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General Wentzell boundary conditions, differential operators and analytic semigroups in $C[0,1]$. [1]

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ABSTRACT: We are concerned with the study of the analyticity of the (C_0) semigroup generated by the realizations of the operators $Au = \alpha u'' + \beta u'$ or $Au = b(au')' + \beta u'$ in $C[0,1]$ with general Wentzell boundary conditions of the type $\lim_{x\to j} Au(x) + \widetilde{b}(x)u'(x) = 0$ for $j = 0,1$ in $C[0,1]$. Here the functions $a, \alpha, \beta, b, \widetilde{b}$ are assumed to be in $C[0,1]$, with a, $\alpha \in C^1(0,1)$, $a(x) > 0$, $\alpha(x) > 0$, in $(0,1)$, $b(x) > 0$ in $[0,1]$ and a, or α, possibly degenerate at the endpoints, i.e. a, or α, allowed to vanish at 0 and 1.

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1. Introduction

Inspired by their previous results concerning generation and analyticity for (C_0) semigroups generated by degenerate elliptic second order differential operators with Wentzell boundary conditions, the authors started a systematic investigation of analogous problems for some classes of linear or nonlinear, possibly degenerate at the boundary, second order differential operators with *general Wentzell boundary conditions* in different function spaces. General Wentzell boundary condition for such an operator A reads as follows

$$Au + \beta \frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on} \quad \partial\Omega$$

where Ω is a bounded subset of $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$, β, γ are nonnegative continuous functions on $\partial\Omega$ with $\beta > 0$ and n is the unit outer normal. In [9] we proved generation results in $C[0,1]$ which extended substantially earlier results dealing with Dirichlet, Neumann, Robin and Wentzell boundary conditions. Subsequently generation and regularity in $C(\overline{\Omega})$, in suitable $L^p(\Omega, \mu)$

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spaces and in $H^1(\Omega)$ were obtained in the papers [10][1], [12] and [13], while in [11] the wave equation with *Wentzell boundary conditions* (i.e. $\beta = \gamma = 0$ on $\partial\Omega$) for $\Omega = (0, 1)$ was considered. After these results, in a couple of years, an increasing and wide interest for the above topics led to significant and widespread progress, by using different approaches and techniques: see e.g. Warma [20], Arendt, Metafune, Pallara and Romanelli [1], Xiao and Liang [21], Engel [5], [6], Vogt and Voigt [19], and Batkai and Engel [2]. In particular, it is worthwhile to point out that the recent results by Xiao and Liang [21] and Batkai and Engel [2] revealed that, thanks to a suitable abstract framework, analyticity with general Wentzell boundary condition for second order (in time) equations in $C[0, 1]$ can be obtained as a byproduct of the study of cosine families with general Wentzell boundary conditions. For wave equations, see also Gal, G. R. Goldstein and J. A. Goldstein [15]. Despite so many different efforts, the problem of analyticity in the degenerate case remains not completely solved even in $C[0, 1]$. Here our aim is to fill this lack, at least in the cases

$$Au := \alpha u'' + \beta u' \qquad \text{or} \qquad Au := b(au')' + \beta u'$$

with general Wentzell boundary conditions of the type

$$\lim_{x\to j} Au(x) + \tilde{b}(x)u'(x) = 0 \qquad j = 0, 1 \tag{1}$$

provided that $a, \alpha \in C[0, 1] \cap C^1(0, 1)$, $\beta, b, \tilde{b} \in C[0, 1]$, with $a > 0$, or $\alpha > 0$, in $(0, 1)$, $b > 0$ in $[0, 1]$ and a, or α, possibly degenerate (i.e. a, or α, allowed to vanish at the endpoints 0 and 1). Our method relies in the idea of replacing the considered operator by a related new operator with *pure* Wentzell boundary conditions (i.e. $Au(j) = 0$ at $j = 0, 1$) which generates an analytic semigroup, and therefore into coming back to the original operator by a suitable perturbation which saves analyticity. To this aim in Section 1 we present analyticity results stated in previous papers, while Section 2 is devoted to the proof of the main results.

2. Preliminary results

In order to make more clear the tools involved in the proof of the main theorem, we recall two theorems, one due to Campiti and Metafune [[3], Theorem 4.2] and the other to Favini and Romanelli [[14], Theorems 2.2 and 2.5] (see also Favini, G. R. Goldstein, J. A. Goldstein and Romanelli [8]).

