Existence of a Renormalized Solution of Nonlinear Parabolic Equations with Lower Order Term and General Measure Data

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ABSTRACT: We give an existence result of a renormalized solution for a class of nonlinear parabolic equations
\[ \frac{\partial b(u)}{\partial t} - \text{div}(a(x, t, \nabla u)) + H(x, t, \nabla u) = \mu, \]
where the right side is a general measure, \( b \) is a strictly increasing \( C^1 \)-function, \( -\text{div}(a(x, t, \nabla u)) \) is a Leray–Lions type operator with growth \( |\nabla u|^{p-1} \) in \( \nabla u \) and \( H(x, t, \nabla u) \) is a nonlinear lower order term which satisfy the growth condition with respect to \( \nabla u \).

Key Words: Nonlinear parabolic equations, Existence, Renormalized solutions, Lower order term, Measure data.

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1. Introduction

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), \( (N \geq 1) \), \( T > 0 \) and let \( Q_T := \Omega \times (0, T) \). We prove the existence of a renormalized solution for a class of nonlinear parabolic equations of the type:
\[ \frac{\partial b(u)}{\partial t} - \text{div}(a(x, t, \nabla u)) + H(x, t, \nabla u) = \mu \quad \text{in} \ Q_T, \]
\[ b(u)(t = 0) = b(u_0) \quad \text{in} \ \Omega, \]
\[ u = 0 \quad \text{on} \ \partial \Omega \times (0, T). \]

In Problem (1.1)-(1.3) the framework is the following: the data \( \mu \) is a general measure, \( b \) is a strictly increasing \( C^1 \)-function, the operator \( -\text{div}(a(x, t, \nabla u)) \) is a Leray–Lions operator which is coercive and grows like \( |\nabla u|^{p-1} \) with respect to \( \nabla u \) and \( H(x, t, \nabla u) \) is a nonlinear lower order term. In the case where \( b(u) = u, H = 0, \) and the growth condition holds, then a renormalized solution exists.
the right hand side is a bounded measure, the existence of a distributional solution was proved in [3], but due the lack of regularity of solution, the distributional formulation is not strong enough to provide uniqueness (see [27] for a counter example in the elliptic case). To overcome this difficulty the notion of renormalized solutions firstly introduced by R.J. DiPerna and P.-L.Lions in [8] for the study of Boltzmann equation was adapted to parabolic equations and elliptic equations with \( L^1 \) data. When \( \mu \) is measure data that does not charge the sets of zero \( p \)-capacity (the so called diffuse measure, see the definition in the section 2 below) a notion of renormalized solution for Problem (1.1)-(1.3) was introduced in [11] for \( b(u) = u \) and \( H = 0 \). Similar result was proved in [5] when \( H = 0 \). In [22] the existence of renormalized solution was proved in the case where \( b(u) = u \) and \( H = 0 \) and \( \mu \) is a general measure (see also [6]). In [10] a similar notion of entropy solution is also defined and proved to be equivalent to the renormalized one. In this paper we use a new definition of renormalized solution which is adopted in [25] and [24] for the study of parabolic \( p \)-Laplacian equations with general measure data.

The paper is organized as follows. In section 2 we give some preliminaries on the concept of \( p \)-capacity. Section 3 will be devoted to set our main assumptions and definition of renormalized solution and the statement of the existence result, while in Section 4 we give the proof of our main result.

2. Preliminaries on parabolic capacity

We introduce the notion of \( p \)-capacity associated to our problem (for further details see [21], [11]). Let \( Q_T = \Omega \times (0, T) \) for any fixed \( T > 0 \) and \( 1 < p < \infty \), and let us recall that \( V = W^{1,p}_0(\Omega) \cap L^2(\Omega) \), endowed with its natural norm \( \| \cdot \|_{W^{1,p}_0(\Omega)} + \| \cdot \|_{L^2(\Omega)} \) and

\[
W = \left\{ u \in L^p(0, T; V), u_t \in L^p(0, T; V') \right\},
\]

endowed with its natural norm \( \| \cdot \|_{L^p(0, T; V)} + \| \cdot \|_{L^p(0, T; V')} \), remark that \( W \) is continuously embedded in \( C([0, T], L^2(\Omega)) \), and if \( 1 < p < \infty \), then \( C^\infty_c(Q_T) \) is dense in \( W \). Let \( U \subseteq Q_T \) be an open set, we define the parabolic \( p \)-capacity of \( U \) as

\[
cap_p(U) = \inf \left\{ \| u \|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q_T \right\},
\]

where as usual we set \( \inf \{ \emptyset \} = +\infty \), then for any Borel set \( B \subseteq Q_T \) we define

\[
cap_p(B) = \inf \left\{ \cap_p(U) : U \text{ open set of } Q_T, B \subseteq U \right\}.
\]

We will denote by \( M(Q_T) \) the set of all Radon measures with bounded variation on \( Q_T \), while, as we already mentioned, \( M_0(Q_T) \) the set of all measures with bounded total variation over \( Q_T \) that do not charge the sets of zero \( p \)-capacity, that is if \( \mu \in M_0(Q_T) \), then \( \mu(E) = 0 \), for all \( E \subseteq Q_T \) such that \( \cap_p(E) = 0 \).

In [11] the authors proved the following decomposition theorem:
Theorem 2.1. Let $\mu$ be a bounded measure on $Q_T$. If $\mu \in M_0(Q_T)$ then there exists $(f, g_1, g_2)$ such that $f \in L^1(Q_T)$, $g_1 \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T; V)$ and

$$\int_{Q_T} \phi \, d\mu = \int_{Q_T} f \phi \, dx \, dt + \int_0^T \langle g_1, \phi \rangle \, dt - \int_0^T \langle \phi_t, g_2 \rangle \, dt \quad \phi \in C_c^\infty(Q_T).$$

Such a triplet $(f, g_1, g_2)$ will called decomposition of $\mu$.

Definition 2.2. A sequence of measures $(\mu_n)$ in $Q_T$ is equidiffuse if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every Borel set $E \subseteq Q_T$,

$$\text{cap}_p(E) < \eta \Rightarrow |\mu_n|(E) < \varepsilon \quad \forall n \geq 1.$$

Let $\rho_n$ be a sequence of mollifiers on $Q_T$, the following result is proved in [25]

Proposition 2.3. If $\mu \in M_0(Q_T)$, then the sequence $\rho_n \ast \mu$ is equidiffuse.

If $\mu \in M(Q_T)$, thanks to a well known decomposition result (see for instance [14]), we can split it into a sum (uniquely determined) of its absolutely continuous part $\mu_d$ with respect to $p$-capacity and its singular part $\mu_s$, that is $\mu_s$ is concentrated on a set $E$ of zero $p$-capacity. Hence, if $\mu \in M(Q_T)$, we have

$$\mu = \mu_d + \mu_s = \mu_d + \mu_d^+ - \mu_d^-.$$

3. Assumptions and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: $\Omega$ is a bounded open set on $\mathbb{R}^N$ ($N \geq 1$), $T > 0$ is given and we set $Q_T = \Omega \times (0,T)$

$$b : \mathbb{R} \to \mathbb{R} \quad \text{and} \quad b'(0) = 0,$$

is a strictly increasing $C^1$-function with $b(0) = 0$, and there exists $\gamma > 0$ and $\Lambda > 0$ such that

$$\gamma \leq b'(s) \leq \Lambda, \quad \forall s \in \mathbb{R}. \quad (3.2)$$

$$a : Q_T \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Carathéodory function} \quad (3.3)$$

$$a(x,t,\xi).\xi \geq \alpha|\xi|^p, \quad (3.4)$$

for almost every $(x,t) \in Q_T$, for every $\xi \in \mathbb{R}^N$, where $\alpha > 0$ is a given real number.

