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Why quasi-sets?

Décio Krause

ABSTRACT: Quasi-set theory was developed to deal with collections of indistinguishable objects. In standard mathematics, there are no such kind of entities, for indistinguishability (agreement with respect to *all* properties) entails numerical identity. The main motivation underlying such a theory is of course quantum physics, for collections of indistinguishable ('identical' in the physicists' jargon) particles cannot be regarded as 'sets' of standard set theories, which are collections of *distinguishable* objects. In this paper, a *rationale* for the development of such a theory is presented, motivated by Heinz Post's claim that indistinguishability of quantum entities should be attributed 'right at the start'.

Contents

1. Introduction

I am in glad with the invitation of the editors of this Bulletin to present a paper to the first volume of this new series. I think it might be a nice opportunity to submit to a wider mathematical audience some of the ideas involved in the development of a mathematical theory termed 'quasi-set theory'. Although I cannot provide all the mathematical details of this theory here, to which I suggest the reading of the papers listed in the references (specially [23]), I hope the reader will become aware of some of the insights which have pushed the development of such a theory.

Even without providing a precise idea of how quasi-set theory works, the reader might be convinced that there are some reasons to ask for a mathematical treatment of collections of objects which should be regarded as indiscernible (or indistinguishable) in a sense. Let us begin by posing the problem within a mathematical context.

2. A problem for present day mathematics

During the International Congress of Mathematicians, held in Paris in 1900, the great mathematician David Hilbert presented a list of 23 Problems of Mathematics which in his opinion should occupy the efforts of mathematicians in the century to come [26]. To solve one of the problems become a way of achieving something really important in mathematics, and several Fields medals were awarded for this kind of endeavour, as it is well known. In 1974, the American Mathematical Society sponsored a meeting to evaluate and to explore the consequences of Hilbert Problems. One of the interesting implications of the Congress was that a new list, termed Problems of Present Day Mathematics, was proposed [2].

According to Felix Browder, the editor of the Proceedings, this list was initiated by Jean Dieudonne through correspondence with mathematicians throughout the world. The first problem of this new list deals with foundations of mathematics, and was stated by the mathematician Yuri I. Manin. In the statement of the problem, we find the following passage:

> "We should consider possibilities of developing a totally new language to speak about infinity. (...) I would like to point out that this [the concept of set] is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behaviour. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. In general, a highly probabilistic 'physical infinity' looks considerably more complicated and interesting than a plain infinity of 'things'. " [29]

When Manin says that "[w]e should consider possibilities of developing a totally new language to speak about infinity", he is obviously talking about a *new* 'set' theory, since set theory is usually known as 'the theory of the (actual) infinite'. But why 'sets' of (say) electrons cannot be considered as *sets* of standard set theories? For sure this is due to their indistinguishability. If such collections are not to be regarded as *sets*, which kind of mathematical objects are they? In addition, if we consider the usual way of presentation of a scientific theory is in terms of set-theoretical concepts,[1] how can we deal with indistinguishable objects within such a framework? These are problems closely related to Manin's one. Let us approach them first sketching how classical logic and standard set theories deal with the concept of identity.

[1] I shall not discuss here other alternatives like category theory, higher-order logics, mereology etc., but keep with the most usual mathematical framework, namely, *set theory*.

3. Identity in standard mathematical frameworks

Classical logic and standard mathematics encompass a concept of identity which resembles Leibniz's dictum that there are no objects which differ *solo numero*. If they are distinct objects, then there exists a qualitative property which makes the difference. This dictum in encapsuled in his Principle of the Identity of Indiscernibles,[2] which in second order language has been written as

$$\forall F(F(x) \leftrightarrow F(y)) \rightarrow x = y \tag{1}$$

where x and y are individual variables and F is a variable ranging over the collection of the properties of individuals. Here, $x = y$ stands for numerical identity, that is, if $x = y$ is true, then there are no 'two' distinct objects, but just one, which can be referred to indistinctly by either x or y. Intuitively, PII says that if x and y agree with respect to all their properties (in this case they are indistinguishable, or indiscernible), then they cannot be distinct objects at all. PII is a theorem of second order logic.

In first order logic, usually the predicate $=$ of identity is taken as a primitive symbol, subjected to well known postulates (Reflexivity: $\forall x(x = x)$ and Substitutivity: $\forall x \forall y(x = y \rightarrow A(x) \rightarrow A(y))$, with the usual restrictions [31]). The intended interpretation of the binary symbol '=' is the diagonal of the domain (call it D), that is, the set $\Delta_D = \{\langle x, x\rangle : x \in D\}$.[3] In set theory, treated as a first order theory, as is usual, these axioms must be accompanied by the Extensionality Axiom, which says that sets with the same elements are identical: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$. There are some variants for this axiom if the theory admits *Urelemente*, but we shall not give all the details here. The main lesson is that both classical logic and standard mathematics are 'Leibnizian' in the sense that there are not indistinguishable objects, except if we restrict the indistinguishability to a reduced number of properties or course (in this case we should talk of '*relative* indistinguishability'), or if we work within a certain structure, as we shall see below, but there is not of indistinguishability *tout court*).

Having said that, let us see now some of the problems involved with these concepts into the scope of the discussions on the indistinguishability of quantum objects (henceforth, we will sometimes refer to them simply as 'quanta').

4. Semantics for languages of microphysics

Yuri Manin has also suggested that quantum mechanics has no its 'own' language; generally we use a fragment of standard functional analysis (hence, the language of set theory) to express its concepts [30, p. 84]. If Manin is right, for sure we need to explain what we mean by 'semantics of languages of microphysics'.

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[2] We are of course not guessing that the formula (1) interprets Leibniz's principle in the context of his philosophy. It is just an expression given in present day symbology which is associated to his principle by resemblance.

