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A comparative study of the control of two beam models

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ABSTRACT: We study the well posedness and the controllability of a realistic beam model. It turns out that for some values of the papameters it is controllable, while for other values it is not even well posed. In order to solve this problem we also give a general abstract necessary and sufficient condition of well posedness of linear distributed systems.

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1. On the well posedness of linear systems

Consider the linear evolutionary problem

$$y' = Ay + f, \quad y(0) = y_0 \tag{1}$$

in a complex Hilbert space H. Assume that H has a Riesz basis e_1, e_2,... formed by eigenvectors of A, corresponding to complex eigenvalues λ_1, λ_2, ...:

$$Ae_k = \lambda_k e_k, \quad k = 1, 2, \dots$$

We are going to study the well posedness of this problem. It is rather well-known that the condition

$$C := \sup \Re\lambda_k < \infty$$

is sufficient for the well-posedness. It seems to be less known that this condition is also necessary. For the reader's convenience we prove both results. Fix $T > 0$ arbitrarily.

Definition 1.1 *Given $f \in L^2(0,T;H)$ and $y_0 \in H$, a solution of (1) is a function $y \in C([0,T],H)$ satisfying $y(0) = y_0$ and such that*

$$\int_0^T y\varphi' + (Ay + f)\varphi \; dt = 0$$

for all test functions $\varphi \in C_c^\infty(0,T)$.

Theorem 1

(a) If $C < \infty$, then the problem (1) has a unique solution for every $y_0 \in H$ and $f \in L^2(0,T;H)$. It is given by the series

$$y(t) = \sum_{k=1}^{\infty} \Big(c_k e^{\lambda_k t} + \int_0^t e^{\lambda_k (t-s)} f_k(s)\ ds \Big) e_k \tag{2}$$

converging in $C([0,T],H)$, where the coefficients c_k and $f_k(s)$ are given by the expansions of y_0 and $f(s)$ according to the Riesz basis (e_k):

$$y_0 = \sum_{k=1}^{\infty} c_k e_k,$$
$$f(s) = \sum_{k=1}^{\infty} f_k(s) e_k.$$

(The last expansion holds true for almost every $s \in (0,T)$.)

(b) If $C = \infty$, then there exists $y_0 \in H$ such that the problem (1) has no solution (defined on the whole interval $[0,T]$) for any $f \in L^2(0,T;H)$.

Remark 1 *The proof will show that if $C = \infty$ but*

$$\sum_{k=1}^{\infty} |c_k|^2 e^{2\Re \lambda_k T} < \infty$$

and

$$\int_0^T \Big(\sum_{k=1}^{\infty} e^{2\Re \lambda_k T} |f_k(s)|^2 \Big)^{1/2} ds < \infty,$$

then there still exists a unique solution $y \in C([0,T],H)$.

Proof:[Proof of part (a)] Assume that (1) has a solution y and expand it into a series according to the Riesz basis (e_k): we obtain for each $t \in [0,T]$ an expansion

$$y(t) = \sum_{k=1}^{\infty} y_k(t) e_k.$$

Using the definition of the solution, we obtain for each k that

$$y_k'(t) = \lambda_k y_k + f_k, \quad y_k(0) = c_k,$$

whence

$$y_k(t) = c_k e^{\lambda_k t} + \int_0^t e^{\lambda_k (t-s)} f_k(s)\ ds.$$

This proves the uniqueness of the solution and also the validity of the series representation (2).

It remains to verify that under the assumption on y_0 and f the series (2) converges in $C([0,T],H)$. One can then easily show that its sum is a solution of (1). Since the space $C([0,T],H)$ is complete, it suffices to prove that the partial sums of the series

$$\sum_{k=1}^{\infty} c_k e^{\lambda_k t} e_k$$

and

$$\sum_{k=1}^{\infty} \int_0^t e^{\lambda_k (t-s)} f_k(s)\ ds\ e_k$$

form two Cauchy sequences in $C([0,T],H)$.

Since (e_k) is a Riesz basis in H, there exists a constant $c > 0$ such that for every $x \in H$, its expansion $x = \sum x_k e_k$ satisfies the inequalities

$$c^{-1} \sum |x_k|^2 \le \|x\|_H^2 \le c \sum |x_k|^2.$$

We shall frequently use these inequalities in the sequel.

