \ge \frac{1}{4}p, \quad \frac{1}{4} \le t \le \frac{3}{4},$$

and it follows from (H_4) that

$$\begin{aligned} (Au)(\sigma) &= \int_0^1 \int_0^1 G(\sigma,\tau)G(\tau,s)f(s,u(s))dsd\tau \\ &\ge \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\sigma,\tau)G(\tau,s)f(s,u(s))dsd\tau \\ &> \lambda p \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\sigma,\tau)G(\tau,s)dsd\tau \\ &= p = ||u||. \end{aligned}$$

This shows that

$$||Au|| > ||u||, \ \forall u \in \partial K_p.$$

Therefore, from the first part of Lemma 1, we conclude that $i(A, K_p, K) = 0$.

Proof of Theorem 1. According to Lemma 5, we have that

$$i(A, K_p, K) = 1. \tag{7}$$

Suppose that (H_1) holds. Since $f_0 > \lambda_1$, one can find $\varepsilon > 0$ and $0 < r_0 < p$ so that

$$f(t,u) \ge (\lambda_1 + \varepsilon)u, \ \forall t \in [0,1], \ 0 \le u \le r_0. \tag{8}$$

Let $r \in (0, r_0)$. Then for $u \in \partial K_r$ we have $u(t) \ge \frac{1}{4}r$ for $t \in [\frac{1}{4}, \frac{3}{4}]$, and so

$$\begin{aligned} (Au)(\sigma) &= \int_0^1 \int_0^1 G(\sigma,\tau)G(\tau,s)f(s,u(s))dsd\tau \\ &\ge \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\sigma,\tau)G(\tau,s)f(s,u(s))dsd\tau \\ &\ge (\lambda_1+\varepsilon) \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\sigma,\tau)G(\tau,s)u(s)dsd\tau \\ &\ge \frac{(\lambda_1+\varepsilon)r}{4} \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\sigma,\tau)G(\tau,s)dsd\tau, \end{aligned}$$

from which we see that $\inf_{u \in \partial K_r} ||Au|| > 0$, and therefore, hypothesis (i) of Lemma 3 holds. Next we show that $\mu Au \ne u$ for any $u \in \partial K_r$ and $\mu \ge 1$. If this is not true, then there exist $u_0 \in \partial K_r$ and $\mu_0 \ge 1$ such that $\mu_0 Au_0 = u_0$. Note that $u_0(t)$ satisfies

$$u_0^{(4)}(t) = \mu_0 f(t, u_0(t)), \ \ 0 \le t \le 1, \tag{9}$$

and the boundary condition (2). Multiply equation (9) by $\phi_1(t)$ and integrate from 0 to 1, using integration by parts in the left side, to obtain

$$\begin{aligned}\lambda_1 \int_0^1 u_0(t)\phi_1(t)dt &= \mu_0 \int_0^1 \phi_1(t)f(t,u_0(t))dt \\ &\geq \int_0^1 \phi_1(t)f(t,u_0(t))dt \\ &\geq (\lambda_1+\varepsilon)\int_0^1 \phi_1(t)u_0(t)dt.\end{aligned}$$

Since $u_0(t) \geq \min\{t, 1-t\}||u_0||$, we have $\int_0^1 \phi_1(t)u_0(t)dt > 0$, and so from the above inequality we see that $\lambda_1 \geq \lambda_1 + \varepsilon$, which is a contradiction. Hence A satisfies the hypotheses of Lemma 3 in K_r. By Lemma 3, we have

$$i(A, K_r, K) = 0. \tag{10}$$

On the other hand, since $f_\infty > \lambda_1$, there exist $\varepsilon > 0$ and $H > 0$ such that

$$f(t,u) \geq (\lambda_1+\varepsilon)u, \ \forall t \in [0,1],\ u \geq H. \tag{11}$$

Let $C = \max\limits_{0\leq u\leq H}\max\limits_{0\leq t\leq 1}|f(t,u) - (\lambda_1+\varepsilon)u| + 1$, and it is clear that

$$f(t,u) \geq (\lambda_1+\varepsilon)u - C, \ \forall t \in [0,1],\ u \geq 0. \tag{12}$$