[3] As it is well known, the first order axioms cannot characterize this diagonal without ambiguity. See [31], [24].

But even without a detailed explanation, we may speculate on the formal characterization (in both syntactic and semantic terms) of such a language, which we believe can be found. So, let us suppose that we have presented a suitable language for microphysics by usual means, involving logical and mathematical notation and concepts (which is a pertinent supposition), that is, with primitive symbols and formation rules, and that we intend to define the corresponding semantic concepts as we usually do. What kind of problem will we be faced with? Such an analysis was done by M. L. Dalla Chiara and G. Toraldo di Francia some time ago [10],[4] and the points they have amphasised are of course of fundamental importance for whatever discussion on structure and semantics of physical theories. So, let us mention some of the basic ideas here in order to see how quasi-set theory may enter in this discussion.

Suppose we have a language $\mathcal{L}$ (at least of first order) and let $\mathfrak{A}$ be a suitable structure where $\mathcal{L}$ is interpreted. So $\mathfrak{A}$ involves a domain D of *individuals* and an interpretation (or denotation) function ρ which assigns appropriate meaning to the non-logical constants of $\mathcal{L}$.[5] Of course we can think for generality of $\mathfrak{A}$ as determining a set of possible worlds for each particular physical situation under analysis, so that in this case we would have a set of domains D_i and a collection of corresponding interpretation functions ρ_i. The particular case above happens when the set of possible worlds is a singleton. Then, as recalled by Dalla Chiara and Toraldo di Francia, the following situations can be considered in 'standard' semantics:

(i) Any *property* P of $\mathcal{L}$ (an unary predicate) is related to a subset $P^* \subseteq D_i$, while n-ary predicates are related to subsets of D_i^n as usual.

(ii) For each individual $d^* \in D_i$, the language can be extended to a language containing a *name* d and an extended interpretation function ρ' such that $\rho'(d) = d^*$.

(iii) If $\mathcal{L}$ is at least of second order, then Leibniz' Principle of the Identitity of Indiscernibles holds (individuals can be distinguished by at least one property):[6]

$$\forall x \forall y (x \neq y \to \exists F(F(x) \land \neg F(y)). \tag{2}$$

(iv) If $\mathfrak{A}$ refers to a set of possible worlds (physical situations), as in the usual Kripke semantics we can suppose the existence of a number of *world-relations*; a particular one may corresponds to a time-order relation as follows: $i < j$ iff the situation i temporally precedes the situation j. Of special interest is the relation $\approx$, termed the *trans-world identity relation*, defined on $U = \cup_i D_i$ satisfying the following conditions:

[4] See also [8], [9], [11], [39].

[5] In the particular case of physical theories, it is reasonable to suppose that the domain D sum up a certain mathematical construction in terms of other sets. In this case, we should talk of a 'species of structures' in the sense of Bourbaki, but we are obviously making things easy; see [7, Chap. 4].

[6] This is another formulation of the same principle given by equation (1).

(a) For any $d^* \in D_i$ and any D_j there exists at most one $\widehat{d^*} \in D_j$ such that $d^* \approx \widehat{d^*}$

(b) If d is an individual constant of $\mathcal{L}$ (a name) which names individuals in two distinct worlds, say $d^* \in D_i$ and $\widehat{d^*} \in D_j$, then $d^* \approx \widehat{d^*}$. In particular, if $i < j$, the trans-world identity relation between D_i and D_j is usually called *genidentity relation*.

In what concerns quantum physics, the interesting remark is that all these standard set-theoretical semantic situations are violated. Dalla Chiara and Toraldo di Francia provided a detailed analysis of the motives why such standard (set-theoretical) semantics fails, but here we will only sketch the main ideas of their argumentation, yet without the technical details. The first motive is that we should be able to construct a suitable language containing monadic predicates for expressing 'meaningful properties' of the physical systems (in particular, of elementary particles), so as to consider *names* $d, d', \ldots$ which should be associated to these physical systems at different times. Since indistinguishable quanta cannot be named, for in general we cannot distinguish a physical system (elementary particle) from another, our language cannot be extended with 'names' as indicated at item (ii) above and so we cannot define a suitable interpretation function ρ that univocally determines an element of D. As remarked by these mentioned authors, "the problem is not 'whether or not *we are allowed* to introduce names $a_1, \ldots, a_n$ for the n subsystems $(\ldots)$' but rather 'whether or not *we are able* to introduce a reasonable denotation function ρ for such names' ".

Furthermore, a physical system can be regarded as represented by a pure state ψ, as usual, and time evolution of the system is governed by the Schrödinger equation. The problem is that we cannot say that ψ determines a *set* of n elements in the standard set theoretical sense, which contradicts condition (i) above. So see why, let us introduce the concept of vagueness in this context.[7] Philosophers usually say that vagueness is a feature of our languages, and not of the world. For instance, the predicate 'intelligent' is vague, for we might be in doubt either our friend John is or not intelligent. But John is a well defined physical object (a man), an individual we know very well. That is, John is to be regarded as 'sharp', while the predicate 'intelligent' is vague. Then, it has been proposed that a suitable semantics for such vague predicates should be developed not within standard set theories, but we should use fuzzy set theory instead. The motive is that within the standard framework all predicates are 'sharp' in the sense that they are associated to a subset of the domain, called its *extension* (as indicated at (i) above) so that whatever element of this domain belongs or not to the extension of the predicate, which means that any individual should be classified as intelligent or as not intelligent (the *tertium non datur* applies here). It is easy to see why fuzzy sets provide a more adequate semantics, for in using fuzzy sets, we can express in a certain way our ignorance about John's precise location in the rank of intelligent men. I will not explain the details here, but let me remark that fuzzy sets can be useful just for

