

\textbf{\textit{g}Δ^*_\mu – Closed Sets in Generalized Topological Spaces}

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ABSTRACT: In this paper, we introduce some new classes of generalized closed sets called \(\Delta^*_\mu – g\)-closed, \(\Delta^*_\mu – g\),\(\mu\)-closed and \(g\Delta^*_\mu – g\)-closed sets, which are related to the classes of \(g\mu\)-closed sets, \(g\lambda\mu\)-closed sets and \(\lambda\mu – g\)-closed sets. We investigate their properties as well as the relations among these classes of generalized closed sets.

Key Words: Generalized topology, \(\lambda\mu\)-closed, \(\Delta^*_\mu\)-closed, \(\Delta^*_\mu – g\)-closed, \(\Delta^*_\mu – g\mu\)-closed, \(g\Delta^*_\mu\)-closed sets.

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\section{1. Introduction}

In 1997, \'{A}.Császár \cite{2} introduced the concept of a generalization of topological spaces, which is called a generalized topological space. A subset \(\mu\) of \(\text{exp}(X)\) is called a generalized topology \cite{4} on \(X\) if \(\emptyset \in \mu\) and \(\mu\) is closed under arbitrary union. Elements of \(\mu\) are called \(\mu\)-open sets. The complement of a \(\mu\)-open set is said to be \(\mu\)-closed. A set \(X\) with a GT \(\mu\) on it is called a generalized topological space (briefly GTS) and is denoted by \((X, \mu)\).

For a subset \(A\) of \(X\), we denote by \(c_{\mu}(A)\) the intersection of all \(\mu\)-closed sets containing \(A\) and by \(i_{\mu}(A)\) the union of all \(\mu\)-open sets contained in \(A\). Then \(c_{\mu}(A)\) is the smallest \(\mu\)-closed set containing \(A\) and \(i_{\mu}(A)\) is the largest \(\mu\)-open set contained in \(A\). A point \(x \in X\) is called a \(\mu\)-cluster point of \(A\) if for every \(U \in \mu\) with \(x \in U\) we have \(A \cap U \neq \emptyset\). \(c_{\mu}(A)\) is the set of all \(\mu\)-cluster points of \(A\) \cite{4}. A GTS \((X, \mu)\) is called a quasi-topological space \cite{3} if \(\mu\) is closed under finite intersections. A subset \(A\) of \(X\) is said to be \(\pi\)-regular \cite{5} (resp. \(\sigma\)-regular) if \(A = i_{\mu}c_{\mu}(A)\) (resp. \(A = c_{\mu}i_{\mu}(A)\)).

\textbf{Definition 1.1.} \cite{6} If \((X, \mu)\) is a GTS and \(A \subseteq X\), then the set \(\wedge_{\mu}(A)\) is defined as follows:

\[\wedge_{\mu}(A) = \begin{cases} \cap\{G : A \subseteq G, G \in \mu\} & \text{if there exists } G \in \mu \text{ such that } A \subseteq G; \\ X & \text{otherwise.} \end{cases}\]

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Definition 1.2. [6] In a GTS $(X, \mu)$, a subset $B$ is called a $\wedge_{\mu}$-set if $B = \wedge_{\mu}(B)$.

Definition 1.3. [1] A subset $A$ of a GTS $(X, \mu)$ is called a $\lambda_{\mu}$-closed set if $A = T \cap C$, where $T$ is a $\Lambda_{\mu}$-set and $C$ is a $\mu$-closed set. The complement of a $\lambda_{\mu}$-closed set is called a $\lambda_{\mu}$-open set. We set $\lambda_{\mu}O(X, \mu) = \{U : U$ is $\lambda_{\mu}$-open in $(X, \mu)\}$.

Definition 1.4. [10] Let $(X, \mu)$ be a GTS. A subset $A$ of $X$ is called a $^*\wedge_{\mu}$-set if $A = \ ^*\wedge_{\mu}(A)$, where $^*\wedge_{\mu}(A) = \cap\{U : A \subseteq U, U \in \lambda_{\mu}O(X, \mu)\}$.

Definition 1.5. [9] Let $(X, \mu)$ be a GTS. A subset $A$ of $X$ is called a $\Delta_{\mu}$-set if $\Lambda_{\mu}(A) = ^*\Lambda_{\mu}(A)$.

Definition 1.6. [9] A subset of a GTS $(X, \mu)$ is called a $\Delta^*_{\mu}$-closed set if $A = T \cap F$, where $T$ is a $\Delta_{\mu}$-set and $F$ is a $\mu$-closed set. The complement of a $\Delta^*_{\mu}$-closed set is said to be $\Delta^*_{\mu}$-open.

Definition 1.7. A subset $A$ of GTS $(X, \mu)$ is said to be $g_{\mu}$-closed [11] (resp. $g - \lambda_{\mu}$-closed [8], $\lambda_{\mu} - g$-closed [8]) if $c_{\mu}(A) \subseteq U$ (resp. $c_{\lambda_{\mu}}(A) \subseteq U$, $c_{\mu}(A) \subseteq U$) whenever $A \subseteq U$ and $U$ is $\mu$-open (resp. $U$ is $\mu$-open, $U$ is $\lambda_{\mu}$-open) in $(X, \mu)$.