Choose $R > R_0 := \max\{4H, p\}$. Let $u \in \partial K_R$. Since $u(t) \geq \frac{1}{4}||u|| > H$ for $t \in [\frac{1}{4}, \frac{3}{4}]$, from (11) we see that

$$f(t,u(t)) \geq (\lambda_1+\varepsilon)u(t) \geq \frac{1}{4}(\lambda_1+\varepsilon)||u||, \ \forall t \in [\frac{1}{4}, \frac{3}{4}].$$

Essentially the same reasoning as above yields $\inf_{u\in\partial K_R}||Au|| > 0$. Next we show that if R is large enough, then $\mu Au \neq u$ for any $u \in \partial K_R$ and $\mu \geq 1$. In fact, if there exist $u_0 \in \partial K_R$ and $\mu_0 \geq 1$ such that $\mu_0 A u_0 = u_0$, then $u_0(t)$ satisfies equation (9) and boundary condition (2). Multiply equation (9) by $\phi_1(t)$ and integrate (use (12)) to obtain

$$\begin{aligned}\lambda_1 \int_0^1 u_0(t)\phi_1(t)dt &= \mu_0 \int_0^1 f(t,u_0(t))\phi_1(t)dt \\ &\geq (\lambda_1+\varepsilon)\int_0^1 u_0(t)\phi_1(t)dt - C\int_0^1 \phi_1(t)dt.\end{aligned}$$

Consequently, we obtain that

$$\int_0^1 u_0(t)\phi_1(t)dt \leq \frac{C}{\varepsilon}\int_0^1 \phi_1(t)dt. \tag{13}$$

We also have

$$\begin{aligned}\int_0^1 u_0(t)\phi_1(t)dt &\geq ||u_0||\int_0^1 \min\{t, 1-t\}\phi_1(t)dt \\ &\geq ||u_0||\int_0^1 t(1-t)\phi_1(t)dt,\end{aligned}$$

and this together with (13) yields

$$||u_0|| \leq \frac{C\int_0^1 \phi_1(t)dt}{\varepsilon\int_0^1 t(1-t)\phi_1(t)dt} =: \bar{R}. \tag{14}$$

Let $R > \max\{\bar{R}, R_0\}$. Then for any $u \in \partial K_R$ and $\mu \geq 1$ we have $\mu Au \neq u$. Hence hypothesis (ii) of Lemma 3 also holds. By Lemma 3,

$$i(A, K_R, K) = 0. \tag{15}$$

In view of (7),(10) and (15), we obtain from the additivity property of the fixed-point index that

$$i(A, K_R \setminus \bar{K_p}, K) = -1,$$
$$i(A, K_p \setminus \bar{K_r}, K) = 1.$$

Thus, A has fixed points u_1 and u_2 in $K_p \setminus \bar{K_r}$ and $K_R \setminus \bar{K_p}$, respectively, which means $u_1(t)$ and $u_2(t)$ are positive solution of BVP (1)-(2) and $0 < ||u_1|| < p < ||u_2||$.

Remark 3. *Note to deduce the existence of* u_1 *in Theorem 1 we need only assume* $(P1)$, $(H3)$ *and* $f_0 > \lambda_1$. *A similar remark applies to* u_2.

Proof of Theorem 2. According to Lemma 3, we have that

$$i(A, K_p, K) = 0. \tag{16}$$

Suppose that (H_2) holds. Since $f^0 < \lambda_1$, one can find $\varepsilon > 0$ and $0 < r_0 < p$ so that

$$f(t, u) \leq (\lambda_1 - \varepsilon)u, \ \forall t \in [0, 1], \ 0 \leq u \leq r_0. \tag{17}$$

Let $r \in (0, r_0)$. We now prove that $\mu Au \neq u$ for any $u \in \partial K_r$ and $0 < \mu \leq 1$. If this is not true, then there exist $u_0 \in \partial K_r$ and $0 < \mu_0 \leq 1$ such that $\mu_0 Au_0 = u_0$. Then $u_0(t)$ satisfies equation (9) and boundary condition (2). Multiply equation (9) by $\phi_1(t)$ and integrate (use (17)) to obtain

$$\begin{aligned} \lambda_1 \int_0^1 u_0(t)\phi_1(t)dt &= \mu_0 \int_0^1 \phi_1(t) f(t, u_0(t))dt \\ &\leq (\lambda_1 - \varepsilon) \int_0^1 \phi_1(t) u_0(t)dt. \end{aligned}$$