[7] This is not considered by the mentioned authors, but of course can be related to their ideas.

expressing the situation when a *well defined individual* (like our friend John) has or not a certain (vague) property –such a semantics express a certain 'epistemological ignorance' only. But in quantum physics, there are certain 'predicates' which are *sharp* in the sense that physicist know quite well the conditions an individual should obey to have the property ascribed by the predicate, but there are 'vague objects' instead, which induce the consideration of a kind of 'ontological ignorance' in this realm.[8] This shows that the relationship between the predicates themselves (which stand for the *intensions* of certain concepts) and their corresponding *extensions* (the set of the individuals which have the property ascribed by the predicate) becomes distinct from standard semantics. In such cases, it seems clear that we should ask for a semantics which deal with 'imprecise' or 'vague' objects, and indistinguishable objects of course are good candidates for that. As an example of this situation, let us quote a passage from [12]:

> "[P]hysical kinds and compound systems in QM [quantum mechanics] seem to share some features that are characteristic of intensional entities. Further, the relation between intensions and extensions turns out to behave quite differently from the classical semantic situations. Generally, one cannot say that a quantum intensional notion uniquely determines a corresponding extension. For instance, take the notion of *electron*, whose intension is well defined by the following physical property: mass $= 9.1 \times 10^{-28}$g, electron charge $= 4.8 \times 10^{-10}$e.s.u., spin $= 1/2$. Does this property determine a corresponding *set*, whose elements should be all and only the physical objects that satisfy our property at a certain time interval? The answer is negative. In fact, physicists have the possibility of recognizing, by theoretical or experimental means, whether a given physical system is an electron system or not [as we have said, the predicate is *sharp*]. If yes, they can also enumerate all the quantum states available within it. But they can do so in a number of different ways. For example, take the spin. One can choose the x-axis and state how many electrons have spin up and how many have spin down. However, we could instead refer to the z-axis or any other direction, obtaining *different collections* of quantum states, all having the same cardinality. This seems to suggest that microobject systems present an irreducibly intensional behaviour: generally they do not determine precise extensions and are not determined thereby."

Situations like these ones obviously violate condition (i) of standard semantics, that is, that one developed within classical set theories, and reinforce the problem posed by Manin seen above, for such collections of quanta should not be regarded as 'sets' (as in standard set theories) but as legitimate collections of indistinguishable objects instead. Concerning (iii), it has been shown elsewhere that PII is violated in the quantum domain (and there are of course 'set-theoretical' versions of this

[8] This idea was developed in a series of papers [14], [15], [16], [18], where the details are presented.

principle), for we may consider (absolutely) indistinguishable quanta (having all their quantum parameters in common) which of course are not the very same object (see [13], [19]). As for (iv), we would need much space for providing the details; so, we shall just say that Dalla Chiara and Toraldo di Francia have shown that in this 'land of anonymity', as they refer to the quantum world, standard forms of Kripke semantics fail, for there are no trans-world identity and the notion of rigid designator cannot be applied [39], among other interesting 'deviations' from classical situations.

All of this suggest that we should ask for a mathematical theory encompassing 'collections' which could stand for 'sets' of indistinguishable objects, where no names can be used, no individuation of these objects can be given, but even so they should be considered in aggregates, having a cardinal number, although not an associated ordinal. This is what quasi-set theory intends to do. The applications of this theory to the above situations have been suggested in some of the papers listed in the references, but there is still much to be done in this direction.[9]

But instead of showing just now how we could provide the grounds for such use of quasi-sets, let us first turn to the nature of quantum entities to see more on the motivations for a better understanding of Manin's problem.

5. The standard ways of dealing with indistinguishability

Within classical mathematics, it is possible to consider indistinguishable objects, but this has a price. From the physicist's point of view, the price can be payed without any restriction, for physics works. This is more or less what happens with a mathematician who proves $\alpha \to \beta$ by assuming α as an hypothesis and (using it) deduces β, without any questioning on what lies behind this 'innocent' procedure (the Deduction Theorem, of course). This is a problem for the philosopher, one may say, or to the mathematician interested in foundations. So, let us discuss it a little bit.

George Mackey, in his very important book *The Mathematical Foundations of Quantum Mechanics* [28, pp. 109ff], treats the problem of indiscernibility in a way that can be understood by means of an analogy, suggested by Mackey himself (the mathematical details will be not mentioned). Suppose Peter and Paul think of two numbers whose sum is 7 and whose squares sum 25. It is quite easy to solve the two equations $x+y=7$ and $x^2+y^2=25$ to show that one of then has thought of 3 while the another one has thought of 4. But there is no way of knowing who thought of which number. This is what happens with elementary particles, Mackey seen to suggest. According to him, they are indistinguishable in this sense. Mackey still suggests that "identity manifests itself through the appearance of anti-symmetric subspaces ..." (op. cit., p. 111); here, suffices to say that anti-symmetric states can have just one quanta, and this should be the motive for then to have identity, at least is what seems.

[9] Even the cardinal of a quasi-set may change with time, as shown in [17]. This is motivated by Toraldo di Francia's discussion on virtual particles [37], [38].

Such a solution (of course given by him mathematically) might be criticised as follows. Firstly, *if* we add to the two equations above the information that Peter has thought of an odd number, it will be easy to know that Peter's number is 3. But in quantum physics there are no such 'hidden' information (variables) at disposal, according to the standard interpretations which refuse hidden variables. The physical laws which 'define' the nature of quantum objects give *all* their essential properties: they are *nomological objects*, given by physical law [37]. Furthermore, in the anti-symmetric case, we should remember that what are distinguishable are the states, and not the quanta (Schrödinger insisted on this point [34]). So, the situation is of course not as Mackey suggests, despite his way of dealing with the problem satisfies the needs of physics.