$$|a(x,t,\xi)| \leq \beta(L(x,t) + |\xi|^{p-1}), \quad (3.5)$$

for almost every $(x,t) \in Q_T$, for every $\xi \in \mathbb{R}^N$, where $\beta > 0$ is a given real number, $L$ is a non negative function in $L^p(Q_T)$,

$$[a(x,t,\xi) - a(x,t,\xi')]|\xi - \xi'| > 0. \quad (3.6)$$
Let $H : Q_T \times \mathbb{R}^N \to \mathbb{R}$ be Carathéodory function such that for a.e. $(x, t) \in Q_T$ and for every $\xi \in \mathbb{R}^N$, the growth condition
\begin{equation}
|H(x, t, \xi)| \leq g(x, t)|\xi|^\delta, \tag{3.7}
\end{equation}
is satisfied, with $\delta = \frac{p(N+1)-N}{N+2}$ and $g$ belongs to $L^{N+2,1}(Q_T)$.
\begin{equation}
\mu \in \mathcal{M}(Q_T), \tag{3.8}
\end{equation}
\begin{equation}
u_0 \text{ is an element of } L^1(\Omega). \tag{3.9}
\end{equation}

We use in the present paper the two Lorentz spaces $L^{q,1}(Q_T)$ and $L^{q,\infty}(Q_T)$, see for example ([18], [19]) for references about Lorentz spaces $L^{q,s}$. If $f^*$ denotes the decreasing rearrangement of a measurable function $f$,
\begin{equation}
f^*(r) = \inf \left\{ s \geq 0 : \text{meas}\{ (x, t) \in Q_T : |f(x, t)| > s \} < r \right\}, \quad r \in [0, \text{meas}(Q_T)], \tag{3.10}
\end{equation}
$L^{q,1}(Q_T)$ is the space of Lebesgue measurable functions such that
\begin{equation}
\left\| f \right\|_{L^{q,1}(Q_T)} = \int_0^{\text{meas}(Q_T)} f^* r^{\frac{1}{q}} \frac{dr}{r} < \infty,
\end{equation}
while $L^{q,\infty}(Q)$ is the space of Lebesgue measurable functions such that
\begin{equation}
\left\| f \right\|_{L^{q,\infty}(Q_T)} = \sup_{r > 0} r \left[ \text{meas}\{ (x, t) \in Q_T : |f(x, t)| > r \} \right]^{\frac{1}{q}} < \infty.
\end{equation}
If $1 < q < \infty$ we have the generalized Hölder inequality
\begin{equation}
\forall f \in L^{q,\infty}(Q_T), \forall g \in L^{r',1}(Q_T) \text{ such that } \frac{1}{q} + \frac{1}{r'} = 1,
\int_Q |fg| \leq \left\| f \right\|_{L^{q,\infty}(Q_T)} \left\| g \right\|_{L^{r',1}(Q_T)}, \tag{3.10}
\end{equation}

Now we give the definition of renormalized solution of Problem (1.1) – (1.3).

**Definition 3.1.** A measurable function $u$ is a renormalized solution of Problem (1.1) – (1.3) if
\begin{equation}
T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega)), \text{ for every } k > 0, \quad H(x, t, \nabla u) \in L^1(Q_T), \tag{3.11}
\end{equation}
and if there exists a sequence of measures $\Gamma^k \in \mathcal{M}(Q_T)$ such that:
\begin{equation}
\Gamma^k \to \mu_u \text{ tightly as } k \to \infty, \tag{3.12}
\end{equation}
\begin{equation}
B_k(u) - \text{div} \left( a(x, t, \nabla T_k(u)) \right) + H(x, t, \nabla T_k(u)) = \mu_d + \Gamma^k \text{ in } \mathcal{D}'(Q_T), \tag{3.13}
\end{equation}
where $B_k(s) = \int_0^s T'_k(r)b'(r) dr$, $\forall s \in \mathbb{R}$.
Remark 3.2. Observe that (3.2) and (3.11) imply that each term in (3.13) is well defined and that (3.13) implies that $B_k(u)_t - \text{div}\left(a(x,t,\nabla T_k(u))\right) + H(x,t,\nabla T_k(u))$ is a bounded measure, then we have

$$B_k(u)_t - \text{div}\left(a(x,t,\nabla T_k(u))\right) + H(x,t,\nabla T_k(u)) = \mu_d + \Gamma_k \text{ in } \mathcal{M}(Q_T).$$

A remark on the assumption (3.2) is also necessary. As one could check later, since the data is a measure $\mu$, we are forced to assume $\gamma \leq b'(s) \leq \Lambda$. We conjecture that this assumption is only technical and could be removed in order to deal with more general elliptic-parabolic problems (see for instance [1], [7]).

In order to prove the existence result we give the following Lemma

Lemma 3.3. Let $u$ be a measurable function satisfying $T_k(u) \in L^p(0,T; W^{1,p}_0(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ for every $k > 0$ such that:

$$\sup_{t \in (0,T)} \int_\Omega |T_k(u)|^2 \, dx + \int_{Q_T} |\nabla T_k(u)|^p \, dx \, dt \leq M_k \quad \forall k > 0,$$

where $M$ is a positive constant. Then

$$\|u\|_{\frac{p}{p-1}} \leq C M \left(\frac{N+1}{N(p-1)}\right)^\frac{1}{p} |Q_T|^{\frac{1}{N(p-1)}},$$

$$\|\nabla u\|_{\frac{p}{p-1}} \leq C M \left(\frac{N+1}{N(p-1)}\right)^\frac{1}{p} |Q_T|^{\frac{1}{N(p-1)}},$$

where $C$ is a constant which depends only on $N$ and $p$.

Proof. See [13] and [12].

4. Existence result

Let us introduce the following regularization of the data: for $n \geq 1$ fixed

$$u_0^n \in C_c^\infty(Q_T), \text{ such that } u_0^n \to u_0 \text{ in } L^1(\Omega),$$

$$\mu^n \in C^\infty(Q_T), \mu^n = \mu_d^n + \mu_s^n,$$

where $\mu_d^n = \rho_n \ast \mu_d$ and $\mu_s^n = \rho_n \ast \mu_s^+ - \rho_n \ast \mu_s^- = \lambda_s^+ - \lambda_s^-$. Moreover we have

$$\|\mu^n\|_{L^1(Q_T)} \leq |\mu|_{\mathcal{M}(Q_T)},$$

and

$\mu^n$ converges to $\mu$ in the narrow topology of measures.

Let us now consider the following regularized problem

$$u^n \in L^p(0,T; W^{1,p}_0(\Omega)),$$
\[ \int_0^T \frac{\partial v^n}{\partial t} \varphi \, dt + \int_{Q_T} a(x,t,\nabla u^n) \nabla \varphi \, dx \, dt + \int_{Q_T} H(x,t,\nabla u^n) \varphi \, dx \, dt = \int_{Q_T} \mu^n \varphi \, dx \, dt \]

\[ \forall \varphi \in L^p(0,T; W^{1,p}_0(\Omega)) \cap L^\infty(Q_T), \]

\[ b(u^n)(t = 0) = b(u^n_0) \text{ in } \Omega, \tag{4.4} \]

where \( v^n = b(u^n) \).

As a consequence, proving existence of a weak solution \( u^n \in L^p(0,T; W^{1,p}_0(\Omega)) \) of (4.3)-(4.4) is classical (see for instance [15]).

Now we give the following proposition which gives some compactness results.