Theorem 1 *Assume that $\alpha, \beta \in C[0, 1]$, where $\alpha > 0$ in $(0, 1)$, $\frac{1}{\alpha} \in L^1(0, 1)$, and β is Hölder continuous at $x = 0, 1$. Then the operator $(L, D(L))$ given by*

$$Lu := \alpha u'' + \beta u', \tag{2}$$

$$D(L) := \{u \in C[0, 1] \cap C^2(0, 1) \,:\, Lu \in C[0, 1],\ \lim_{x\to j} Lu(x) = 0,\ for\, j = 0, 1\} \tag{3}$$

generates an analytic semigroup on $C[0, 1]$.

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Theorem 2 *Let $\alpha, \beta \in C[0,1]$ are such that $\alpha > 0$ on $(0,1)$, $\sqrt{\alpha} \in C^1[0,1]$, $\alpha(0) = 0 = \alpha(1)$, and $\frac{\beta}{\sqrt{\alpha}} \in C[0,1]$.*

Then the operator $(G, D(G))$ given by

$$\begin{aligned} Gu &:= \alpha u'' + \beta u', \\ D(G) &:= \{u \in C[0,1] \cap C^2(0,1) \,:\, Gu \in C[0,1],\, \lim_{x\to j} Gu(x) = 0,\, \textit{for}\, j = 0,1\} \end{aligned}$$

generates an analytic semigroup on $C[0,1]$.

3. Main Results

Let us start by considering an operator of the type $Au := \alpha u''$ where the coefficient α may degenerate at the boundary, but with degeneracy of low order.

Theorem 3 *Let $\alpha, \beta \in C[0,1]$ verify the following assumptions:*

$$\alpha > 0 \textit{ in } (0,1), \tag{4}$$

$$\textit{the mapping } x \to \int_0^x \frac{dt}{\alpha(t)} \textit{ (makes sense and) is Hölder continuous on } [0,1] \textit{ with exponent } k \in (0,1), \tag{5}$$

$$\beta \textit{ is Hölder continuous at } x = 0,1. \tag{6}$$

Then the operator $(A, D(A))$ given by

$$\begin{aligned} Au &:= \alpha u'' \\ D(A) &:= \{u \in C[0,1] \cap C^2(0,1) \,:\, Au \in C[0,1],\, \lim_{x\to j} Au(x) + \beta(x)u'(x) = 0, \\ &\quad\ \textit{for}\, j = 0,1\} \end{aligned}$$

generates an analytic semigroup on $C[0,1]$.

Proof Let us observe that assumption (5) implies $\frac{1}{\alpha} \in L^1(0,1)$ and, consequently, by Theorem 1, the operator $(L, D(L))$ given by

$$\begin{aligned} Lu &:= \alpha u'' + \beta u' \\ D(L) &:= D(A) \end{aligned}$$

generates an analytic semigroup on $C[0,1]$. Now, let $Bu := -\beta u'$ with domain $D(B) := D(L)$; by suitably adapting the well-known argument that the first derivative operator is bounded with respect to the second derivative operator in various function spaces (see e.g. [[7], pp. 170-171]), we will show that $(B, D(B))$ is L-bounded with L-bound equal to zero. Hence, by Hille's theorem [[17], p. 190], $A = L + B$ will generate an analytic semigroup on $C[0,1]$.

First we show that every $u \in C[0,1] \cap C^2(0,1)$ with $\alpha u'' + \beta u' \in C[0,1]$ has its first derivative u' in $C[0,1]$. Indeed, from $\frac{1}{\alpha} \in L^1(0,1)$ it follows that the function

$$W(x) := exp(-\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\, dt)$$

satisfies

$$0 < c_o \le W(x) \le c_1,$$

for suitable positive constants c_o, c_1. Since

$$f := \alpha W(\frac{u'}{W})' \in C[0,1],$$

we deduce

$$\frac{u'(x)}{W(x)} - \frac{u'(\frac{1}{2})}{W(\frac{1}{2})} = \int_{\frac{1}{2}}^{x} \frac{f(t)}{\alpha(t)W(t)}\, dt$$

and, consequently,

$$u'(x) = W(x)\left[\frac{u'(\frac{1}{2})}{W(\frac{1}{2})} + \int_{\frac{1}{2}}^{x} \frac{f(t)}{\alpha(t)W(t)}\, dt\right].$$

Therefore our assertion holds.