Most of the proposals for dealing with indistinguishable objects involve the introduction of some symmetry condition.[10] In short, we *need* to label quanta in order to write the relevant equations (as Schrödinger's) but then we select adequate solutions (or alternatively, vectors in the relevant Hilbert space) which obey symmetry conditions, which make the desired result that 'permutations are not regarded as observable'. In my opinion, this is a mathematical trick. From the philosophical (and foundational) point of view, it should be interesting to pursue Manin's suggestion of looking for a theory which consider indistinguishable entities taken as such from the bottom. This idea is endorsed by Heinz Post, who have said that indistinguishability should be attributed to elementary quanta "right at the start" [33]. So, we may say (although here so roughly) that these 'standard solutions' have a common feature, which can be summed up by what we call *Weyl's strategy*: the mathematical considerations are always supplied by some extra postulate which enable us to work *as if* the relevant objects *were* indistinguishable, but they are not. Let us be more explicit on this point.

In considering 'aggregates of individuals' for discussions on the foundations of quantum theory, Hermann Weyl intended to treat the case where the elements of a certain collection may be in certain 'states' but only the quantity of them in each one of these states could be known [40, App. B]. This is of course what happens in quantum physics. As he says,

> "[i]n physics one aims at making division into classes so fine that no refinement is possible; in other words, one aims at a *complete* description of state. Two individuals in the same 'complete state' are indiscernible by any intrinsic characters –although they may not be the same thing". (Op. cit., p. 245.)

To realize that, Weyl considers a *set* S (let us emphasize that S is a set, hence a collection of *discernible* objects) with n elements, say $x_{p_1}, \ldots, x_{p_n}$, endowed with an equivalence relation $\sim$. The intuitive interpretation is that $a \sim b$ means that a and b are of the same *kind*, or *nature*, and in this case they are said to belong to the same *state*. The equivalence classes $C_1, \ldots, C_k$ of the quotient set $S/_\sim$ stand for these 'states'. An *aggregate* S is "a set of elements each of which is in

[10] For details, see [20].

a definite state; hence, the term aggregate is used in the sense of 'set of elements with equivalence relation' " (op. cit., p. 239). So, an aggregate is a pair $\langle S, \sim \rangle$, where $\sim$ is an equivalence relation on the non-empty set S. A certain *individual state* of the aggregate is then achieved when "it is known, for each of the n marks p, to which of the k classes the element marked p belongs". If the elements of S are distinct from one another, then of course there are k^n possible individual states of the aggregate, but, as Weyl remarked,

> "[i]f, however, no artificial differences between elements are introduced by the labels p and merely the intrinsic differences of state are made use of, then the aggregate is completely characterized by assigning to each class C_i $(i = 1, \ldots, k)$ the number n_i of elements of S that belong to each class C_i. These numbers, the sum of which equals n, describe what may conveniently be called the *visible or effective state* of the system S. Each individual state of the system is connected with an effective state, and any two individual states are connected with the same effective state if and only if one may be carried into the other by a permutation of the labels" (op. cit., pp. 239-40).

In other words, since each equivalence class has a cardinal n_i, $i = 1, \ldots, k$, the effective state of the aggregate is characterized by the *ordered decomposition* $n_1 + \cdots + n_k = n$. Then, if the individuality of the elements of S is forgotten for a moment and only this ordered decomposition is considered, we arrive at a formula which expresses the number of different effective states, which is the well known formula for Bose-Einstein statistics, namely,

$$\frac{(n+k-1)!}{n!(k-1)!}. \tag{3}$$

Although adequate for mathematical purposes, Weyl's suggestion of considering a *set* endowed with an equivalence relation does not deal with indistinguishability *solo numero* of the elements of S, but only mimics indistinguishability, for the basic objects are not taken as indistinguishable from the start, as required by Post (who follows the intuitions involving quantum objects). Of course, in order to arrive at the above formula, that is, to the situation where only the permutation of the labels are given (according to the above quotation), one has to suppose that certain elements of some class C_i were permuted with elements of a class C_j $(i \neq j)$, preserving their cardinalities, so that the ordered decomposition keeps the same. But the permutation of course changes the classes, due to the Axiom of Extensionality of set theory; in other words, after the permutation (of discernible elements), the 'states' are no longer the same! (more on this below). For the mathematical description of physics, perhaps this is not important, for it satisfies what Weyl called the Principle of Relativity, according to which "[o]nly relations and statements [we should say, physical laws] have objective significance as are not affected by any change in the choice of the labels p".[11]

[11] Op. cit., p. 240. The effective state of an aggregate helps in stating that "the principle of

This line of thought is the basis for the use of group theory in quantum mechanics, and we should recall that Weyl was one of the founders of this application.[12] But philosophically we require a more precise mathematical mechanism which enables us to consider not only that the permutations of the particles (the above 'change of labels') do not change physical laws, but a general procedure which expresses the fact, mentioned by R. Penrose, that

> "[a]ccording to quantum mechanics, any two electrons must necessarily be completely identical [in the physicist's jargon, that is, indistinguishable], and the same holds for any two protons and for any two particles whatever, of any particular kind. This is not merely to say that there is no way of telling the particles apart; the statement is considerably stronger than that. If an electron in a person's brain were to be exchanged with an electron in a brick, then the state of the system would be *exactly the same state* as it was before, not merely indistinguishable from it! The same holds for protons and for any other kind of particle, and for the whole atoms, molecules, etc. If the entire material content of a person were to be exchanged with the corresponding particles in the bricks of his house then, in a strong sense, nothing would be happened whatsoever. What distinguishes the person from his house is the *pattern* of how his constituents are arranged, not the individuality of the constituents themselves" [32, p. 32].