**Proposition 4.1.** Let \( u^n \) and \( v^n \) be defined as before. Then

\[ \| |\nabla u^n|^p | \|_{L^{\frac{N+1}{N}}(Q_T)} \leq C, \tag{4.5} \]

\[ \| u^n \|_{L^\infty(0,T; L^1(\Omega))} \leq C, \tag{4.6} \]

\[ \int_Q |\nabla T_k(u^n)|^p \, dx \, dt \leq Ck, \tag{4.7} \]

\[ \int_Q |\nabla T_k(v^n)|^p \, dx \, dt \leq Ck, \tag{4.8} \]

\( u^n \) is bounded in \( L^q(0,T; W^{1,q}_0(\Omega)) \) \( \forall 1 < q < p - \frac{N}{N+1} \). \tag{4.9}

Moreover, there exists a measurable function \( u \) and \( v = b(u) \) such that \( T_k(u) \) and \( T_k(v) \) belong to \( L^p(0,T; W^{1,p}_0(\Omega)) \), and \( u \) belongs to \( L^\infty(0,T; L^1(\Omega)) \), up to a subsequence, for any \( k > 0 \) and for any \( 1 < q < p - \frac{N}{N+1} \) we have

\[ u^n \rightarrow u \text{ a.e. on } Q_T \text{ weakly in } L^q(0,T; W^{1,q}_0(\Omega)) \text{ and strongly in } L^1(Q_T), \tag{4.10} \]

\[ T_k(u^n) \rightarrow T_k(u) \text{ weakly in } L^p(0,T; W^{1,p}_0(\Omega)) \text{ and a.e. in } Q_T, \tag{4.11} \]

\[ T_k(v^n) \rightarrow T_k(v) \text{ weakly in } L^p(0,T; W^{1,p}_0(\Omega)) \text{ and a.e. in } Q_T. \tag{4.12} \]

**Proof.** The proof of this Proposition relies on standard techniques for problems of type (4.3)-(4.4). Let \( k > 0 \), we take \( T_k(u^n)^0 \chi(0,t) \) as test function in (4.3) for every \( t \in (0,T) \) and we have

\[ \int_\Omega \overline{B_k(u^n)}(t) \, dx + \int_{Q_T} a(x,t,\nabla u^n) \nabla T_k(u^n) \, dx \, dt \]

\[ \leq \int_{Q_T} |H(x,t,\nabla u^n)||T_k(u^n)| \, dx \, dt + \int_{Q_T} \mu^n T_k(u^n) \, dx \, dt + \int_\Omega \overline{\omega_k(u^n_0)} \, dx, \tag{4.13} \]
where $\overline{B}_k(s) = \int_0^s T_k(r)b'(r)\,dr$.

Using (3.4) and (3.7) we obtain

$$\int_\Omega \overline{B}_k(u^n)(t)\,dx + \alpha \int_{Q_{t_1}} |\nabla T_k(u^n)|^p\,dxdt \leq k\left(\int_{Q_{t_1}} |g(x,t)||\nabla u^n|^\delta\,dxdt + ||\mu^n||_{L^1(Q_T)} + ||b(u^n_0)||_{L^1(\Omega)}\right),$$

if we take the supremum for $t \in (0, t_1)$, where $t_1 \in (0, T)$ will be chosen later, by (3.2) we have

$$\frac{\gamma}{2} \sup_{t \in (0, t_1)} \int_\Omega |T_k(u^n)|^2\,dx + \alpha \int_{Q_{t_1}} |\nabla T_k(u^n)|^p\,dxdt \leq k\left(\int_{Q_{t_1}} |g(x,t)||\nabla u^n|^\delta\,dxdt + ||\mu^n||_{L^1(Q_T)} + ||b(u^n_0)||_{L^1(\Omega)}\right),$$

and thanks to the generalized Hölder inequality we obtain

$$\frac{\gamma}{2} \sup_{t \in (0, t_1)} \int_\Omega |T_k(u^n)|^2\,dx + \alpha \int_{Q_{t_1}} |\nabla T_k(u^n)|^p\,dxdt \leq k\left(\int_{Q_{t_1}} |g(x,t)||\nabla u^n|^\delta\,dxdt + ||\mu^n||_{L^1(Q_T)} + ||b(u^n_0)||_{L^1(\Omega)}\right),$$

(4.14)

where $M = \max_{Q_{t_1}} \|g\|_{L^{N+2,1}(Q_{t_1})} + \|\mu^n\|_{L^1(Q_T)} + \|b(u^n)\|_{L^1(\Omega)}$,

by Lemma 3.3 we obtain

$$\|\nabla u^n|^\delta\|_{L^{N^{-\frac{N}{N+2}}} (Q_{t_1})} = \|\nabla u^n|^{p-1}\|_{L^{\frac{(N+2)(N-1)}{N+2} (Q_{t_1})}} \leq C(||\nabla u^n|^\delta\|_{L^{N^{-\frac{N}{N+2}}} (Q_{t_1})} + \|g\|_{L^{N+2,1}(Q_{t_1})} + ||\mu^n||_{L^1(Q_T)} + ||b(u^n)\|_{L^1(\Omega)}),$$

(4.15)

If we choose $t_1$ such that

$$1 - C\|g\|_{L^{N+2,1}(Q_{t_1})} > 0,$$

(4.16)

holds, then we have

$$\|\nabla u^n|^\delta\|_{L^{N^{-\frac{N}{N+2}}} (Q_{t_1})} \leq C,$$

(4.17)

which yields (4.5).

Since $\overline{B}_k(s) \geq \gamma \int_0^s T_1(r)\,dr \geq \gamma(|s| - 1) \forall s \in \mathbb{R}$, we obtain

$$\|u^n\|^\infty_{L^\infty(0, t_1; L^1(\Omega))} \leq \frac{1}{\gamma} M + \text{meas}(\Omega).$$
From (4.17) it follows that
\[ \|u^n\|_{L^\infty(0,t_1;L^1(\Omega))} \leq C. \quad (4.18) \]

Now we use the same technique as in ([23]). We consider a partition of the interval
\([0,T]\) into a finite number of intervals \([0,t_1], [t_1,t_2], \ldots, [t_{n-1}, T]\) such that for each
\([t_{i-1}, t_i]\) the condition (4.16) holds.

In this way in each cylinder \(\Omega \times [t_{i-1}, t_i]\) we obtain a priori estimates of type (4.5)
and (4.6). From (4.14) and (4.17) with \(T\) in place of \(t_1\) we obtain (4.7).

By using (4.6) and (4.7), and thanks to L. Boccardo and T. Gallouët (see [3])
we obtain (4.9). By (3.2), (4.9), and since \(\mu^n\) is bounded in \(L^1(Q_T)\), one obtain
that \(\partial u^n/\partial t\) is bounded in \(L^1(0,T;W^{-1,q'}(\Omega))\) for every \(q' < 1 + p/(p-1)(N+1),\) using
a standard compactness arguments (see [26]) yield (4.10), (4.11) and (4.12).

Let us introduce for \(k \geq 0\) fixed, the time regularization of the function \(T_k(v)\).
This kind of regularization has been first introduced by R. Landes. More recently,
it has been exploited to solve a few nonlinear evolution problems with \(L^1\) or measure
data. This specific time regularization of \(T_k(v)\) (for fixed \(k \geq 0\)) is defined as follows.
Let \((v^n_\nu)_n\) in \(L^\infty(\Omega) \cap W^{1,p}(\Omega)\) such that \(\|v^n_\nu\|_{L^\infty(\Omega)} \leq k\), for all \(\nu > 0\), and
\(v^n_\nu \to T_k(b(u_0))\) a.e. in \(\Omega\) with \(\frac{\partial u^n}{\partial t}\|v^n_\nu\|_{L^p(\Omega)} \to 0\) as \(\nu \to +\infty\).