Lemma 1.1. [7] For a GTS $(X, \mu)$ and $S, T \subseteq X$, the following properties hold:
(i) $i_{\mu}(S \cap T) \subseteq i_{\mu}(S) \cap i_{\mu}(T)$.
(ii) $c_{\mu}(S) \cup c_{\mu}(T) \subseteq c_{\mu}(S \cup T)$.

Remark 1.8. [7] In general, for subsets $S$ and $T$ of a GTS $(X, \mu)$, $i_{\mu}(S \cap T) \supseteq i_{\mu}(S) \cap i_{\mu}(T)$ is not true.

Lemma 1.2. [5] Let $(X, \mu)$ be a quasi-topological space. Then $c_{\mu}(A \cup B) = c_{\mu}(A) \cup c_{\mu}(B)$ for every $A$ and $B$ of $X$.

Lemma 1.3. [1,6,9] For a subset of a GTS $(X, \mu)$, the following implication hold:

$\mu$-open $\Rightarrow$ $\Lambda_{\mu}$-set $\Rightarrow$ $\Delta_{\mu}$-set $\Downarrow$

$\mu$-closed $\Rightarrow$ $\lambda_{\mu}$-closed $\Rightarrow$ $\Delta^*_{\mu}$-closed $\Downarrow$

For $A \subseteq X$, we denote by $c_{\Delta^*_{\mu}}(A)$ [9] (resp. $c_{\lambda_{\mu}}(A)$ [1]) the intersection of all $\Delta^*_{\mu}$-closed (resp. $\lambda_{\mu}$-closed) subsets of $X$ containing $A$. Then we have

$c_{\Delta^*_{\mu}}(A) \subseteq c_{\lambda_{\mu}}(A) \subseteq c_{\mu}(A)$

for every $A \subseteq X$.

The purpose of this present paper is to define some new classes of generalized closed sets called $\Delta^*_{\mu} - g$-closed, $\Delta^*_{\mu} - g_{\mu}$-closed and $g_{\Delta^*_{\mu}}$-closed and to obtain some basic properties of these closed sets. Further, we establish the relation between these classes of sets.
2. $\Delta^*_\mu - g-$closed sets

In this section, we introduce the notion of $\Delta^*_\mu - g-$closed sets and discuss its properties.

**Definition 2.1.** Let $(X, \mu)$ be a GTS. A subset $A$ of $X$ is called a $\Delta^*_\mu - g-$closed set if $c_{\mu}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $\Delta^*_\mu-$open set in $X$. The complement of a $\Delta^*_\mu - g-$closed set is called a $\Delta^*_\mu - g-$open set.

**Theorem 2.2.** Every $\mu-$closed set is a $\Delta^*_\mu - g-$closed set.

**Proof:** Let $A$ be a $\mu-$closed set and $U$ be any $\Delta^*_\mu-$open set containing $A$. Since $A$ is $\mu-$closed, we have $c_{\mu}(A) = A$. Therefore $c_{\mu}(A) \subseteq U$. Thus $A$ is $\Delta^*_\mu - g-$closed.

Example 2.3. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\{c\}$ is $\Delta^*_\mu - g-$closed but not $\mu-$closed.

Theorem 2.4 shows that every $\Delta^*_\mu - g-$closed set is a $g_\mu-$closed set (a $g - \lambda_\mu-$closed set, a $\lambda_\mu - g-$closed set) and Example 2.5 shows that converses are not true.

**Theorem 2.4.** Let $(X, \mu)$ be a GTS. Then the following hold:

(i) Every $\Delta^*_\mu - g-$closed set is a $g_\mu-$closed set.

(ii) Every $g_\mu-$closed set is a $g - \lambda_\mu-$closed set.

(iii) Every $\Delta^*_\mu - g-$closed set is a $\lambda_\mu - g-$closed set.

(iv) Every $\lambda_\mu - g-$closed set is $g - \lambda_\mu-$closed.

**Proof:**

(i) Let $A$ be a $\Delta^*_\mu - g-$closed set and $U$ be any $\mu-$open set containing $A$ in $(X, \mu)$. Since every $\mu-$open set is $\Delta^*_\mu-$open, we have $U$ is $\Delta^*_\mu-$open. Since $A$ is $\Delta^*_\mu - g-$closed, $c_{\mu}(A) \subseteq U$. Therefore $A$ is $g_\mu-$closed.

(ii) Let $A$ be a $g_\mu-$closed set and $U$ be any $\mu-$open set containing $A$ in $(X, \mu)$. Since $A$ is $g_\mu-$closed, $c_{\mu}(A) \subseteq U$. Since $c_{\lambda_\mu}(A) \subseteq c_{\mu}(A)$, $c_{\lambda_\mu}(A) \subseteq U$ and hence $A$ is $g - \lambda_\mu-$closed.