For $n > m$ we have

$$\Big\| \sum_{k=m+1}^{n} c_k e^{\lambda_k t} e_k \Big\|_H^2 \le c \sum_{k=m+1}^{n} |c_k|^2 e^{2\Re \lambda_k t} \le c e^{2CT} \sum_{k=m+1}^{n} |c_k|^2.$$

We conclude by noting that the last expression does not depend on $t \in [0,T]$ and that it converges to zero as $m, n \to \infty$ because

$$\sum_{k=1}^{\infty} |c_k|^2 \le c \|y_0\|_H^2 < \infty.$$

Similarly, for $n > m$ we also have

$$\begin{aligned}
\Big\| \sum_{k=m+1}^{n} \int_0^t e^{\lambda_k (t-s)} f_k(s)\ ds\ e_k \Big\|_H &\le \int_0^t \Big\| \sum_{k=m+1}^{n} e^{\lambda_k (t-s)} f_k(s) e_k \Big\|_H\ ds \\
&\le \sqrt{c} \int_0^t \sqrt{\sum_{k=m+1}^{n} e^{2\Re \lambda_k (t-s)} |f_k(s)|^2}\ ds \\
&\le \sqrt{c} e^{CT} \int_0^T \sqrt{\sum_{k=m+1}^{n} |f_k(s)|^2}\ ds \\
&\le c e^{CT} \int_0^T \Big\| \sum_{k=m+1}^{n} f_k(s) e_k \Big\|_H\ ds.
\end{aligned}$$

The last expression does not depend on the particular choice of $t \in [0,T]$. Moreover, thanks to our assumption $f \in L^1(0,T;H)$, it tends to zero as $m, n \to \infty$.

This completes the proof. □

[Proof of part (b)] Choose a subsequence (λ_{k_ℓ}) satisfying

$$\Re\lambda_{k_\ell} > e^\ell$$

for $\ell = 1, 2, \dots$, and set

$$y_0 = \sum_{\ell=1}^{\infty} \frac{1}{\ell} e_{k_\ell}.$$

Since $\sum 1/\ell^2 < \infty$, we have $y_0 \in H$. We claim that for $f \in L^2(0,T;H)$ given arbitrarily, the problem (1) has no solution defined on the whole interval $[0,T]$.

Assume on the contrary that for some $f \in L^2(0,T;H)$ there exists a solution $y \in C([0,T];H)$. Then, repeating the beginning of the proof of part (a) above, y is given by the series (2) with $c_{k_\ell} = 1/\ell$ for all ℓ and with $c_k = 0$ otherwise.

Observe that since

$$\int_0^T |f_k(s)|^2 \ ds \le \int_0^T \sum |f_k(s)|^2 \ ds = \|f\|^2_{L^2(0,T;H)} < \infty,$$

there exists a constant C such that

$$\int_0^T |f_k(s)|^2 \ ds \le C \quad \text{for all} \quad k.$$

Since $y(T) \in H$, it follows from (2) that

$$\sum_{\ell=1}^{\infty} \Big| \frac{e^{\lambda_{k_\ell} T}}{\ell} + \int_0^T e^{\lambda_{k_\ell}(T-s)} f_{k_\ell}(s) \ ds \Big|^2 \le c \|y(T)\|_H^2 < \infty. \tag{3}$$

Hence all but finitely many terms of the sum are smaller than 1. Since

$$\begin{aligned}
&\Big| \frac{e^{\lambda_{k_\ell} T}}{\ell} + \int_0^T e^{\lambda_{k_\ell}(T-s)} f_{k_\ell}(s) \ ds \Big| \\
&\qquad \ge e^{\Re\lambda_{k_\ell} T} \Big(\frac{1}{\ell} - \Big(\int_0^T e^{-2\Re\lambda_{k_\ell} s} \ ds \Big)^{1/2} \|f_{k_\ell}\|_{L^p(0,T)} \Big) \\
&\qquad \ge e^{\Re\lambda_{k_\ell} T} \Big(\frac{1}{\ell} - \frac{C}{(2\Re\lambda_{k_\ell})^{1/2}} \Big),
\end{aligned}$$

we conclude that for all sufficiently large ℓ we have

$$e^{\Re\lambda_{k_\ell} T} \Big(\frac{1}{\ell} - \frac{C}{(2\Re\lambda_{k_\ell})^{1/2}} \Big) < 1$$

so that

$$C > (2\Re\lambda_{k_\ell})^{1/2} \Big(\frac{1}{\ell} - \frac{1}{e^{\Re\lambda_{k_\ell} T}} \Big)$$