For fixed \(k \geq 0\) and \(\nu > 0\), let us consider the unique solution \(T_k(v)_\nu \in L^\infty(Q_T) \cap
L^p(0,T;W^{1,p}_1(\Omega))\) of the monotone problem:
\[ \frac{\partial T_k(v)_\nu}{\partial t} + \nu(T_k(v)_\nu - T_k(v)) = 0 \text{ in } \mathcal{D}'(Q_T), \]
\[ T_k(v)_\nu(t = 0) = v^n_\nu \text{ in } \Omega. \]

The behavior of \(T_k(v)_\nu\) as \(\nu \to +\infty\) is investigated in [17] and we just recall here that:
\[ T_k(v)_\nu \to T_k(v) \text{ strongly in } L^p(0,T;W^{1,p}_1(\Omega)) \text{ a.e. in } Q_T \text{ as } \nu \to +\infty \]
with \(\|T_k(v)_\nu\|_{L^\infty(\Omega)} \leq k\) for any \(\nu > 0\), and \(\frac{\partial T_k(v)_\nu}{\partial t} \in L^p(0,T;W^{1,p}_1(\Omega)).\)

We will denote \(\omega(n,\nu,k,\varepsilon)\) any quantity that vanishes as the parameters go to their
limit point with in the same order in which they appear, that is, for example
\[ \lim_{\varepsilon \to 0} \lim_{k \to \infty} \lim_{\nu \to \infty} \lim_{n \to \infty} |\omega(n,\nu,k,\varepsilon)| = 0. \]

We give the following result which has been proved in [2].

**Lemma 4.2.** Let \(v^n\) be a sequence in \(L^p(0,T;W^{1,p}_1(\Omega)) \cap C^0([0,T];L^2(\Omega)),\)
and \((v^n)_\epsilon \in L^p(0,T;W^{-1,q'}(\Omega)),\) suppose that \(v^n\) converges almost everywhere in \(Q_T\)
to a function \(v\) such that \(T_k(v) \in L^p(0,T;W^{1,p}_1(\Omega))\) for every \(k > 0\). then we have
\[ \int_0^T \left| \frac{\partial v^n}{\partial t}, T_k(v^n - T_k(v)_\nu) \right| dt \geq \omega(n,\nu,k,\varepsilon). \]
Proposition 4.3. The sequence \((\nabla u^n)\) converges to \(\nabla u\) a.e. in \(Q_T\).

Proof. Adopting the method used in \([2]\), we prove that for some \(\theta > 0\), one has up to subsequences still denoted by \(u^n\) (for simplicity of notation, we will omit the dependence of \(a\) on \(x\) and \(t\)),

\[
\left[ (a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u) \right]^\theta \to 0 \text{ a.e. in } Q. \tag{4.19}
\]

Note that (4.19) will be true if we show that

\[
\int_{Q_T} \left[ (a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u) \right] dxdt = \omega(n) \tag{4.20}
\]

The same argument in \([16]\) and under assumptions on \(a(x,t,\xi)\) implies that \(\nabla u^n\) converges to \(\nabla u\) a.e. in \(Q_T\). Thanks to Proposition 4.1, the following estimate holds

\[\text{meas}\{ |v| \geq k\} = \omega(k),\]

We can write

\[
\int_{Q_T} \left[ (a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u) \right] dxdt = \int_{\{|v|\geq k\}} \left[ (a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u) \right] dxdt + \int_{\{|v|<k\}} \left[ (a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u) \right] dxdt = I_{n,k} + J_{n,k}.
\]

Since \(u^n\) is bounded in \(L^q(0,T;W^{1,q}_0(\Omega))\) for \(q < p - \frac{N}{N+1}\), we can choose \(\theta < \frac{q}{p} < 1\), so that using Hölder inequality, we obtain

\[|I_{n,k}| \leq c \text{meas}\{ |v| \geq k\}^{1-\theta p/q},\]

and then \(I_{n,k} = \omega(k)\). Now we set

\[\Psi_{n,k} = \left( a(\nabla u^n) - a(\nabla u) \right)(\nabla u^n - \nabla u)\chi_{\{|v|<k\}},\]

and we have

\[
\int_{Q_T} \left[ a(\nabla u^n) - a(\nabla u) \right] dxdt \leq \int_{Q_T} \Psi_{n,k}^\theta \chi_{\{|v|\leq T_k(v)|\leq \epsilon\}} + \int_{Q_T} \Psi_{n,k}^\theta \chi_{\{|v|=T_k(v)|>\epsilon\}} + \omega(k), \tag{4.21}
\]

since \(\Psi_{n,k}^\theta\) is bounded in \(L^{q/\theta p}(Q_T)\) independently of \(n\) and \(k\), \(\chi_{\{|v|=T_k(v)|>\epsilon\}}\) converges to \(\chi_{\{|v|=T_k(v)|>\epsilon\}}\) almost everywhere in \(Q_T\) as \(n\) tends to \(+\infty\) (see \([2]\),
Lemma 3.2) and \( \chi_{\{|v-T_k(v)_\nu|>|\varepsilon|\}} \) converges to zero almost everywhere in \( Q_T \) as \( \nu \) and \( k \) tends to \( +\infty \) we obtain

\[
\int_{Q_T} \Psi_{\nu,k} \chi_{\{|v^n-T_k(v)_\nu|>|\varepsilon|\}} = \omega(n,\nu,k),
\]

using Hölder inequality, (4.21) becomes

\[
\int_{Q_T} \left[ \left( a(\nabla u^n) - a(\nabla u) \right) (\nabla u^n - \nabla u) \right] dx \leq \text{meas}(Q_T)^{1-\delta} \left( \int_{Q_T} \Psi_{\nu,k} \chi_{\{|v^n-T_k(v)_\nu|\leq \varepsilon\}} \right)^{\theta} + \omega(n,\nu,k).
\]

Then it remains to prove that

\[
\int_{Q_T} \Psi_{\nu,k} \chi_{\{|v^n-T_k(v)_\nu|\leq \varepsilon\}} = \omega(n,\nu,k,\varepsilon). \tag{4.22}
\]

By assumption (3.2) we can write

\[
\int_{Q_T} \Psi_{\nu,k} \chi_{\{|v^n-T_k(v)_\nu|\leq \varepsilon\}} \leq \frac{1}{\gamma} \left( \int_{Q_T} b'(u^n) a(\nabla u^n) \left( \nabla u^n - \nabla u \chi_{\{|v|\leq k\}} \right) \chi_{\{|v^n-T_k(v)_\nu|\leq \varepsilon\}} \right) - \frac{1}{\gamma} \left( \int_{Q_T} b'(u^n) a(\nabla u \chi_{\{|v|\leq k\}}) \left( \nabla u^n - \nabla u \chi_{\{|v|\leq k\}} \right) \chi_{\{|v^n-T_k(v)_\nu|\leq \varepsilon\}} \right) \tag{4.23}
\]

By Proposition 4.1 and since \(|T_k(v)_\nu| \leq k\) we obtain

\[
\int_{Q_T} b'(u^n) a(\nabla u \chi_{\{|v|\leq k\}}) \left( \nabla u^n - \nabla u \chi_{\{|v|\leq k\}} \right) \chi_{\{|v^n-T_k(v)_\nu|\leq \varepsilon\}} \tag{4.24}
\]

For \( \varepsilon < 1 \) and thanks to Proposition 4.1 we obtain

\[
A_1 = \int_{Q_T} a(\nabla u \chi_{\{|v|\leq k\}}) \nabla T_\varepsilon(v^n-T_k(v)_\nu)
\]
$$= \int_{Q_T} a(\nabla u_\{\{|v|\leq k\}|}) \nabla T_\varepsilon(T_{k+1}(v^n) - T_k(v))$$
$$= \int_{Q_T} a(\nabla u_\{\{|v|\leq k\}|}) \nabla T_\varepsilon(T_{k+1}(v) - T_k(v)) + \omega(n),$$

and the strong convergence of $\nabla T_k(v)$ to $\nabla T_k(v)$ in $(L^p(Q_T))^N$ leads to

$$A_1 = \int_{Q_T} a(\nabla u_\{\{|v|\leq k\}|}) \nabla T_\varepsilon(T_{k+1}(v) - T_k(v)) + \omega(n, v)$$
$$= \omega(n, v).$$