Moreover every $u \in D(L)$ verifies the inequality

$$|u''(x)| \le \frac{1}{\alpha(x)}\|\alpha u'' + \beta u'\|_\infty + \frac{|\beta(x)|}{\alpha(x)}|u'(x)|, \quad x \in (0,1),$$

hence $u'' \in L^1(0,1)$. Let

$$\beta_0 := \max_{x\in[0,1]} |\beta(x)|$$

and fix a (large) positive integer n. For sake of brevity, denote by

$$\|u\|_{J_i,\infty}, \quad j = 0, 1, ..., n-1$$

the supremum norm in the space $C(J_i)$, where

$$J_i = [\frac{i}{n}, \frac{i+1}{n}] \quad i = 0, 1, ..., n-1.$$

Take $u \in D(A)$, $x \in J_0$, $s \in [0, \frac{1}{3n}]$ and $t \in [\frac{2}{3n}, \frac{1}{n}]$. From

$$|u'(x)| = |u'(x_0) + \int_{x_0}^{x} u''(y)\, dy| \le 3n(|u(s)| + |u(t)|) + \int_{J_0} |u''(y)|\, dy$$

for a suitable $x_0 \in J_0$, it follows that

$$\begin{aligned} |u'(x)| \quad &\le 6n\|u\|_{J_0,\infty} + \textstyle\int_0^{\frac{1}{n}} |\alpha(y)u''(y) + \beta(y)u'(y)| \frac{dy}{\alpha(y)} \\ &\quad + \textstyle\int_0^{\frac{1}{n}} |\beta(y)u'(y)| \frac{dy}{\alpha(y)}. \end{aligned} \tag{7}$$

Therefore, in view of (5), there exists a suitable constant $C > 0$ such that for any $x \in J_0$, we have

$$|\beta(x)u'(x)| \le 6n\beta_0\|u\|_{J_0,\infty} + \beta_0 C\frac{1}{n^k}\|Lu\|_{J_0,\infty} + \beta_0 C\frac{1}{n^k}\|\beta u'\|_{J_0,\infty}, \tag{8}$$

and this yields

$$(1-\beta_0\frac{C}{n^k})\|\beta u'\|_{J_0,\infty} \leq 6n\beta_0\|u\|_{J_0,\infty} + \beta_0 C\frac{1}{n^k}\|Lu\|_{J_0,\infty}. \tag{9}$$

On the other hand, we can repeat the argument on J_1, if $x \in J_1$ and obtain that

$$|\beta(x)u'(x)| \leq 6n\beta_0\|u\|_{J_1,\infty} + \beta_0 C(\frac{2}{n}-\frac{1}{n})^k\|Lu\|_{J_1,\infty} + \beta_0\frac{C}{n^k}\|\beta u'\|_{J_1,\infty}.$$

Thus, in general, for $i = 0, 1, ..., n-1$ and $x \in J_i$, we have

$$|\beta(x)u'(x)| \leq 6n\beta_0\|u\|_{J_i,\infty} + \beta_0\frac{C}{n^k}\|Lu\|_{J_i,\infty} + \beta_0\frac{C}{n^k}\|\beta u'\|_{J_i,\infty}.$$

Taking n sufficiently large so that $1-\beta_0\frac{C}{n^k} > 0$, we deduce

$$\sup_{x\in J_i}|\beta(x)u'(x)|(1-\beta_0\frac{C}{n^k}) \leq 6n\beta_0\|u\|_{J_i,\infty} + \beta_0\frac{C}{n^k}\|Lu\|_{J_i,\infty}.$$

Therefore

$$\|Bu\|_\infty \leq \frac{6n\beta_0}{1-\beta_0\frac{C}{n^k}}\|u\|_\infty + \frac{\beta_0\frac{C}{n^k}}{1-\beta_0\frac{C}{n^k}}\|Lu\|_\infty. \tag{10}$$

Since we can choose $n \in \mathbf{N}$ arbitrarily large, our claim is proved. ▮

Remark 1 *Notice that, if in the previous theorem we assume $\beta(0) \leq 0$ and $\beta(1) \geq 0$, then the semigroup generated by $(A, D(A))$ is a Feller semigroup (i.e. it is contractive and positive) according to [[9], Theorem 1.1].*

The above result can be extended to operators of more general type. Indeed the following results hold.