But, in the usual strategies, the (in principle) 'identifiable' characteristics of the elements of S are masked by a trick of 'forgetting' that they are elements of a set and only their role as elements of certain equivalence classes are taken into account. This is the 'Weyl's strategy'. The selection of symmetric and anti-symmetric solutions of the Schrödinger equation or, alternatively, symmetric and anti-symmetric vectors in certain Hilbert spaces, have the same aim.[13]

An alternative approach is to work within a certain mathematical structure, so that indistinguishable objects can be defined, say, as those elements of the domain which are invariant by the automorphisms of the structure [24]. But even in this case there is no *truly indistinguishability* (or 'philosophical' indistinguishability), for these elements can always be distinguished *from the outside* of the structure, say in the well-founded universe $\langle V, \in \rangle$. When we look to the real aspect of the ground elements of the structure, we realise that they are *sets*. Even if we admit atoms in our pantheon, these usually 'indistinguishable' *Urelemente* (they are invariant by automorphisms) can be seen as *sets* of the same rank in $\langle V, \in \rangle$ (hence being always distinguishable). In short: 'standard' set theories like ZF, ZFC, NBG, KM and the like encompass a Cantorian concept of set; as said Cantor, a set is "a collection into a whole of *distinct* objects of or intuition or of our thought" (my emphasis)

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relativity finds expression in the postulate of invariance with respect to the group of all permutations" (ibid.).

[12] The another being Wigner, as it is well known [3].

[13] The role of permutational symmetries is explained in details in [20].

[4, p. 85]. So, there are no truly indistinguishable objects in set theory, for every one of them can be individualised by its singleton (it is a theorem of all these theories that $x \neq y \rightarrow \exists z(x \in z \wedge y \notin z)$ – take $z = \{x\}$); so it is not possible to consider Post's claim within such a framework.

So, after having an idea of why to look for a 'set' theory of collections of indistinguishable objects makes sense, let us see why quantum objects have such a strange behaviour. We shall consider something on this point next.

6. Quantum non-individuality

In this section I shall sketch the main ideas which render quantum particles as 'non-individuals' and explain how this non-individuality has been understood within quasi-set theory.[14] There is a huge literature on the topic of non-individuality to which we cannot do justice here. But let us just pose the problem once more with few technical details.

We usually agree that people, rocks and chairs can be regarded as 'individuals'. The problem of how this 'individuality' is to be understood is an old problem in philosophy. A first attempt in answering this question runs in the direction of saying that they are individuals because they can be distinguished from one another, and this distinguishability is usually understood in terms of differences in the properties of the objects. But, is it possible for two pencils, say, to have all their characteristics in common? And, if not, why not? It became famous that Leibniz said that this is not possible; his famous 'Principle of the Identity of Indiscernibles' says that two objects which are indistinguishable, in the sense of possessing all properties in common, cannot, in fact, be two objects at all, as we have seen earlier. But this is a metaphysical principle, and it seems to be of course true for our everyday objects, since (at least we believe) they will always possess some distinguishing property, some scratch that will distinguish them even from similar objects of the same shape, colour, etc. But the problem continues: why is it not possible for two such objects – our two pencils – to possess not only the same shape and colour, but even the same scratches? If this were possible, how they can be regarded as 'two' individual pencils and not as just one? In answer this issue we might point to some property which cannot be shared by them, such as location in space-time. Clearly –or so it would seem– our two pencils cannot occupy the same space at the same time, since they are impenetrable.

Pushing this line of thought, one might seek for some ground for this impenetrability in the stuff, the substance, of which the objects are, in some sense, 'made' off. This brings a different answer to our initial question: the individuality of the objects is then to be understood not in terms of some property or set of properties which also render them distinguishable, but in something else, underlying or 'transcending' their properties, such as some form of substantial substratum, famously characterised by John Locke as the 'something we know not what' (since it cannot be described in terms of properties; for details in all these points, see [17]).

[14] The full explanations are given in [17].

The philosophical literature on this topic presents various attempts to answer these and related questions and provides a discussion on what is often referred to as the 'principle' of individuality. Let us consider now a particular but related question: can the above considerations involving macroscopic objects be extended to the fundamental objects posited by current physical theories, such as electrons, protons, neutrons etc.? If these ones can be indistinguishable and also are not 'impenetrable', for their state functions can overlap, how our physical theories do not count them as just one? Can they be regarded as individuals, like our pencils? Is there some principle of individuation in this case?

In trying to answer these questions we must turn to the relevant physics, namely quantum theory, and, in particular, we must consider what this physics tells us about the aggregate behaviour of these objects, as Hans Reichenbach has taught us. Do they behave, in aggregate, like rocks or people? Since this behaviour is described by the relevant statistical mechanics, we shall of course need to consider the implications of them.

Let us consider only particles of the same 'kind', which possess the same 'intrinsic' or state-dependent properties (termed 'indistinguishable' –although physicists seem to prefer to call them 'identical') and then we take into account the distribution of these particles over states –say two particles over two one-particle states– and it is also assumed that each resulting arrangement is accorded equal probability. This generates four possibilities:

$$(1)\ |a_1^A\rangle \otimes |a_2^A\rangle \qquad (2)\ |a_1^B\rangle \otimes |a_2^B\rangle$$

$$(3)\ |a_1^A\rangle \otimes |a_2^B\rangle \qquad (4)\ |a_1^B\rangle \otimes |a_2^A\rangle.$$

In this Dirac's 'bra' and 'ket' notation, the superscripts (labels) A and B in the states mean that they can be distinguished, since different states are characterised by different state functions. The subscripts were introduced just to specify which particles are in which states.