By Proposition 4.1 we have $b'(u^n)$ converges to $b'(u)$ almost everywhere in $Q_T$, since $a(\nabla u_\{\{|v|\leq k\}|})$ belongs to $(L^p(Q_T))^N$, $\nabla T_k(v)$ and $\nabla T_k(v)$ belong to $(L^p(Q_T))^N$, the Lebesgue’s convergence theorem leads to

$$|A_2| \leq \int_{Q_T} |a(\nabla u_\{\{|v|\leq k\}|})| \nabla T_\varepsilon(T_k(v)) - b'(u^n)b'(u)^{-1}\nabla T_k(v)|,$$
$$\leq \int_{Q_T} |a(\nabla u_\{\{|v|\leq k\}|})| \nabla T_\varepsilon(T_k(v)) - \nabla T_k(v)| + \omega(n),$$

and by the strong convergence of $\nabla T_k(v)$ to $\nabla T_k(v)$ in $(L^p(Q_T))^N$ we obtain

$$A_2 = \omega(n, v).$$

On the other hand we have

$$\int_{Q_T} b'(u^n) a(\nabla u^n) (\nabla u^n - \nabla u_\{\{|v|\leq k\}|}) \chi_{\{v^n - T_k(v)\leq \varepsilon\}}$$
$$= \int_{Q_T} a(\nabla u^n) \nabla (v^n - T_k(v)) \chi_{\{v^n - T_k(v)\leq \varepsilon\}}$$
$$+ \int_{Q_T} a(\nabla u^n) (\nabla T_k(v) - b'(u^n)b'(u)^{-1}\nabla T_k(v)) \chi_{\{v^n - T_k(v)\leq \varepsilon\}}.$$

We deal with the second term on the right side of (4.25), by assumption (3.1) it is clear that $\{|v^n| \leq k + \varepsilon\} \subset \{|u^n| \leq k = \max \{b^{-1}(k + \varepsilon), |b^{-1}(-k - \varepsilon)|\}$ and by H"older inequality we have

$$\left| \int_{Q_T} a(\nabla u^n) (\nabla T_k(v) - b'(u^n)b'(u)^{-1}\nabla T_k(v)) \chi_{\{v^n - T_k(v)\leq \varepsilon\}} \right|$$
$$\leq \|a(\nabla T_k(v))\|_{L^{p'}(Q_T)} \|\nabla T_k(v) - b'(u^n)b'(u)^{-1}\nabla T_k(v)\|_{L^p(Q_T)},$$

the almost everywhere convergence of $b'(u^n)$ to $b'(u)$ and Lebesgue’s convergence theorem imply that $b'(u^n)b'(u)^{-1}\nabla T_k(v)$ converges to $\nabla T_k(v)$ strongly in $(L^p(Q_T))^N$, since $\|a(\nabla T_k(v))\|_{L^{p'}(Q_T)}$ is bounded in $L^p(Q_T)$ we obtain

$$\left| \int_{Q_T} a(\nabla u^n) (\nabla T_k(v) - b'(u^n)b'(u)^{-1}\nabla T_k(v)) \chi_{\{v^n - T_k(v)\leq \varepsilon\}} \right|$$
\[ \leq C \| \nabla T_k(v)_\nu - \nabla T_k(v) \|_{L^p(Q_T)} + \omega(n), \]
and the strong convergence of \( \nabla T_k(v)_\nu \) to \( \nabla T_k(v) \) in \((L^p(Q_T))^N\) leads to
\[
\int_{Q_T} b'(u^n) a(\nabla u^n) \left( b'(u^n)^{-1} \nabla T_k(v)_\nu - b'(u)^{-1} \nabla T_k(v) \right) \chi_{\{|v^n - T_k(v)_\nu| \leq \epsilon\}} = \omega(n, \nu).
\]
Hence (4.23), (4.24) and (4.25) imply that
\[
\int_{Q_T} \Psi_{n,k} \chi_{\{|v^n - T_k(v)_\nu| \leq \epsilon\}} \leq \int_{Q_T} a(\nabla u^n) \nabla (v^n - T_k(v)_\nu) \chi_{\{|v^n - T_k(v)_\nu| \leq \epsilon\}} + \omega(n, \nu).
\]

Now we use the equation solved by \( u^n \). Taking \( T_\varepsilon(v^n - T_k(v)_\nu) \) in (4.3) we obtain
\[
\int_0^T \frac{\partial u^n}{\partial t} T_\varepsilon(v^n - T_k(v)_\nu) \, dt + \int_{Q_T} a(\nabla u^n) \nabla T_\varepsilon(v^n - T_k(v)_\nu) \, dxdt
\]
\[
+ \int_{Q_T} H(x,t,\nabla u^n) T_\varepsilon(v^n - T_k(v)_\nu) \, dxdt = \int_{Q_T} \mu^n T_\varepsilon(v^n - T_k(v)_\nu) \, dxdt.
\]
By property of \( \mu^n \) we have
\[
\left| \int_{Q_T} \mu^n T_\varepsilon(v^n - T_k(v)_\nu) \, dxdt \right| \leq \varepsilon \| \mu^n \|_{L^1(Q_T)} \leq \varepsilon \mu|M(Q_T)|.
\]
By generalized Hölder inequality we have
\[
\left| \int_{Q_T} H(x,t,\nabla u^n) T_\varepsilon(v^n - T_k(v)_\nu) \, dxdt \right| \leq \varepsilon \| g \|_{L^{N+2,1}(Q_T)} \| \nabla u^n |^\delta \|_{L^{\frac{N+2}{\delta}}(Q_T)}.
\]
By Lemma 4.2 we obtain
\[
\int_{Q_T} a(\nabla u^n) \nabla T_\varepsilon(v^n - T_k(v)_\nu) \, dxdt \leq \varepsilon \left( C \| g \|_{L^{N+2,1}(Q_T)} + |\mu|M(Q_T) \right).
\]
Hence
\[
\int_{Q_T} a(\nabla u^n) \nabla T_\varepsilon(v^n - T_k(v)_\nu) \, dxdt \leq \omega(n, \nu, \varepsilon).
\] (4.26)
Then by (4.26) we obtain (4.22) and therefore (4.20) and (4.19). \( \square \)

\textbf{Remark 4.4.} Let us observe that from Proposition 4.3 we have \( H(x,t,\nabla u^n) \) converges to \( H(x,t,\nabla u) \) a.e. in \( Q_T \) and by Proposition 4.1 \( H(x,t,\nabla u^n) \) is equi-integrable in \( L^1(Q_T) \). Indeed if \( E \) is a measurable set of \( Q_T \), due the growth assumption (3.7) on \( H \), estimate (4.5) yields that
\[
\int_E |H(x,t,\nabla u^n)| \, dxdt \leq \int_E g(x,t)|\nabla u^n|^\delta \, dxdt.
We conclude that $H(x,t,\nabla u^n)$ is equi-integrable in $L^1(Q_T)$. Then by Vitali’s theorem we deduce that $H(x,t,\nabla u^n)$ converges to $H(x,t,\nabla u)$ strongly in $L^1(Q_T)$.

Let also remark that from Proposition 4.3, assumption (3.5) on $a$ and Vitali’s theorem, we deduce that $a(x,t,\nabla u^n)$ is strongly compact in $L^1(Q_T)$.

Now we define the space $S$ by

$S = \{ z \in L^p(0,T;W^{1,p}_0(\Omega)), z_t \in L^p(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T) \},$

endowed with its natural norm $\| \cdot \|_{L^p(0,T;W^{1,p}_0(\Omega))} + \| \cdot \|_{L^p(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)}$ and its sub-space $W_1$ as

$W_1 = \{ z \in L^p(0,T;W^{1,p}_0(\Omega)) \cap L^\infty(Q_T), z_t \in L^p(0,T;W^{-1,p'}(\Omega)) + L^{1}(Q_T) \},$

endowed with its natural norm

$\| \cdot \|_{L^p(0,T;W^{1,p}_0(\Omega))} + \| \cdot \|_{L^\infty(Q_T)} + \| \cdot \|_{L^p(0,T;W^{-1,p'}(\Omega)) + L^{1}(Q_T)}$,

for any $p > 1$.