Corollary 3A *Let us consider α, β, b in $C[0,1]$ such that α satisfies assumptions (4) and (5) and, in addition, suppose that*

$$\beta + b \quad \textit{is Hölder continuous at} \quad x = 0, 1. \tag{11}$$

Then the operator $(C, D(C))$ given by

$$\begin{aligned} Cu &:= \alpha u'' + \beta u', \\ D(C) &:= \{u \in C[0,1] \cap C^2(0,1) \,:\, Cu \in C[0,1],\, \lim_{x\to j} Cu(x) + b(x)u'(x) = 0, \\ &\quad \textit{for } j = 0, 1\} \end{aligned}$$

generates an analytic semigroup on $C[0,1]$.

Proof According to Theorem 1, the operator $L_1 u = \alpha u'' + (\beta + b)u'$ with domain $D(L_1) = \{u \in C[0,1] \cap C^2(0,1) : L_1 u \in C[0,1], L_1 u(j) = 0$ for $j = 0, 1\}$ generates an analytic semigroup on $C[0,1]$. Now, in analogy with the previous proof, we will show that the operator $B_1 u := -bu'$ with domain $D(B_1) := D(L_1)$ is L_1-bounded with L_1-bound equal to zero. Indeed, arguing as in Theorem 3, it is still true that every $u \in C[0,1] \cap C^2(0,1)$ with $L_1 u \in C[0,1]$ has its first derivative u' in $C[0,1]$ and its second derivative u'' in $L^1(0,1)$. Let

$$\tilde{\beta_0} := \max_{x\in[0,1]} |(\beta + b)(x)|, \qquad b_0 := \max_{x\in[0,1]} |b(x)|.$$

By similar arguments as in Theorem 3) we can find a suitable positive constant C such that

$$\begin{aligned} \|bu'\|_\infty \quad &\le 6nb_0\|u\|_\infty + \tfrac{b_0 C}{n^k}\|L_1 u\|_\infty \\ &\quad + \tfrac{b_0 C}{n^k}\|(\beta+b)u'\|_\infty. \end{aligned} \tag{12}$$

On the other hand, for n sufficiently large, we rewrite the inequality (10) where β is replaced by $\beta + b$, Bu by $(\beta+b)u'$ and L by L_1. We obtain that

$$\|(\beta+b)u'\|_\infty \quad \le \frac{6n\tilde{\beta_0}}{1-\frac{\tilde{\beta_0}C}{n^k}}\|u\|_\infty + \frac{\frac{\tilde{\beta_0}C}{n^k}}{1-\frac{\tilde{\beta_0}C}{n^k}}\|L_1 u\|_\infty. \tag{13}$$

Then, plugging (13) into (12) gives

$$\begin{aligned} \|bu'\|_\infty \quad \le &\left[6nb_0 + \frac{b_0 C}{n^k}\left(\frac{6n\tilde{\beta_0}}{1-\frac{\tilde{\beta_0}C}{n^k}}\right)\right]\|u\|_\infty + \\ &\left[\frac{b_0 C}{n^k}\left(1+\frac{\frac{\tilde{\beta_0}C}{n^k}}{1-\frac{\tilde{\beta_0}C}{n^k}}\right)\right]\|L_1 u\|_\infty. \end{aligned}$$

Since we can choose $n \in \mathbf{N}$ sufficiently large, the above calculations imply our assertion.

Corollary 3B *Let us consider $a, b, \beta_1, \widetilde{b}$ in $C[0,1]$ such that $a \in C^1[0,1]$ satisfies assumptions (4) and (5) and $b(x) > 0$ for any $x \in [0,1]$. In addition, suppose that*

$$a'b + \beta_1 + \widetilde{b} \quad \textit{is Hölder continuous at} \quad x = 0, 1. \tag{14}$$

Then the operator $(C_1, D(C_1))$ given by

$$\begin{aligned} C_1 u \quad &:= \quad b(au')' + \beta_1 u', \\ D(C_1) \quad &:= \quad \{u \in C[0,1] \cap C^2(0,1) \,:\, C_1 u \in C[0,1],\, \lim_{x\to j} C_1 u(x) + \widetilde{b}(x)u'(x) = 0, \\ &\qquad \textit{for } j = 0, 1\} \end{aligned}$$

generates an analytic semigroup on $C[0,1]$.