But this is impossible because the right hand side tends to infinity by the choice of λ_{k_ℓ}. □

2. Control of Kirchhoff beams

Given three numbers $a, b, c > 0$, consider for a given (time) $T > 0$ the following problem:

$$\begin{cases} au_{tt} + b^2 u_{xxxx} - cu_{ttxx} = g \quad \text{in} \quad (0,\pi) \times (0,T), \\ u(0,t) = u(\pi,t) = 0 \quad \text{for} \quad 0 < t < T, \\ u_{xx}(0,t) = u_{xx}(\pi,t) = 0 \quad \text{for} \quad 0 < t < T, \\ u(x,0) = u^0(x) \text{ and } u_t(x,0) = u^1(x) \quad \text{for} \quad 0 < x < \pi. \end{cases} \tag{4}$$

We may rewrite it in the form (1) by setting

$$y = \begin{pmatrix} u \\ u_t \end{pmatrix}, \ f = \begin{pmatrix} 0 \\ (aI - c\Delta)^{-1} g \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & I \\ -b^2 (aI - c\Delta)^{-1} \Delta^2 & 0 \end{pmatrix}.$$

Set

$$D^j := \{v \in H^j(0,\pi) \ : \ v(0) = v(\pi) = 0\}, \quad j = 1, 2$$

and

$$D^j := \{v \in H^j(0,\pi) \ : \ v(0) = v(\pi) = v''(0) = v''(\pi) = 0\}, \quad j = 3, 4$$

for brevity. Equivalently, D^j is the completion of the vector space generated by the functions $\sin kx$, $k = 1, 2, \dots$, with respect to the scalar product

$$(u, v)_j := \sum_{k=1}^{\infty} k^j u_k v_k,$$

where

$$u(x) = \sum_{k=1}^{\infty} u_k \sin kx \quad \text{and} \quad v(x) = \sum_{k=1}^{\infty} v_k \sin kx.$$

(Note that this definition enables us to define a Hilbert space D^j for each real j.

Consider the Hilbert space $H := D^4 \times D^3$. One can readily verify that $f \in L^1(0,T;H)$ if and only if $g \in L^1(0,T;D^1)$. Furthermore, setting

$$e_k^{\pm} = \begin{pmatrix} 1 \\ \lambda_k^{\pm} \end{pmatrix} \frac{\sin kx}{k^4} \quad \text{and} \quad \lambda_k^{\pm} = \frac{\pm ibk^2}{\sqrt{a + ck^2}}$$

for $k = 1, 2, \dots$, we have a Riesz basis $(e_k^{\pm})$ of H, formed by eigenvectors of A with the eigenvalues $\lambda_k^{\pm}$.

Since the eigenvalues $\lambda_k^{\pm}$ are purely imaginary for all sufficiently large k, applying Theorem 1 we conclude that our problem is well posed. Hence for every

$$u^0 \in D^4, \quad u^1 \in D^3 \quad \text{and} \quad g \in L^1(0,T;D^1),$$

the problem (4) has a unique solution satisfying

$$u \in C([0,T];D^4) \cap C^1([0,T];D^3) \cap C^2([0,T];D^2).$$

We are going to prove the following exact internal controllability result:

Theorem 2 *Given $u^0 \in D^4$ and $u^1 \in D^3$ arbitrarily, there exists a function $g \in L^\infty(0,T;D^1)$ such that the solution of* (4) *satisfies*

$$u(x,T) = u_t(x,T) = 0 \quad \textit{for all} \quad 0 < x < \pi. \tag{5}$$

Proof: Let us write the solution of (4) in an explicit form. Developing u, g, u^0 and u^1 into Fourier series according to the orthogonal basis $(\sin kx)$ of $L^2(0,\pi)$ (which is also an orthogonal basis in $H^1_0(0,\pi)$ and in $(H^2 \cap H^1_0)(0,\pi)$), we have

$$\begin{aligned} u(x,t) &= \sum u_k(t)\sin kx, \\ g(x,t) &= \sum g_k(t)\sin kx, \\ u^0(x) &= \sum u^0_k(t)\sin kx, \\ u^1(x) &= \sum u^1_k(t)\sin kx, \end{aligned}$$

where k runs from 1 to ∞ in all sums. Substituting these expressions into (4), we obtain for each k the initial-value problem

$$\begin{cases} au_k'' + ck^2u_k'' + b^2k^4u_k = g_k \quad \text{in} \quad (0,T), \\ u_k(x,0) = u^0_k(x) \text{ and } u_k'(x,0) = u^1_k(x) \quad \text{for} \quad 0 < x < \pi. \end{cases} \tag{6}$$