In classical statistical mechanics (Maxwell-Boltzmann), (3) and (4) are counted as distinct and given equal weight in the assignment of probabilities. This informally means that the situation where we have one particle in each state is given a weight of two, corresponding to the two arrangements or complexions that may be formed by a permutation of the particles. Since they are 'indistinguishable' in the above sense, the difference in the states imply that the particles are to be regarded as *individuals* in a sense, that is, the difference in the states should be attributed to something 'transcending' the particles' properties, sometimes spelled out in terms of some underlying 'haecceity' or 'primitive thisness' or, more typically, the spatio-temporal location of the particles.

In the quantum case the situation goes as follows.[15] situations, (3) and (4) must be counted as one and the same, while the arrangement of one particle in each state is given a weight of one. This is standardly taken to reflect the fact

[15] We shall restrict our attention to the two standard forms of statistics –Bose-Einstein and Fermi-Dirac.

that arrangements obtained by a permutation of the particles do not feature in the relevant counting in quantum statistics.

More formally, states of quantum systems (single or many-particle systems) are represented by unitary vectors Ψ in a Hilbert space $\mathcal{H}$. For many particle systems the Hilbert space is constructed by forming the tensor product of the component particles' Hilbert spaces. For a system consisting of two indistinguishable particles, the Hilbert space is $\mathcal{H}_{\mathsf{two}} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where the subscripts '1' and '2' label the particles, and $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ for they are indistinguishable. If the particles are in the pure states ϕ and ψ respectively, then the composite system is in the (pure) state $\Psi = \phi \otimes \psi$. The observables $\hat{O}$ of a quantum system are represented by Hermitian operators acting upon that system's Hilbert space. A permutation of the particles over states is represented by an operator and these 'permutation operators' form a group known ever since Hermann Weyl as the Permutation Group [3]. These permutation operators are projections (hence have eigenvalues ± 1) and act upon Ψ as follows: (1) $\hat{P}_{id}(\Psi) = (\phi \otimes \psi)$ and (2) $\hat{P}_{\phi\psi}(\Psi) = (\psi \otimes \phi)$.

The Hamiltonian operator, $\hat{H}_\Psi = \hat{H}(\phi \otimes \psi)$, of the composite system is symmetric with respect to ϕ and ψ. Hence, $\hat{H}_\Psi$ is invariant under the action of the permutation group of permutations of the composite particles' labels, that is, $[\hat{H}, \hat{P}] = 0$, for any $\hat{P}$. The 'fact' that particle permutations are not counted is understood in terms of there being no measurement that we could perform which would result in a discernible difference between permuted (final) and unpermuted (initial) states. This is represented in the formalism by insisting that every physical observable $\hat{O}$ commutes with every permutation operator $\hat{P}$, that is, $[\hat{O}, \hat{P}] = 0$, $\forall \hat{O} \forall \hat{P}$. Expressing this formally, we have the so-called 'Indistinguishability Postulate' (IP), that is, for any arbitrary state ψ, Hermitian operator $\hat{O}$, and permutation operator $\hat{P}$,

$$\langle \psi \mid \hat{O} \mid \psi \rangle = \langle \hat{P}\psi \mid \hat{O} \mid \hat{P}\psi \rangle = \langle \psi \mid \hat{P}^{-1}\hat{O}\hat{P} \mid \psi \rangle. \tag{4}$$

Since (IP) allows for the possibility of forms of quantum statistics which are different from the 'standard' Bose-Einstein and Fermi-Dirac kind, if one wants to restrict the formalism to the latter kinds only, then a further condition, known as the 'Symmetrisation Postulate' (SP) must be applied (this corresponds to a form of 'symmetry conditions' as indicated above) [20]. Put simply this dictates that, states of indistinguishable particle systems must be either symmetrical or antisymmetrical under the action of the permutation operators (corresponding to the Bose-Einstein and Fermi-Dirac cases respectively). The difference between (SP) and (IP) can be expressed as follows (ibid.): (SP) expresses a restriction on the states for all observables, $\hat{O}$; whereas (IP) expresses a restriction on the observables, $\hat{O}$, for all states.

Now, (IP) seems to run counter to the point of regarding the particles as individuals and labelling them; from the point of view of the statistics, the particle labels are otiose. The implication, then, is that the particles can no longer be considered to be individuals, that they are, in some sense, 'non-individuals'. This conclusion expresses what S. French calls 'the Received View', an idea that came from Schrödinger, Born, Heisenberg, Weyl, Hesse and Post at least (see [17] for historical facts): classical particles are individuals but quantum particles are not. Post, for instance, drew on the distinction between form and substance, arguing that what quantum statistics indicates is the ontological primacy of the former over the latter. The interesting remark to be made here is that we can go beyond mere metaphors and underpin the Received View with an appropriate logico-mathematical framework, namely, quasi-set theory.[16]

7. Quasi-sets: general ideas

As we have said, intuitively speaking a quasi-set is a collection of indistinguishable (but not identical) objects. This of course is not a strict 'definition' of a quasi-set, acting more or less as Cantor's 'definition' mentioned above, giving no more than an intuitive account of the concept. But we should realise that it seems reasonable, due to the above argumentation (which of course does not cover all the situations presented by modern physics),[17] to search for a mathematical theory which considers, without dodges, collections of *truly* indistinguishable objects. In characterizing such collections (quasi-sets), we have followed Erwing Schrödinger's opinion that the concept of identity cannot be applied to elementary particles and developed the theory by posing that the expression $x = y$ is not generally a well-formed formula (and likewise for the negation $x \neq y$). This enable us to consider logico-mathematical systems in which identity and indistinguishability are separated concepts; that is, these concepts do not reduce to one another as in standard set theories.

In particular, forms of logic –called Schrödinger logics– have been introduced for which '$a = a$' cannot be inferred for certain objects a [5], [6]. For all other entities classical logic is maintained.[18] Correspondingly, quasi-set theory is the 'set-theoretical' version of this idea.