Since $u_0(t) \geq \min\{t, 1-t\}||u_0||$, we have $\int_0^1 \phi_1(t)u_0(t)dt > 0$, and so from the above inequality we see that $\lambda_1 \leq \lambda_1 - \varepsilon$, which is a contradiction. By Lemma 2, we have

$$i(A, K_r, K) = 1. \tag{18}$$

On the other hand, since $f^\infty < \lambda_1$, there exist $\varepsilon > 0$ and $H > p$ such that

$$f(t, u) \leq (\lambda_1 - \varepsilon)u, \ \forall t \in [0, 1], \ u \geq H.$$

Let $C = \max\limits_{0 \leq u \leq H} \max\limits_{0 \leq t \leq 1} |f(t, u) - (\lambda_1 - \varepsilon)u| + 1$, and it is clear that

$$f(t, u) \leq (\lambda_1 - \varepsilon)u + C, \ \forall t \in [0, 1], \ u \geq 0. \tag{19}$$

We can show that there exists $R > H > p$ such that $\mu Au \neq u$ for any $u \in \partial K_R$ and $0 < \mu \leq 1$; we omit the details, since they are similar to those in the proof of Theorem 1. Thus, we obtain

$$i(A, K_R, K) = 1. \tag{20}$$

In view of (16),(18) and (20), we obtain

$$i(A, K_R \setminus \bar{K_p}, K) = 1,$$
$$i(A, K_p \setminus \bar{K_r}, K) = -1.$$

Thus, A has fixed points u_1 and u_2 in $K_p \setminus \bar{K_r}$ and $K_R \setminus \bar{K_p}$, respectively, which means $u_1(t)$ and $u_2(t)$ are positive solution of BVP (1)-(2) and $0 < ||u_1|| < p < ||u_2||$.

The proofs of Theorem 3 and 4 follow the ideas in the proofs of Theorems 1 and 2.

4. Appendix and Example

In this section, we shall give the computations for η, σ and λ. Note that

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Thus we have

$$\begin{aligned} & \int_0^1 \int_0^1 G(\tau,\tau)G(\tau,s)dsd\tau \\ = \ & \int_0^1 G(\tau,\tau)[\int_0^\tau G(\tau,s)ds + \int_\tau^1 G(\tau,s)ds]d\tau \\ = \ & \int_0^1 G(\tau,\tau)[\int_0^\tau s(1-\tau)ds + \int_\tau^1 \tau(1-s)ds]d\tau \\ = \ & \int_0^1 \tau(1-\tau) \times \frac{\tau(1-\tau)}{2} d\tau = \frac{1}{60}, \end{aligned}$$

and so

$$\eta = [\int_0^1 \int_0^1 G(\tau,\tau)G(\tau,s)dsd\tau]^{-1} = 60.$$

On the other hand, we have

$$g(\tau) := \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,s)ds = \begin{cases} \int_{\frac{1}{4}}^{\frac{3}{4}} \tau(1-s)ds = \frac{\tau}{4}, & \tau \in [0, \frac{1}{4}], \\ \int_{\frac{1}{4}}^{\tau} s(1-\tau)ds + \int_{\tau}^{\frac{3}{4}} \tau(1-s)ds & \\ \qquad = -\frac{\tau^2}{2} + \frac{\tau}{2} - \frac{1}{32}, & \tau \in [\frac{1}{4}, \frac{3}{4}], \\ \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-\tau)ds = \frac{1-\tau}{4}, & \tau \in [\frac{3}{4}, 1]. \end{cases}$$