Let us solve (6) for u_k. Since the characteristic equation

$$(a + ck^2)\lambda^2 + b^2k^4 = 0$$

has two distinct roots

$$\lambda_k^\pm = \pm \frac{ibk^2}{\sqrt{a+ck^2}} =: \pm\gamma_k i, \tag{7}$$

applying the variation of constants formula we find that

$$u_k(t) = u^0_k \cos\gamma_k t + \frac{u^1_k}{\gamma_k}\sin\gamma_k t + \frac{1}{a+ck^2}\int_0^t \frac{\sin\gamma_k(t-s)}{\gamma_k} f_k(s)\ ds. \tag{8}$$

Let us try to find controls of the form

$$g_k(t) = \alpha_k \cos\gamma_k(T-t) + \beta_k \sin\gamma_k(T-t) \tag{9}$$

with suitable coefficients α_k and β_k. Substituting this expression into (8), the conditions (5) are equivalent to $u_k(T) = u_k'(T) = 0$ for each k, which leads to the system of linear equations

$$\gamma_k u^0_k \cos\gamma_k T + u^1_k \sin\gamma_k T + \frac{1}{a+ck^2}\int_0^T \sin\gamma_k s(\alpha_k\cos\gamma_k s + \beta_k \sin\gamma_k s)\ ds = 0$$

and

$$
\begin{aligned}
&-\gamma_k u_k^0 \sin \gamma_k T + u_k^1 \cos \gamma_k T \\
&\qquad\qquad + \frac{1}{a+ck^2} \int_0^T \cos \gamma_k s (\alpha_k \cos \gamma_k s + \beta_k \sin \gamma_k s)\ ds = 0.
\end{aligned}
$$

This system can be solved uniquely for α_k and β_k because its determinant

$$
\begin{vmatrix}
\int_0^T \sin \gamma_k s \cos \gamma_k s\ ds & \int_0^T \sin^2 \gamma_k s\ ds \\
\int_0^T \cos^2 \gamma_k s\ ds & \int_0^T \cos \gamma_k s \sin \gamma_k s\ ds
\end{vmatrix}
$$

is strictly negative. Indeed, this follows from the Cauchy–Schwarz inequality because the functions $\sin \gamma_k s$ and $\cos \gamma_k s$ are linearly independent in $L^2(0,T)$.

Let us investigate the asymptotic behavior of α_k and β_k for large k. From (7) we deduce that

$$
\gamma_k \asymp \frac{b}{\sqrt{c}} k =: dk.
$$

Here and in the sequel the notation $a_k \asymp b_k$ means the existence of two positive constants c_1 and c_2 such that

$$
c_1 a_k \le b_k \le c_2 a_k
$$

for all k. Therefore we have

$$
\begin{aligned}
\int_0^T \sin \gamma_k s \cos \gamma_k s\ ds &= \frac{1-\cos 2\gamma_k T}{4\gamma_k} \to 0, \\
\int_0^T \sin^2 \gamma_k s\ ds &= \frac{T}{2} - \frac{\sin 2\gamma_k T}{4\gamma_k} \to \frac{T}{2}, \\
\int_0^T \cos^2 \gamma_k s\ ds &= \frac{T}{2} + \frac{\sin 2\gamma_k T}{4\gamma_k} \to \frac{T}{2}.
\end{aligned}
$$

Hence

$$
\alpha_k \asymp \tilde{\alpha}_k \quad \text{and} \quad \beta_k \asymp \tilde{\beta}_k
$$

where $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ denote the solutions of the system

$$
\begin{aligned}
k^3 u_k^0 \cos dkT + k^2 u_k^1 d^{-1} \sin dkT + (2d)^{-1} T \tilde{\beta}_k &= 0, \\
-dk^3 u_k^0 \sin dkT + k^2 u_k^1 \cos dkT + (2)^{-1} T \tilde{\alpha}_k &= 0.
\end{aligned}
$$