Quasi-set theory $\mathfrak{Q}$ ([23]) allows two kinds of Urelemente: the m-atoms, whose intended interpretation are the quanta, and the M-atoms, which stand for macroscopic objects, to which classical logic is supposed to apply.[19] Quasi-sets are the collections obtained by applying ZFU-like (Zermelo-Fraenkel plus *Urelemente*) axioms to a basic domain composed of m-atoms, M-atoms and aggregates of them.[20] The theory still admits a primitive concept of quasi-cardinal which intuitively stands for the 'quantity' of objects in a collection. The main idea is that the quasi-cardinal of a quasi-set cannot be associated (in the sense of this association being something described in the 'classical' part of $\mathfrak{Q}$) to a particular ordinal due to the (absolute) indistinguishability of the m-atoms, and this is the motive for taking this concept

[16] But this of course will not be made here; see [17].

[17] The interested reader should have a look in the papers by Dalla Chiara listed in our references.

[18] This is a characteristic of these systems, but in principle it is possible to define Schrödinger logics associated with other non-classical logics. In [6], a system encompassing modal operators is considered.

[19] But see the previous footnote were we have suggested that other kinds of logic could also be used.

[20] A similar remark is in order here. Perhaps for some applications it would be interesting to have, say, 'quasi-classes', and use NBG-like axioms instead. This shall be mentioned again below.

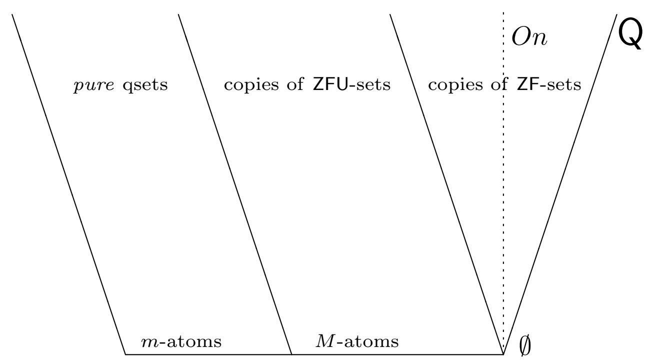

Figure 1: The Quasi-Set Universe

as primitive (but see below). This point notwithstanding, it is possible to define a translation from the language of ZFU into the language of $\mathfrak{Q}$ in such a way so that there is a 'copy' of ZFU in $\mathfrak{Q}$ (the 'classical' part of $\mathfrak{Q}$). In this copy, all the usual mathematical concepts can be defined, and the 'sets' (in reality, the '$\mathfrak{Q}$-sets') turn out to be those quasi-sets whose transitive closure (this concept is like the usual one) does not contain m-atoms (see the Figure 1).[21]

In $\mathfrak{Q}$ there may exist quasi-sets whose elements are m-atoms only, called 'pure' quasi-sets whose elements are indistinguishable (in the sense of partaking the primitive indistinguishability relation $\equiv$) and the axioms provide the grounds for saying that nothing in the theory can distinguish the elements of such an x from one another. Within the theory the idea that there is more than one entity in x is expressed by an axiom which states that the quasi-cardinal of the power quasi-set of x (the concept of subquasi-set is like that of standard set theory)[22] has quasi-cardinal $2^{qc(x)}$, where $qc(x)$ is the *quasi-cardinal* of x (which is a cardinal obtained in the 'copy' of ZFU just mentioned). Now, what exactly this supposition means?

Consider the three protons and the four neutrons in the nucleus of a ^{7}Li atom. As far as quantum mechanics goes, nothing distinguishes these *three* protons. If we regard these protons as forming a quasi-set, its quasi-cardinal should to be 3, and there is no apparent contradiction in saying that there are also 3 subquasi-sets

[21] So, we can make sense to the primitive concept of quasi-cardinal of a quasi-set x (written $qc(x)$) as being a cardinal defined in the 'classical' part of the theory. The reason to take qc as a primitive concept will appear below, when we make reference to the distinction between cardinals and ordinals.

[22] This is what makes a basic difference with fuzzy sets. In fuzzy set theory, as it is well-known, the counter-domains of the characteristic functions are not $\{0, 1\}$, but $[0, 1]$.

with 2 elements each, despite we can't distinguish their elements, and so on. So, it is reasonable to postulate that the quasi-cardinal of the power quasi-set of x is $2^{qc(x)}$. Whether we can distinguish among these subquasi-sets is a matter which does not concern logic.

In other words, we may consistently (with the axiomatics of $\mathfrak{Q}$) reason as if there are three entities in our quasi-set x, but x must be regarded as a collection for which it is not possible to discern its elements as individuals. The theory does not enable us to form the corresponding singletons. The grounds for such kind of reasoning has been delineated by Dalla Chiara and Toraldo di Francia as partly theoretical and partly experimental. Speaking of electrons instead of protons, they note that in the case of the helium atom we can say that there are two electrons because, *theoretically*, the appropriate wave function depends on six coordinates and thus "... we can therefore say that the wave function has the same degrees of freedom as a system of two classical particles" (op. cit., p. 268).[23] Dalla Chiara and Toraldo di Francia have also noted that, "[e]xperimentally, we can ionize the atom (by bombardment or other means) and extract two separate electrons ..." (ibid.).

Of course, the electrons can be counted as two only at the moment of measurement; as soon as they interact with other electrons (in the measurement apparatus, for example) they enter into entangled states once more. It is on this basis that one can assert that there are two electrons in the helium atom or six in the 2p level of the sodium atom or (by similar considerations) three protons in the nucleus of a ^{7}Li atom (and it may be contended that the 'theoretical' ground for reasoning in this way also depends on these experimental considerations, together with the legacy of classical metaphysics). On this basis it is stated the axiom of 'weak extensionality' of $\mathfrak{Q}$, which says that those quasi-sets that have the same quantity of elements of the same sort (in the sense that they belong to the same equivalence class of indistinguishable objects) are indistinguishable.