Proof It suffices to observe that the operator

$$C_1 u := abu'' + (a'b + \beta_1)u'$$

with domain $D(C_1)$ satisfies the assumptions of the previous Corollary.

Now, let us consider the case of the coefficient α with degeneracy of higher order at 0 and 1.

Theorem 4 *Let α, β, b be in $C[0,1]$ such that*

$$\alpha > 0 \in (0,1), \sqrt{\alpha} \in C^1[0,1], \alpha(0) = 0 = \alpha(1), \tag{15}$$

$$\tfrac{\beta+b-\alpha'}{\alpha} \in C[0,1]. \tag{16}$$

Then the operator $(A, D(A))$ given by

$$\begin{aligned} Au &:= \alpha u'' + \beta u', \\ D(A) &:= \{u \in C[0,1] \cap C^2(0,1) \,:\, Au \in C[0,1],\ \lim_{x\to j} Au(x) + b(x)u'(x) = 0, \\ &\quad \textit{for } j = 0,1\} \end{aligned}$$

generates an analytic semigroup on $C[0,1]$.

Proof Define

$$W_1(x) := exp\left(-\int_{\frac{1}{2}}^{x} \frac{\beta(t)+b(t)}{\alpha(t)}\,dt\right), \quad x \in (0,1)$$

and observe that

$$W_1(x) = \frac{\alpha(\frac{1}{2})}{\alpha(x)}\Phi(x) \qquad x \in (0,1)$$

where

$$\Phi(x) := exp\left(-\int_{\frac{1}{2}}^{x} \frac{\beta(t)+b(t)-\alpha'(t)}{\alpha(t)}\,dt\right), \quad x \in (0,1).$$

Therefore there exist two positive constants C_0, C_1 such that

$$C_0 \le \alpha(x)W_1(x) \le C_1. \tag{17}$$

It is straightforward to check that the operator $Lu := \alpha u'' + (\beta+b)u' = \alpha W_1 \left(\frac{u'}{W_1}\right)'$ with domain $D(L) := D(A)$ satisfies assumptions of Theorem 2 and hence generates an analytic semigroup on $C[0,1]$. On the other hand, the operator $B_0 u := \frac{u'}{W_1}$ with domain

$$D(B_0) := \{u \in C[0,1] \cap C^1(0,1) : \lim_{x\to j} \frac{u'(x)}{W_1(x)} \in \mathbf{C}, \quad j = 0,1\}$$

generates a (C_0) group on $C[0,1]$. Indeed, the mapping

$$\varphi(x) := \int_{\frac{1}{2}}^{x} W_1(t)\,dt$$

is strictly increasing and differentiable, and as in [14] (see also [8]), this allows to deduce that the operator $(B_0, D(B_0))$ is similar to the operator $(B_\infty, D(B_\infty))$ given by

$$\begin{aligned} B_\infty v &:= v', \\ D(B_\infty) &:= \{v \in C(\overline{\mathbf{R}}) : v \quad \text{differentiable}, \quad v' \in C(\overline{\mathbf{R}})\} \end{aligned}$$

where
$$C(\overline{\mathbf{R}}) := \{v \in C(\mathbf{R}) : \lim_{y\to\tau} v(y) \in \mathbf{C}, \text{ for } \tau \in \{+\infty, -\infty\}\}.$$

Consequently, according to [[7], Chapter II Corollary 4.9], the operator $(B_0^2, D(B_0^2))$ generates an analytic semigroup on $C[0,1]$. Here we have that

$$\begin{aligned} D(B_0^2) &= \{u \in C[0,1] \cap C^2(0,1) : \lim_{x\to j} \frac{u'(x)}{W_1(x)} \in \mathbf{C}, \\ &\quad \lim_{x\to j} \frac{1}{W_1(x)} \left(\frac{u'}{W_1}\right)'(x) \in \mathbf{C} \quad \text{for} \quad j = 0,1\} \\ &= \{u \in C[0,1] \cap C^2(0,1) : \lim_{x\to j} \frac{u'(x)}{W_1(x)} = 0, \\ &\quad \lim_{x\to j} \frac{1}{W_1(x)} \left(\frac{u'}{W_1}\right)'(x) = 0 \quad \text{for} \quad j = 0,1\} \end{aligned}$$

and, arguing as in [14], or in [8], it follows that $D(B_0^2) = D(L)$.