We conclude that

$$
|\alpha_k| + |\beta_k| \asymp k^2 (|k u_k^0| + |u_k^1|).
$$

Using (9), there exists thus a constant $C > 0$ such that

$$
|g_k(t)|^2 \le (|\alpha_k| + |\beta_k|)^2 \le 2C^2 k^4 (|k u_k^0|^2 + |u_k^1|^2)
$$

for all k and for all $0 \le t \le T$, and then

$$\begin{aligned}
\|g(t)\|^2_{H^1_0(0,\pi)} &= \int_0^\pi \Big|\sum g_k(t) k \cos kx\Big|^2 \ dx\\
&= \sum |g_k(t)|^2 k^2 \int_0^\pi \cos^2 kx \ dx\\
&= \frac{\pi}{2} \sum |g_k(t)|^2 k^2\\
&\le C^2 \pi \sum k^4 (|k^2 u_k^0|^2 + |k u_k^1|^2)\\
&= C^2 \pi \big(\|u^0\|^2_{D^4} + \|u^1\|^2_{D^3}\big)
\end{aligned}$$

for all $0 \le t \le T$. This proves that g satisfies (6). $\square$

One can show that the choice (9) leads to the control of minimal norm. Indeed, putting $R_k(s) = \sin \gamma_k (T-s)$ for brevity, for any control $g_k(t)$ we deduce from the equalities $u_k(T) = u_k'(T) = 0$ that

$$\begin{aligned}
|u_k^0 \cos \gamma_k T + \frac{u_k^1}{\gamma_k} \sin \gamma_k T| &= \frac{1}{|(a+ck^2)\gamma_k|} \Big| \int_0^T R_k(s) g_k(s) \ ds \Big|\\
&\le \frac{1}{|(a+ck^2)\gamma_k|} \|R_k\| \cdot \|g_k\|
\end{aligned}$$

whence

$$\begin{aligned}
|u_k^0 \gamma_k \cos \gamma_k T + u_k^1 \sin \gamma_k T| &= \frac{1}{|a+ck^2|} \Big| \int_0^T R_k(s) g_k(s) \ ds \Big|\\
&\le \frac{1}{|a+ck^2|} \|R_k\| \cdot \|g_k\|,
\end{aligned}$$

and

$$\begin{aligned}
|-u_k^0 \gamma_k \sin \gamma_k T + u_k^1 \cos \gamma_k T| &= \frac{1}{|a+ck^2|} \Big| \int_0^T \gamma_k^{-1} R_k'(s) g_k(s) \ ds \Big|\\
&\le \frac{1}{|a+ck^2|} \|R_k\| \cdot \|g_k\|,
\end{aligned}$$

so that

$$|u_k^0|^2 |\gamma_k|^2 + |u_k^1|^2 = |a+ck^2|^{-2} (\|R_k\|^2 + \gamma_k^{-2} \|R_k'\|^2) \cdot \|g_k\|^2.$$

Since

$$\|R_k\|^2 + \gamma_k^{-2} \|R_k'\|^2 = T,$$

it follows that

$$\|g_k\|^2 \ge T^{-1} |a+ck^2|^2 (|u_k^0|^2 |\gamma_k|^2 + |u_k^1|^2),$$

with equality if and only if g_k is a multiple of R_k by the condition of equality in the Cauchy–Schwarz inequality.

3. A beam model of bending waves. The well-posed case

Given three positive numbers a, b, c, consider the problem

$$\begin{cases} u_{tttt} + 2au_{tt} + b^2 u_{xxxx} - 2cu_{ttxx} = g \text{ in } (0,\pi) \times (0,T), \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 \text{ for } 0 < t < T, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) \text{ for } 0 < x < \pi, \\ u_{tt}(x,0) = u^2(x),\ u_{ttt}(x,0) = u^3(x) \text{ for } 0 < x < \pi \end{cases} \tag{10}$$

for some given time $T > 0$. Let us rewrite it in the form (1) by setting

$$y = \begin{pmatrix} u \\ u_t \\ u_{tt} \\ u_{ttt} \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ 0 \\ 0 \\ g \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -b^2\Delta^2 & 0 & -2(aI - c\Delta) & 0 \end{pmatrix}.$$

Introducing the same Hilbert spaces D^j as in the preceding section, now consider the Hilbert space $H := D^4 \times D^3 \times D^2 \times D^1$. We have $f \in L^1(0,T;H)$ if and only if $g \in L^1(0,T;D^1)$. Furthermore, denoting for each positive integer k by $\lambda_{k1}, \dots, \lambda_{k4}$ the four numbers given by the formula

$$\pm\sqrt{-(a+ck^2) \pm \sqrt{a^2 + 2ack^2 + (c^2 - b^2)k^4}}$$

and setting

$$e_{kj} = \begin{pmatrix} 1 \\ \lambda_{kj} \\ \lambda_{kj}^2 \\ \lambda_{kj}^3 \end{pmatrix} \frac{\sin kx}{k^4},$$

one can readily verify that every e_{kj} is an eigenvector of A with the eigenvalue λ_{kj} and that these vectors form a Riesz basis in H.