This axiom has interesting consequences. As we have said, there is no space here for the details, but let us mention just one of them which is related to the above discussion on the non observability of permutations in quantum physics, which is one of the most basic facts regarding indistinguishable quanta (recall Penrose's quotation given above). In standard set theories, if $w \in x$, then of course $(x-\{w\})\cup\{z\} = x$ iff $z = w$. That is, we can 'exchange' (without modifying the original arrangement) two elements iff they are *the same* elements, by force of the axiom of extensionality. But in $\mathfrak{Q}$ we can prove the theorem below, where z' (and similarly w') stand for a quasi-set with quasi-cardinal 1 whose only element is indistinguishable from z (respectively, from w –the reader shouldn't think that this element *is identical to either* z or w, for the relation of equality shouldn't apply here; the set theoretical operations can be understood according to their usual definitions):

[23] This might be associated to the legacy of Schrödinger, who says that this kind of formulation "gets off on the wrong foot" by initially assigning particle labels and then permuting them before extracting combinations of appropriate symmetry [34].

Theorem 1 [Unobservability of Permutations] *Let x be a finite quasi-set such that x does not contains all indistinguishable from z, where z is an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists w' such that*

$$(x - z') \cup w' \equiv x$$

Supposing that x has n elements, then if we 'exchange' their elements z by correspondent indistinguishable elements w (set theoretically, this means performing the operation $(x - z') \cup w'$), then the resulting quasi-set remains *indistinguishable* from the original one. In a certain sense, it is not important whether we are dealing with x or with $(x - z') \cup w'$. This of course gives a 'set-theoretical' sense to Penrose's claim mentioned above. So, within $\mathfrak{Q}$ we can express that 'permutations are not observable', without necessarily introducing ad hoc postulates like IP (equation (4)) above.

The theory has other applications, for instance in deriving quantum statistics without the needs of postulating certain symmetry conditions [25], but these developments shall be not mentioned here. To end this outline, let us make reference only to a certain detail involving the relationship between ordinals and cardinals, which is relevant for the above discussion, as we have seen.

Jonathan Lowe has characterised individuality in terms of countability [27]. This is one of the common ways of doing that. In considering it, a question arises: how are we to understand a 'countable plurality' of individuals about which we cannot deny that it is indeterminate whether they are identical to themselves or not? In particular, in what sense can such a plurality be said to be countable? Countability is precisely what is problematic about quantum entities. Teller, for example, has emphasised that 'quanta' cannot be counted but only *aggregated*, and invokes the analogy of money in a bank account to exemplify his claims [36]. In this sense, I can say that I have 100 reais (the Brazilian currency) in my current account but I can't go in and point to a particular real and say, 'that's mine'. This offers a further perspective on the way in which quantum particles might be regarded as non-individuals.

How might we represent this distinction between a 'countable plurality' and an aggregate? One way is to note that it corresponds to which holds between ordinality and cardinality respectively. Hence an 'aggregate' of quanta would have the latter but not the former, since we can say on experimental and theoretical grounds that there are, say, seven electrons in an atom, but we cannot count them, in the sense of putting them in a series and establishing an ordering. This distinction between ordinality and cardinality is interesting in several aspects; recently, Michel Bitbol has suggested that in understanding Jean-Louis Destouches' transcendental structuralism, one should acknowledge such a distinction (and he mentioned the possible use of quasi-sets in this realm [1]). But we should realise that there are some technical details involved in this question.

The main reason is that $\mathfrak{Q}$ is based on ZF-like axioms, and the concept of cardinal is introduced *a la* von Neumann, as particular ordinals. So, if a quasi-set has a quasi-cardinal (which is a cardinal, as given by the axioms of the theory), then it is an ordinal, and hence the quasi-set *has* an associated ordinal, although this being contrary to the basic idea that indistinguishable m-atoms cannot be ordered. The explanation of this fact is as follows, and resembles in much the solution given by T. Skolem to his famous 'paradox': the mapping (a 'quasi-function') which 'orders' a quasi-set of indistinguishable m-atoms is not something that can be described by the axioms of $\mathfrak{Q}$. In other words, it may exist but *outside* the universe Q (see Figure 1 again). In fact, we can prove that no order relation can be defined on a quasi-set of indistinguishable m-atoms [23].

Anyway, it remains the problem of finding a 'more natural' way of expressing that a quasi-set has a (quasi-)cardinal but not an ordinal. One of the possibilities could be to use NBG-like axioms instead of axioms based on ZF, and then to introduce the concept of cardinal in the Frege-Russell sense, by means of the so-called 'Tarski's axiom' (the axiom says that two sets have the same cardinal iff they are equinumerous; in this case, as it is well known, except for the cardinal 0, which is the cardinal of the empty set, all cardinals are proper classes [35]). But this is still something to be done, perhaps by someone who intend to push Bitbol's suggestion of studying Destouches' philosophy of physics.

8. Final remarks

Although quasi-set theory has been dealt with in some directions, of course there remain lots of details to be cleaned up, as the cardinal/ordinal discussion shows. Also in what respects applications we can find interesting insights; Bitbol's suggestion is just one of them, but it seems that something of this kind could be useful in the foundational aspects of quantum field theory as well, since in this field one do not deal with 'individuals' at all [36]. Anyway, we certainly can't anticipate anything in this sense.

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Décio Krause
Departamento de Filosofia
Universidade Federal de Santa Catarina
E-mail: dkrause@uol.com.br