In order to conclude our proof we have only to show that the operator $Bu := -bu'$ with domain $D(A)$ is L-bounded with L-bound equal to zero. According to (17) and above considerations, for any $\epsilon > 0$ there exists a suitable $k > 0$ such that

$$\begin{aligned} \|bu'\|_\infty &\le \|bW_1\|_\infty \|\frac{u'}{W_1}\|_\infty \\ &\le \|bW_1\|_\infty \left(\epsilon \|\frac{1}{W_1}\left(\frac{u'}{W_1}\right)'\|_\infty + \frac{k}{\epsilon}\|u\|_\infty\right) \\ &\le \|bW_1\|_\infty \left(\epsilon \|Lu\|_\infty \|\frac{1}{\alpha W_1^2}\|_\infty + \frac{k}{\epsilon}\|u\|_\infty\right). \end{aligned}$$

Hence the assertion holds true.

Corollary 4A *Let $a, \beta_1, b, \widetilde{b}$ be in $C[0,1]$ such that*

$$a > 0 \in (0,1), \sqrt{a} \in C^1[0,1], a(0) = 0 = a(1), \tag{18}$$

$$b > 0 \in [0,1], b \in C^1[0,1], \frac{ba' + \beta_1 + \widetilde{b} - (ba)'}{a} \in C[0,1]. \tag{19}$$

Then the operator $(A, D(A))$ given by

$$\begin{aligned} Au &:= b \quad (au')' + \beta_1 u', \\ D(A) &:= \{u \in C[0,1] \cap C^2(0,1) : Au \in C[0,1], \lim_{x\to j} Au(x) + \widetilde{b}(x)u'(x) = 0, \\ &\quad \textit{for } j = 0,1\} \end{aligned}$$

generates an analytic semigroup on $C[0,1]$.

Proof It suffices to apply the previous Theorem to the operator

$$Au = bau'' + (ba' + \beta_1)u', \quad u \in D(A).$$

Example Assume that $\alpha(x) := m(x)[x^i(1-x)^\lambda]$, for $x \in [0,1]$, where

$$m \in C[0,1], \quad m(x) > 0, \qquad x \in [0,1].$$

Since for any $x \in [0,1]$ and $0 < \delta < 1$, such that $x + \delta \in [0,1]$, and for any i, λ with $0 \leq i, \lambda < 1$ we have

$$\begin{aligned} \textstyle\int_x^{x+\delta} \frac{dt}{t^i(1-t)^\lambda} = & \int_x^{x+\delta} \frac{(1-t)^{1-\lambda}}{t^i}\,dt + \int_x^{x+\delta} \frac{t^{1-i}}{(1-t)^\lambda}\,dt \\ & \leq \int_x^{x+\delta} \frac{dt}{t^i} + \int_x^{x+\delta} \frac{dt}{(1-t)^\lambda} \\ & = \frac{1}{1-i}[(x+\delta)^{1-i} - x^{1-i}] - \frac{1}{1-\lambda}[(1-x-\delta)^{1-\lambda} - (1-x)^{1-\lambda}] \\ & \leq C(\delta^{1-i} + \delta^{1-\lambda}), \end{aligned}$$

then Theorem 3 and Remark 1, or Corollary 3B, can be applied to the operators

$$Au = \alpha u'' \quad \text{or} \quad Au := (\alpha u')'$$

with general Wentzell boundary conditions of the type (0.1). Hence, the operator A with domain

$$\begin{aligned} D(A) \quad := \{u \in C[0,1] \cap C^2(0,1) :\ & Au \in C[0,1],\ \lim_{x\to j} Au(x) + \widetilde{b}(x) = 0 \\ & \text{for } j = 0,1\} \end{aligned}$$

generates an analytic Feller semigroup on $C[0,1]$, provided that $\widetilde{b} \in C[0,1]$ satisfies suitable additional assumptions.