For large k we have

$$\lambda_{kj} \approx \big(\pm\sqrt{-c \pm \sqrt{c^2 - b^2}}\big)k, \quad j = 1,2,3,4. \tag{11}$$

If $c < b$, then these formulas show that the real parts of the eigenvalues are not bounded from above, so that the problem is ill-posed. On the other hand, if $c \ge b$, then they are bounded from above (indeed, they are purely imaginary apart from maybe a finite number of terms), so that the problem is well posed. More precisely, applying Theorem 1 we obtain that for every given

$$u^0 \in D^4, \quad u^1 \in D^3 \quad u^2 \in D^2, \quad u^3 \in D^1 \quad \text{and} \quad g \in L^1(0,T;D^1),$$

the problem (10) has a unique solution u in

$$\bigcap_{j=0}^{4} C^j([0,T];D^{4-j})$$

with $D^0 := L^2(0,\pi)$. In the rest of this section we restrict ourselves to the case $c > b$. Then the problem is well posed and the four roots λ_{kj} are pairwise distinct for each k. We study the exact internal controllability of this system. We are going to prove the

Theorem 3 *Assume that $c > b$. Given $u^0 \in D^4$, $u^1 \in D^3$, $u^2 \in D^2$ and $u^3 \in D^1$ arbitrarily, there exists $g \in L^\infty(0,T;D^1)$ such that the solution of (10) satisfies*

$$u(x,T) = u_t(x,T) = u_{tt}(x,T) = u_{ttt}(x,T) = 0 \quad \textit{for all} \quad 0 < x < \pi.$$

Proof: Developing u, g, u^0, u^1, u^2, u^3 into Fourier series according to the functions $\sin kx$ as in the preceding section, now we find for each $k = 0, 1, \dots$ the following system:

$$\begin{cases} u_k'''' + 2(a + ck^2)u_k'' + b^2k^4u_k = g_k \quad \text{in} \quad (0,T), \\ u_k(x,0) = u_{k0}(x), \ u_k'(x,0) = u_{k1}(x) \quad \text{for} \quad 0 < x < \pi, \\ u_k''(x,0) = u_{k2}(x), \ u_k'''(x,0) = u_{k3}(x) \quad \text{for} \quad 0 < x < \pi. \end{cases}$$

Since the four roots λ_{kj} are distinct , the solution of this system has the form

$$u_k(t) = \sum_{j=1}^{4} v_{kj} e^{\lambda_{kj} t} + \int_0^t R_k(t-s) g_k(s) \ ds \tag{12}$$

where R_k is the resolvent of the system given by the formula

$$\begin{aligned} R_k(t) =: &\frac{e^{\lambda_{k1}t}}{(\lambda_{k1} - \lambda_{k2})(\lambda_{k1} - \lambda_{k3})(\lambda_{k1} - \lambda_{k4})} \\ &+ \frac{e^{\lambda_{k2}t}}{(\lambda_{k2} - \lambda_{k1})(\lambda_{k2} - \lambda_{k3})(\lambda_{k2} - \lambda_{k4})} \\ &+ \frac{e^{\lambda_{k3}t}}{(\lambda_{k3} - \lambda_{k1})(\lambda_{k3} - \lambda_{k2})(\lambda_{k3} - \lambda_{k4})} \\ &+ \frac{e^{\lambda_{k4}t}}{(\lambda_{k4} - \lambda_{k1})(\lambda_{k4} - \lambda_{k2})(\lambda_{k4} - \lambda_{k3})} =: \sum_{j=1}^{4} R_{kj}(t), \end{aligned}$$

and the complex coefficients v_{kj} depend on the initial data via the linear system