Let us recall that a function $z$ is called $\text{cap}_p$-quasi continuous if for every $\varepsilon > 0$ there exists an open set $F_\varepsilon$ with $\text{cap}(F_\varepsilon) \leq \varepsilon$ such that the restriction of $z$ to $Q_T \setminus F_\varepsilon$ is continuous. The following result shows that every functions in $W_1$ satisfy a capacitary estimate for the parabolic capacity.

**Theorem 4.5.** Let $z \in W_1$, then $z$ admits a unique $\text{cap}_p$-quasi continuous representative. Moreover, we have

$$\text{cap}_p(\{|z| \geq k\}) \leq C \frac{k^p}{k^p + \varepsilon},$$

where

$$[z] = \|z\|_{L^p(0,T;W^{1,p}_0(\Omega))} + \|z_t\|_{L^p(0,T;W^{-1,p'}(\Omega))} + \|z\|_{L^\infty(Q_T)} + \|z_t\|_{L^1(Q_T)} + \|z\|_{L^\infty(0,T;L^2(\Omega))},$$

such that $z_t^1 \in L^p(0,T;W^{-1,p'}(\Omega))$, $z_t^2 \in L^1(Q_T)$ is any decomposition of $z_t$, that is $z_t = z_t^1 + z_t^2$.

**Proof.** See [22], Theorem 3 and Lemma 2. 

Now we prove the following theorem

**Theorem 4.6.** Let $u^n \in L^p(0,T;W^{1,p}_0(\Omega))$ be a solution of Problem (4.3)-(4.4) then

$$\text{cap}_p(\{|u^n| \geq k\}) \leq C \frac{k^p}{k^p + \varepsilon}, \quad \forall k \geq 1$$
Proof. Due to the presence of the lower order term $H$, the approach used in [25] in the proof of Theorem 1.2 does not apply here, to overcome this difficulty we are going to exploit the method used in [22] Theorem 4. Let us first introduce the following function

$$G_k(s) = \begin{cases} 
1 & \text{if } |s| \leq k, \\
k + 1 - |s| & \text{if } k < |s| \leq k + 1, \\
0 & \text{if } |s| > k + 1.
\end{cases}$$

let us denote by $G_k'(s)$ the primitive function of $G_k(s)$. Since we have

$$\int_{Q_T} \left| \nabla T_k(v^n) \right|^p \, dx \, dt \leq C_k,$$

we obtain

$$\int_{Q_T} \left| \nabla G_k(v^n) \right|^p \, dx \, dt \leq C_k. \quad (4.27)$$

Given $\varphi \in C_0^\infty(Q_T)$ and taking $G_k(v^n)\varphi$ as test function in (4.3) we have in the sense of distribution

$$\nabla G_k(v^n) = \text{div} \left( G_k(v^n)a(x,t,\nabla u^n) \right)$$

$$-b'(u^n)a(x,t,\nabla u^n).\nabla u^n \chi_{k < |v^n| \leq k + 1} + b'(u^n)a(x,t,\nabla u^n).\nabla u^n \chi_{k - 1 < |v^n| \leq -k}$$

$$-H(x,t,\nabla u^n)G_k(v^n) + G_k(v^n)\mu_n,$$

therefore by assumption (3.2) and Proposition 4.1, we have

$$\overline{G}_k(v^n) \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q),$$

and

$$\overline{G}_k(v^n) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T),$$

thanks to Theorem 4.5, $\overline{G}_k(v^n)$ has a $\text{cap}_p$-quasicontinuous representative. To conclude the proof is enough to prove the capacitary estimate of $v_n$.

Since $\{|v^n| > k\} = \{\overline{G}_k(v^n) > k\}$, by Theorem 4.5 we obtain

$$\text{cap}_p(\{|v^n| > k\}) \leq \frac{C}{k} \max\{\overline{G}_k(v^n)\}^{\frac{1}{p'}}.$$

Taking $\theta_k(v^n) = T_{k+1}(v^n) - T_k(v^n)$ as test function in (4.3) leads to

$$\int_\Omega \Theta_k(v^n)(T) \, dx + \int_{\{k < |v^n| \leq k + 1\}} b'(u^n)a(x,t,\nabla u^n).\nabla u^n \, dx \, dt$$

$$+ \int_{Q_T} H(x,t,\nabla u^n)\theta_k(v^n) \, dx \, dt = \int_{Q_T} \theta_k(v^n)\mu_n \, dx \, dt + \int_\Omega \Theta_k(b(u^n)) \, dx,$$
where \( \Theta_k(s) = \int_0^s \theta_k(r) \, dr \quad \forall s \in \mathbb{R} \).

Since \( \| \theta_k(v^n) \|_{L^\infty(Q_T)} \leq 1 \) and \( H(x, t, \nabla u^n) \) is strongly compact in \( L^1(Q) \) one obtains

\[
\int_{Q_T} b'(u^n) a(x, t, \nabla u^n) \nabla u^n \chi_{\{k \leq v^n < k+1\}} \, dx \, dt \leq C,
\]

\[
\int_{Q_T} b'(u^n) a(x, t, \nabla u^n) \nabla u^n \chi_{\{-k-1 < v^n \leq -k\}} \, dx \, dt \leq C,
\]

\[
\int_{Q_T} |H(x, t, \nabla u^n)| G_k(v^n) \, dx \, dt \leq C.
\]

Then, from (4.28) it follows that

\[
\| G_k(v^n) \|_{p'_{L^p(0, T; W^{-1, p'}(\Omega))}} \leq C k,
\]

\[
\| G_k(v^n) \|_{L^1(Q)} \leq C,
\]

using the following estimate

\[
\| G_k(v^n) \|_{L^\infty(0, T; L^2(\Omega))} \leq \| G_k(v^n) \|_{L^\infty(Q_T)} \| G_k(v^n) \|_{L^\infty(0, T; L^1(\Omega))},
\]

since \( v^n \) is bounded in \( L^\infty(0, T; L^1(\Omega)) \), we conclude that

\[
cap_{p'}(|v^n| > k) \leq \frac{C}{k} \max\{k^\frac{1}{p'}, k^\frac{1}{p}\}.
\]

\[\Box\]

We have the following technical result

**Lemma 4.7.** Let \( \mu_+ = \mu_+^s - \mu_+^s \in \mathcal{M}(Q_T) \) where \( \mu_+^s \) and \( \mu_+^s \) are concentrated respectively, on two disjoint \( E^+ \) and \( E^- \) of zero p-capacity. Then, for every \( \delta > 0 \), there exist two compact sets \( K_+^\delta \subseteq E^+ \) and \( K^-_\delta \subseteq E^- \) such that

\[
\mu_+^s(E^+ \setminus K_+^\delta) \leq \delta, \quad \mu_+^s(E^+ \setminus K^-_\delta) \leq \delta,
\]

and there exist \( \psi_+^\delta, \psi_-^\delta \in C_c^\infty(Q_T) \), such that

\[
\psi_+^\delta \equiv 1 \text{ and } \psi_-^\delta \equiv 1 \text{ respectively on } K_+^\delta \text{ and } K^-_\delta,
\]

\[
0 \leq \psi_+^\delta, \quad \psi_-^\delta \leq 1,
\]

\[
\text{supp}(\psi_+^\delta) \cap \text{supp}(\psi_-^\delta) \equiv \emptyset.
\]

Moreover

\[
\| \psi_+^\delta \|_{S} \leq \delta, \quad \| \psi_-^\delta \|_{S} \leq \delta,
\]

(4.33)
and in particular, there exists a decomposition of \((\psi^\pm_t)\) and a decomposition of \((\psi^-_t)\) such that

\[
\|\psi^+_t\|_{L^p(0,T;W^{-1,p}(\Omega))} \leq \frac{\delta}{3}, \quad \|\psi^-_t\|_{L^1(Q_T)} \leq \frac{\delta}{3},
\]