In the case $2 \leq i, \lambda$, Theorem 4, or Corollary 4A, applies provided that $\widetilde{b} \in C[0,1]$ satisfies the additional conditions required in the corresponding theorem, thus, as before, we can conclude that the semigroup generated by $(A, D(A))$ with general Wentzell boundary conditions (0.1) is analytic on $C[0,1]$.

References

1. W. Arendt, G. Metafune, D. Pallara and S. Romanelli, *The Laplacian with Wentzell-Robin boundary conditions on spaces of continuous functions*,Semigroup Forum(to apear).

2. A. Batkai and K.-J. Engel - Cosine families generated by operators with generalized Wentzell boundary conditions, *Tübinger Berichte zur Funktionalanalysis*, **11** (2001), 7–19.

3. M. Campiti and G. Metafune - Ventcel's boundary conditions and analytic semigroups, *Arch. Math.*, **70** (1998), 377–390.

4. Ph. Clément and C. Timmermans - On C_o-semigroups generated by differential operators satisfying Ventcel's boundary conditions, *Indag. Math.*, **89** (1986), 379–387.

5. K.-J. Engel Second order differential operators on $C[0,1]$ with Wentzell - Robin boundary conditions Evolution Equations: Proceedings in Honor of J. A. Goldstein's 60^{th} Birthday (G. Ruiz Goldstein, R. Nagel and S. Romanelli eds.) , Lect. Notes in Pure and Applied Mathematics Series M. Dekker , New York (to apear).

6. K.-J. Engel Analyticity of semigroups generated by operators with generalized Wentzell boundary conditions, *preprint*.

7. K.-J. Engel and R. Nagel - One-Parameter Semigroups for Linear Evolution Equations, *Graduate Texts in Mathematics*, (2000) Springer.

8. A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli - On some classes of differential operators generating analytic semigroups, Evolution Equationsand treir applications in Phisical and Life Sciences (G. Lumer and L. Weis eds.) M. Dekker, New York,2000,pp 99-114.

9. A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli -C_o-semigroups generated by second order differential operators with general Wentzell boundary conditions, *Proc. Amer. Math. Soc.*, **128** (2000), 1981–1989.

10. A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli - Generalized Wentzell boundary conditions and analytic semigroups in $C[0,1]$ - Semigroups of Operators: Theory and Applications (Proceedings Newport Beach, CA, 1998) (A. V. Balakrishnan ed.) Progr. Nonlinear Differential Equations Appl., vol. 42, Birkhäuser Verlag, Basel (2000) 125–130

11. A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli - The one dimensional wave equation with Wentzell boundary condition - Differential Equations and Control Theory (S. Aizicovici and N. H. Pavel eds.), M. Dekker, New York,(2002) 139–145.

12. A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli - The heat equation with general Wentzell boundary condition - *J. Evol. Eqns.* (2002) **2** 1–19

13. A. Favini, G. R. Goldstein, J. A. Goldstein, E. Obrecht and S. Romanelli - The Laplacian with generalized Wentzell boundary conditions - Proceedings of EVEQ (2000) (M. Iannelli ed.), Birkhäuser Verlag, to apear.

14. A. Favini and S. Romanelli - Analytic semigroups on $C[0,1]$ generated by some classes of second order differential operators *Semigroup Forum* **56** (1998) 367–372

15. C. Gal, G. R. Goldstein and J. A. Goldstein - Oscillatory boundary conditions for acoustic wave equations *preprint*

16. J.A. Goldstein - Semigroups of Linear Operators and Applications *Oxford University Press*, Oxford, New York (1985)

17. T. Kato - Perturbation Theory for Linear Operators, Springer-Verlag, Berlin Heidelberg New York (1966)

18. A. Lunardi - Analytic Semigroups and Optimal Regularity in Parabolic Problems - Birkhäuser Verlag (1995)

19. H. Vogt and J. Voigt- Wentzell boundary conditions in the context of Dirichlet forms - Diff. & Int. Eqns., to apear.

20. M. Warma - Wentzell-Robin boundary conditions on $C[0,1]$- Semigroup Forum 66 (2003),162-170.

21. T.-J. Xiao and J. Liang - Wave equations with generalized Wentzell boundary conditions - (preprint) (2002)

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