$$\sum_{j=0}^{3} \lambda_{kj}^p v_{kj} = u_{kp}, \quad p = 0, \dots, 3.$$

It is natural to seek controls of the form

$$g_k(t) = \sum_{\ell=1}^{4} g_{k\ell} \overline{R_{k\ell}(T-t)} \tag{13}$$

with suitable coefficients $g_{k\ell}$. Substituting into (12), the “final” conditions

$$u_k^{(p)}(T) = 0, \quad p = 0, \dots, 3$$

are equivalent to the following linear system:

$$\sum_{j=1}^{4} \lambda_{kj}^p v_{kj} e^{\lambda_{kj}T} + \sum_{j,\ell=1}^{4} g_{k\ell} \lambda_{kj}^p \int_0^T R_{kj}(t)\overline{R_{k\ell}(t)}\ dt = 0, p = 0, \dots, 3. \quad (14)$$

The determinant of this system is different from zero by the general theory of linear ordinary differential equations. Furthermore, since the eigenvalues are purely imaginary for all sufficiently large k, using also the asymptotic relations (11) we obtain that

$$k^6 \int_0^T R_{kj}(t)\overline{R_{k\ell}(t)}\ dt \to \begin{cases} r_j & \text{if } j = \ell, \\ 0 & \text{if } j \neq \ell, \end{cases}$$

where the $r_1, \dots, r_4$ are strictly positive numbers depending on a, b, c. Using these relations we deduce from (14) that

$$g_{kj} \approx -\frac{k^6}{r_j} e^{\lambda_{kj}T} v_{kj}$$

if k is sufficiently large. It follows from the definition of the functions R_{kj}, from (11) and from the fact that the real parts of the eigenvalues are bounded from above, that

$$|R_{kj}(t)| \le ck^{-3}$$

for all k, j and $0 < t < T$ with some uniform constant c. Using the last two estimates, we deduce from (13) that

$$|g_k(t)| \le ck^{-3} \sum_{j=1}^{4} |g_{kj}| \le c' k^3 \sum_{j=1}^{4} |v_{kj}|$$

with another constant c', independent of k. Finally, since

$$\|g(t)\|_{D^1}^2 \le c'' \sum_{k=1}^{\infty} k^2 |g_k(t)|^2 \le c'''' \sum_{k=1}^{\infty} k^8 \sum_{j=1}^{4} |v_{kj}|^2 < \infty,$$

we have $g \in L^\infty(0, T; D^1)$ indeed. $\square$

4. The beam model of bending waves. The ill-posed case

If $c < b$, then the problem (10) is generally ill-posed. However, according to the remark following the formulation of Theorem 1, for certain initial data there still exist global solutions. It is then natural to study the existence of controllable states.

Using the notations of the preceding section, we may still assume that the four roots λ_{kj} are pairwise distinct for each k. Let us order them such that

$$\Re\lambda_{k1} \geq \Re\lambda_{k2} \geq \Re\lambda_{k3} \geq \Re\lambda_{k4}$$

for each k. We may repeat the computations of the preceding section, by only changing the asymptotic relations for the integrals in (14). Now we have

$$k^6 e^{\Re\lambda_{k\ell}-\Re\lambda_{kj}} \int_0^T R_{kj}(t)\overline{R_{k\ell}(t)}\ dt \to \begin{cases} r_j & \text{if } j=\ell, \\ 0 & \text{if } j\neq\ell, \end{cases}$$

where the $r_1,\dots, r_4$ are nonzero numbers depending on a, b, c. Using these relations we deduce from (14) that

$$g_{kj} \approx -\frac{k^6}{r_j} e^{\lambda_{kj}T} v_{kj}$$

if k is sufficiently large. It follows from the definition of the functions R_{kj}, from (11) and from the fact that the real parts of the eigenvalues are bounded from above, that

$$|R_{kj}(t)| \leq ck^{-3}$$

for all k, j and $0 < t < T$ with some uniform constant c. Using the last two estimates, we deduce from (13) that

$$|g_k(t)| \leq ck^{-3}\sum_{j=1}^4 |g_{kj}| \leq c'k^3 \sum_{j=1}^4 |v_{kj}|$$

with another constant c', independent of k. Finally, since

$$\|g(t)\|_{D^1}^2 \leq c'' \sum_{k=1}^\infty k^2 |g_k(t)|^2 \leq c'''' \sum_{k=1}^\infty k^8 \sum_{j=1}^4 |v_{kj}|^2 < \infty$$

so that $g \in L^\infty(0,T;D^1)$ indeed.

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