Then it is easy to verify that $g(\tau) = g(1-\tau)$ for $\tau \in [0,1]$. Thus one derives that

$$\begin{aligned} F(t) : \ & = \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,\tau)G(\tau,s)dsd\tau \\ & = \int_0^1 G(t,\tau)g(\tau)d\tau \\ & = \int_0^t \tau(1-t)g(\tau)d\tau + \int_t^1 t(1-\tau)g(\tau)d\tau, \end{aligned}$$

and $F(t) = F(1-t)$ for $t \in [0,1]$. Furthermore, we obtain

$$\begin{aligned} \frac{dF(t)}{dt} & = -\int_0^t \tau g(\tau)d\tau + (1-t)tg(t) + \int_t^1 (1-\tau)g(\tau)d\tau - (1-t)tg(t) \\ & = -\int_0^t \tau g(\tau)d\tau + \int_t^1 (1-\tau)g(\tau)d\tau \\ & = -\int_0^t \tau g(\tau)d\tau + \int_0^{1-t} \tau g(\tau)d\tau, \end{aligned}$$

and thus

$$\left.\frac{dF(t)}{dt}\right|_{t=0} > 0, \quad \left.\frac{dF(t)}{dt}\right|_{t=\frac{1}{2}} = 0, \quad \left.\frac{dF(t)}{dt}\right|_{t=1} < 0.$$

Noting that $\frac{dF(t)}{dt} = 0$ has only one zero point $t = \frac{1}{2}$, $F(t)$ arrives it maximum at $t = \frac{1}{2}$. That is

$$\int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, \tau)G(\tau, s)dsd\tau = \max_{t\in[0,1]} \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(t, \tau)G(\tau, s)dsd\tau,$$

and $\sigma = \frac{1}{2}$. Therefore, we have

$$\begin{aligned}
\lambda &= [\textstyle\int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, \tau)G(\tau, s)dsd\tau]^{-1} \\
&= [\textstyle\int_0^{\frac{1}{2}} \tau(1 - \frac{1}{2})g(\tau)d\tau + \int_{\frac{1}{2}}^{1} \frac{1}{2}(1 - \tau)g(\tau)d\tau]^{-1} \\
&= [\textstyle\int_0^{\frac{1}{2}} \tau g(\tau)d\tau]^{-1} \\
&= [\textstyle\int_0^{\frac{1}{4}} \frac{\tau^2}{4}d\tau + \int_{\frac{1}{4}}^{\frac{1}{2}} \tau(-\frac{\tau^2}{2} + \frac{\tau}{2} - \frac{1}{32})d\tau]^{-1} \\
&= \left\{ \left.\frac{\tau^3}{12}\right|_0^{\frac{1}{4}} + \left.[-\frac{\tau^4}{8} + \frac{\tau^3}{6} - \frac{\tau^2}{64}]\right|_{\frac{1}{4}}^{\frac{1}{2}} \right\}^{-1} \\
&= \left\{ \frac{1}{64\times12} + (-\frac{1}{8\times16} + \frac{1}{6\times8} - \frac{1}{64\times4}) - (-\frac{1}{8\times256} + \frac{1}{6\times64} - \frac{1}{64\times16}) \right\}^{-1} \\
&= [\frac{15}{64\times4\times3} - \frac{21}{64\times4\times8}]^{-1} \\
&= [\frac{57}{64\times8\times4\times3}]^{-1} = \frac{6144}{57} \simeq 108.
\end{aligned}$$

Example. Consider the boundary value problem

$$\begin{aligned}
&u^{(4)}(t) = u^a(t) + u^b(t), \quad 0 < a < 1 < b, \\
&u(0) = u(1) = u''(0) = u''(1) = 0.
\end{aligned} \tag{21}$$

Then the BVP (21) has at least two positive solutions u_1 and u_2 with

$$0 < ||u_1|| < 1 < ||u_2||.$$

To see this we will apply Theorem 1 (or Corollary 1). Set

$$f(t, u) = u^a + u^b.$$

Note

$$\lim_{u\downarrow 0} \frac{f(t, u)}{u} = \infty \quad \text{and} \quad \lim_{u\uparrow\infty} \frac{f(t, u)}{u} = \infty,$$

so (H_1) (or (H_1^*)) holds. Clearly, (P_1) holds. Note $\eta = 60$. Since there exists $p = 1$ such that $0 \leq u \leq p$ implies

$$f(t, u) \leq p^a + p^b = 2 < \eta = \eta p,$$

we have that (H_3) holds. The result is now from Theorem 1 (or Corollary 1).

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