Both \(\psi^+_t\) and \(\psi^-_t\) converges to zero \(\ast\)-weakly in \(L^\infty(Q_T)\), in \(L^1(Q_T)\), and up to subsequences, almost everywhere as \(\delta\) vanishes. Moreover, if \(\lambda^+_n\) and \(\lambda^-_n\) are as in (4.2) we have

\begin{align*}
\int_{Q_T} \psi^-_t d\lambda^+_n &= \omega(n,\delta), \quad \int_{Q_T} \psi^+_t d\mu^+ \leq \delta, \\
\int_{Q_T} \psi^+_t d\lambda^-_n &= \omega(n,\delta), \quad \int_{Q_T} \psi^-_t d\mu^- \leq \delta, \\
\int_{Q_T} (1 - \psi^+_t) d\lambda^+_n &= \omega(n,\delta), \quad \int_{Q_T} (1 - \psi^-_t) d\mu^+ \leq \delta, \\
\int_{Q_T} (1 - \psi^-_t) d\lambda^-_n &= \omega(n,\delta), \quad \int_{Q_T} (1 - \psi^+_t) d\mu^- \leq \delta.
\end{align*}

\(\text{Proof.} \) See [22], Lemma 5. \(\square\)

Now we prove the following theorem

**Theorem 4.8.** Under assumptions (3.1)-(3.8), there exists at least a renormalized solution \(u\) of Problem (1.1)-(1.3).

Let us fix \(\sigma > 0\) and define

\[
S_{k,\sigma}(s) = \begin{cases} 
1 & \text{if } |s| \leq k, \\
0 & \text{if } |s| > k + \sigma, \\
\text{affine} & \text{otherwise.}
\end{cases}
\]

**Proof.** **Step 1.** Estimates in \(L^1(Q_T)\) on the energy term. Using \(h_{k,\sigma}(u^n) = \frac{1}{\sigma}(T_{k+\sigma}(u^n) - T_k(u^n))\) as test function in (4.3) we obtain

\[
\int_{Q_T} B^*_{k,\sigma}(u^n)(T) \, dx + \frac{1}{\sigma} \int \{k < |u^n| \leq k+\sigma\} a(x,t,\nabla u^n) \nabla u^n \, dx dt \\
+ \int_{Q_T} H(x,t,\nabla u^n) h_{k,\sigma}(u^n) \, dx dt = \int_{Q_T} h_{k,\sigma}(u^n) \mu^n \, dx dt + \int_{Q_T} B^*_{k,\sigma}(u^n_0) \, dx,
\]

where \(B^*_{k,\sigma}(s) = \int_0^s b'(r) h_{k,\sigma}(r) \, dr \forall s \in \mathbb{R}\).

So that dropping positive terms

\[
\frac{1}{\sigma} \int \{k < |u^n| \leq k+\sigma\} a(x,t,\nabla u^n) \nabla u^n \, dx dt
\]
\[
\leq \int_{\{|u^n|>k\}} |\mu^n| \, dx \, dt + \int_{\{|u^n|>k\}} |H(x,t,\nabla u^n)| \, dx \, dt + \int_{\{|u^n_0|>k\}} b(u^n_0) \, dx.
\]
which implies, in particular,
\[
\frac{1}{\sigma} \int_{\{|k|\leq k+\sigma\}} a(x,t,\nabla u^n).\nabla u^n \, dx \, dt \leq C. \tag{4.41}
\]

**Step 2.** Equation for the truncations. Given \( \varphi \in C_c^\infty(Q_T) \), taking \( S_{k,\sigma}(u^n)\varphi \) as test function in (4.3), we obtain
\[
B_{S_{k,\sigma}}(u^n)_t - \text{div}\left( S_{k,\sigma}(u^n)a(x,t,\nabla u^n) + H(x,t,\nabla u^n)S_{k,\sigma}(u^n) \right) + \mu^n_t + \mu^n_a S_{k,\sigma}(u^n) + \frac{1}{\sigma} \text{sign}(u^n)a(x,t,\nabla u^n).\nabla u^n \chi_{\{k|\leq|u^n|\leq k+\sigma\}} + \mu^n_a S_{k,\sigma}(u^n) - 1 \in \mathcal{D}'(Q_T), \tag{4.42}
\]
where \( B_{S_{k,\sigma}}^*(s) = \int_0^s b'(r)S_{k,\sigma}(r) \, dr \).

From (4.41), there exists a bounded Radon measure \( \zeta^n_k \) such that, as \( \sigma \) goes to zero
\[
\frac{1}{\sigma} \text{sign}(u^n)a(x,t,\nabla u^n).\nabla u^n \chi_{\{k|\leq|u^n|\leq k+\sigma\}} \to \zeta^n_k \ast \text{weakly in } \mathcal{M}(Q_T).
\]

Taking the limit as \( \sigma \) vanishes in (4.42) it follows that
\[
B_k(u^n)_t - \text{div}\left( a(x,t,\nabla T_k(u^n)) + H(x,t,\nabla T_k(u^n)) \right) + \mu^n_t + \mu^n_a \chi_{\{|u^n|\leq k\}} + \zeta^n_k - \mu^n_a \chi_{\{|u^n|\geq k\}} \in \mathcal{D}'(Q_T),
\]
where \( B_k(s) = \int_0^s T_k'(r)b'(r) \, dr, \forall s \in \mathbb{R} \).

We define the measure \( \Gamma^n_k \) as
\[
\Gamma^n_k = \mu^n_a \chi_{\{|u^n|\leq k\}} + \zeta^n_k - \mu^n_a \chi_{\{|u^n|\geq k\}}.
\]

Notice that
\[
||\Gamma^n_k||_{L^1(Q_T)} \leq C,
\]
so that there exist \( \Gamma^k \in \mathcal{M}(Q) \) such that
\[
\Gamma^n_k \rightharpoonup \Gamma^k \ast \text{weakly in } \mathcal{M}(Q_T).
\]

Therefore, using Proposition 4.1 and Proposition 4.3, in the sense of distribution we have
\[
B_k(u)_t - \text{div}\left( a(x,t,\nabla T_k(u)) + H(x,t,\nabla T_k(u^n)) \right) = \mu_d + \Gamma^k \in \mathcal{D}'(Q_T). \tag{4.43}
\]
Step 3. The limit of $\Gamma^k$. By subtracting (4.43) from the distributional formulation of (4.3) we obtain for any $\varphi \in C_0^\infty(Q_T)$

$$
\int_{Q_T} (v^n - B_k(u)) \varphi_t \, dx \, dt + \int_{Q_T} (a(x, t, \nabla u^n) - a(x, t, \nabla T_k(u)) \nabla \varphi \, dx \, dt (4.44)

+ \int_{Q_T} (H(x, t, \nabla u^n) - H(x, t, \nabla T_k(u))) \varphi \, dx \, dt

= \int_{Q_T} (\mu^n_s - \mu_s) \varphi \, dx \, dt + \int_{Q_T} (\mu^n_s - \Gamma^k) \varphi \, dx \, dt.
$$

Using Proposition 4.1 and Proposition 4.3 we obtain from (4.44) in the sense of distribution

$$
\Gamma^k = \mu_s + \omega(n, k) \text{ in } \mathcal{D}'(Q_T).
$$

To complete the proof we have to show that the previous limit is actually tight. Let us choose without loss of generality $\varphi \in C^1(Q_T)$ (then by density argument we show the result holds with $\varphi \in C(Q_T)$). We have

$$
\int_{Q_T} \Gamma^k \varphi \, dx \, dt = \int_{Q_T} \Gamma^k \Psi_\delta \varphi \, dx \, dt + \int_{Q_T} \Gamma^k (1 - \Psi_\delta) \varphi \, dx \, dt,
$$

where $\Psi_\delta = \psi^+_\delta + \psi^-_\delta$ is chosen as in Lemma 4.7. Thanks to the previous result we can write

$$
\int_{Q_T} \Gamma^k \Psi_\delta \varphi \, dx \, dt = \int_{Q_T} \mu^+_s \Psi_\delta \varphi \, dx \, dt - \int_{Q_T} \mu^-_s \Psi_\delta \varphi \, dx \, dt + \omega(n, k),
$$

we have

$$
\int_{Q_T} \mu^+_s \Psi_\delta \varphi \, dx \, dt = \int_{K^+_s} \mu^+_s \psi^+_\delta \varphi \, dx \, dt + \int_{E^+ \setminus K^+_s} \mu^+_s \psi^+_\delta \varphi \, dx \, dt + \int_{Q_T} \mu^+_s \psi^-_\delta \varphi \, dx \, dt,
$$

since $\psi^+_\delta = 1$ on $K^+_s$ by Lebesgue’s theorem we have

$$
\int_{Q_T} \mu^+_s \Psi_\delta \varphi \, dx \, dt = \int_{Q_T} \mu^+_s \varphi \, dx \, dt + \omega(\delta)
$$

by Lemma 4.7 we obtain

$$
\left| \int_{E^+ \setminus K^+_s} \mu^+_s \psi^+_\delta \varphi \, dx \, dt \right| \leq \delta \| \varphi \|_{L^\infty(Q_T)},
$$

and

$$
\left| \int_{Q_T} \mu^+_s \psi^-_\delta \varphi \, dx \, dt \right| \leq \| \varphi \|_{L^\infty(Q_T)} \int_{Q_T} \psi^-_\delta \varphi \, d\mu^+_s = \omega(\delta).
$$

Then we obtain

$$
\int_{Q_T} \mu^+_s \Psi_\delta \varphi \, dx \, dt = \int_{Q_T} \mu^+_s \varphi \, dx \, dt + \omega(\delta).
$$
Similarly we obtain
\[ \int_{Q_T} \mu_s \Psi_s \varphi \, dx \, dt = \int_{Q_T} \mu_s \varphi \, dx \, dt + \omega(\delta). \]
Hence
\[ \int_{Q_T} \Gamma^k \Psi_s \varphi \, dx \, dt = \int_{Q_T} \varphi \, d\mu_s + \omega(k, \delta). \]
To conclude we have to prove that
\[ \int_{Q_T} \Gamma^k (1 - \Psi_s) \varphi \, dx \, dt = \omega(k, \delta). \]
From the definition of \( \Gamma^k \) we have
\[
\int_{Q_T} (1 - \Psi_s) \varphi \, d\Gamma^k = \lim_{n} \left( \lim_{\sigma} \frac{1}{\sigma} \int_{\{k < |u^n| \leq k + \sigma\}} \text{sign}(u^n) a(x, t, \nabla u^n) \nabla u^n (1 - \Psi_s) \varphi \right)
+ \int_{\{|u^n| \leq k\}} (1 - \Psi_s) \varphi \, d\mu_s^n - \int_{\{|u^n| > k\}} (1 - \Psi_s) \varphi \, d\mu_d^n.
\]
By Proposition 2.3 the sequence \( \mu_d^n \) is equi-diffuse, thanks to assumption \((3.2)\) and Theorem 4.6 we deduce that
\[
\left| \int_{\{|u^n| > k\}} (1 - \Psi_s) \varphi \, d\mu_d^n \right| \leq \| \varphi \|_{L^\infty(Q_T)} \int_{\{|u^n| > k\}} |\mu_d^n| \, dx \, dt = \omega(n, k).
\]
We have
\[
\int_{\{|u^n| \leq k\}} (1 - \Psi_s) \varphi \, d\mu_s^n = \int_{\{|u^n| \leq k\}} (1 - \Psi_s) \varphi \, d\lambda^+_n - \int_{\{|u^n| \leq k\}} (1 - \Psi_s) \varphi \, d\lambda^-_n,
\]
and
\[
\int_{\{|u^n| \leq k\}} (1 - \Psi_s) \varphi \, d\lambda^+_n = \int_{\{|u^n| \leq k\}} (1 - \psi^+_\delta) \varphi \, d\lambda^+_n - \int_{\{|u^n| \leq k\}} \psi^-_\delta \varphi \, d\lambda^+_n,
\]
Thanks to Lemma 4.7 we obtain
\[
\left| \int_{\{|u^n| \leq k\}} (1 - \Psi_s) \varphi \, d\lambda^+_n \right| \leq \| \varphi \|_{L^\infty(Q_T)} \left( \int_{Q_T} (1 - \psi^+_\delta) \, d\lambda^+_n + \int_{Q_T} \psi^-_\delta \, d\lambda^+_n \right) = \omega(n, \delta).
\]
Similarly we obtain
\[
\int_{\{|u^n| \leq k\}} (1 - \Psi_s) \varphi \, d\lambda^-_n = \omega(n, \delta),
\]
and then
\[
\int_{\{|u^n| \leq k\}} (1 - \Psi_s) \varphi \, d\mu^n_s = \omega(n, \delta).
\]
It remains to prove that
\[
\int_{\{k<|u^n|\leq k+\sigma\}} \frac{1}{\sigma} \text{sign}(u^n) a(x,t,\nabla u^n) \cdot \nabla u^n (1 - \Psi_\delta) \varphi \, dx \, dt = \omega(\sigma, n, k, \delta).
\]
we use \(h_{k,\sigma}(u^n) (1-\Psi_\delta)\) as test function in (4.3) we obtain
\[
\int_{Q_T} B^*_h(u^n)(\Psi_\delta) + \int_{\Omega} B_h(u^n)(T) - \int_{\Omega} B^*_{h_k,\sigma}(u_0^n)(1 - \Psi_\delta(0)) \tag{4.45}
\]
\[
+ \frac{1}{\sigma} \int_{\{k<|u^n|\leq k+\sigma\}} a(x,t,\nabla u^n) \cdot \nabla u^n (1 - \Psi_\delta) = \int_{Q_T} a(x,t,\nabla u^n) \nabla \Psi_\delta h_{k,\sigma}(u^n)
\]
\[
+ \int_{Q_T} B\star h(u^n)(1 - \Psi_\delta(h_{k,\sigma}(u^n))(1 - \Psi_\delta)
\]
\[
= \int_{Q_T} h_{k,\sigma}(u^n)(1 - \Psi_\delta) \mu_d^n + \int_{Q_T} h_{k,\sigma}(u^n)(1 - \Psi_\delta) \mu_s^n.
\]
Using assumption (3.2), the convergence in \(L^1(Q_T)\) of \(a(x,t,\nabla u^n), H(x,t,\nabla u^n)\) and the regularity of \(\Psi_\delta\) we obtain
\[
\int_{\Omega} B^*_{h_k,\sigma}(u^n)(T) = \omega(\sigma, n, k),
\]
\[
\int_{Q_T} B^*_{h_k,\sigma}(u^n)(\Psi_\delta) = \omega(\sigma, n, k),
\]
\[
\int_{\Omega} B^*_{h_k,\sigma}(u_0^n)(1 - \Psi_\delta(0)) = \omega(\sigma, n, k),
\]
\[
\int_{Q_T} a(x,t,\nabla u^n) \nabla \Psi_\delta h_{k,\sigma}(u^n) = \omega(\sigma, n, k),
\]
\[
\int_{Q_T} H(x,t,\nabla u^n) h_{k,\sigma}(u^n)(1 - \Psi_\delta) = \omega(\sigma, n, k).
\]
Thanks to Theorem 4.6 and equi-diffuse property of \(\mu_d^n\)
\[
\int_{Q_T} h_{k,\sigma}(u^n)(1 - \Psi_\delta) \mu_d^n = \omega(\sigma, n, k),
\]
finally by Lemma 4.7 we have
\[
\int_{Q_T} h_{k,\sigma}(u^n)(1 - \Psi_\delta) \mu_s^n = \omega(\sigma, n, \delta).
\]
Hence we conclude that \(u\) is a renormalized solution of Problem (1.1)-(1.3). \(\square\)
References


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