(iii) Let $A$ be a $\Delta^*_\mu - g-$closed set and $U$ be a $\lambda_\mu-$open set containing $A$ in $(X, \mu)$. Since every $\lambda_\mu-$open set is $\Delta^*_\mu-$open and $A$ is $\Delta^*_\mu - g-$closed, then $c_{\mu}(A) \subseteq U$. Therefore $A$ is $\lambda_\mu - g-$closed.

(iv) Suppose that $A$ is a $\lambda_\mu - g-$closed set. Let $A \subseteq U$ and $U$ be $\mu-$open. Then $U$ is $\lambda_\mu-$open and $A$ is $\lambda_\mu - g-$closed. Therefore, $c_{\mu}(A) \subseteq U$ and hence $c_{\lambda_\mu}(A) \subseteq c_{\mu}(A) \subseteq U$. Hence $A$ is $g - \lambda_\mu-$closed.

Form Theorem 2.4, we have the following diagram:

\[
\begin{array}{c}
\Delta^*_\mu - g-$closed \\
\downarrow \\
\lambda_\mu - g-$closed
\end{array}
\Rightarrow
\begin{array}{c}
g_\mu-$closed \\
\downarrow \\
g - \lambda_\mu-$closed
\end{array}\]
Example 2.5. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a, b\}, \{b, c\}, X\}$. Then $\{a, c\}$ is both $g_\mu$-closed and $\lambda_\mu - g$-closed but not $\Delta^*_\mu - g$-closed. Further $\{b, c\}$ is $g - \lambda_\mu$-closed but neither $\lambda_\mu - g$-closed nor $g_\mu$-closed.

Theorem 2.6 gives a characterization of $\Delta^*_\mu - g$-closed sets.

Theorem 2.6. Let $(X, \mu)$ be a GTS. A subset $A$ of $X$ is a $\Delta^*_\mu - g$-closed set if and only if $F \subseteq c_\mu(A) \setminus A$ and $F$ is $\Delta^*_\mu$-closed implies that $F$ is empty.

Proof: Let $A$ be $\Delta^*_\mu - g$-closed. Suppose that $F$ is a subset of $c_\mu(A) \setminus A$ and $F$ is $\Delta^*_\mu$-closed. Then $A \subseteq X \setminus F$ and $X \setminus F$ is $\Delta^*_\mu$-open. Since $A$ is $\Delta^*_\mu - g$-closed, we have $c_\mu(A) \subseteq X \setminus F$. Consequently $F \subseteq X \setminus c_\mu(A)$. Hence $F$ is empty. Conversely, Suppose $A \subseteq U$, where $U$ is $\Delta^*_\mu$-open. If $c_\mu(A) \not\subseteq U$, then $c_\mu(A) \cap (X - U)$ is a non-empty $\Delta^*_\mu$-closed subset of $c_\mu(A) \setminus A$. Therefore $A$ is $\Delta^*_\mu - g$-closed.

Theorem 2.7. If $A$ is a $\Delta^*_\mu - g$-closed set in a GTS $(X, \mu)$, then $c_\mu(A) \setminus A$ does not contain any non-empty $\lambda_\mu$-closed $(\mu$-open / $\mu$-closed) subset of $X$.

Proof: Suppose $c_\mu(A) \setminus A$ contains a non-empty $\lambda_\mu$-closed $(\mu$-open / $\mu$-closed) subset of $X$. Since every $\lambda_\mu$-closed $(\mu$-open / $\mu$-closed) set is $\Delta^*_\mu$-closed, a non-empty $\Delta^*_\mu$-closed set is contained in $c_\mu(A) \setminus A$, which is contrary to Theorem 2.6.

Example 2.8 shows that the converse of the above theorem is not true.

Example 2.8. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a\}, \{a, d\}, \{b, c, d\}, X\}$. If $A = \{a, b, d\}$, then $c_\mu(A) \setminus A = \{c\}$, which does not contain any nonempty $\lambda_\mu$-closed $(\mu$-open / $\mu$-closed) set but $A$ is not a $\Delta^*_\mu - g$-closed set.

Theorem 2.9. Let $(X, \mu)$ be a quasi-topological space. Then $A \cup B$ is a $\Delta^*_\mu - g$-closed set whenever $A$ and $B$ are $\Delta^*_\mu - g$-closed sets.

Proof: Let $U$ be a $\Delta^*_\mu$-open set such that $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since $A$ and $B$ are $\Delta^*_\mu - g$-closed, we have $c_\mu(A) \subseteq U$ and $c_\mu(B) \subseteq U$. Hence by Lemma 1.2 $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B) \subseteq U$ and the proof follows.

Example 2.10. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a, b\}, \{b, c\}, X\}$. Then $\mu$ is a GT but not a quasi-topology. If $A = \{a\}$ and $B = \{c\}$, then $A$ and $B$ are $\Delta^*_\mu - g$-closed sets but their union is not a $\Delta^*_\mu - g$-closed set.

Example 2.11. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. If $A = \{b, d\}$ and $B = \{a, c, d\}$, then $A$ and $B$ are $\Delta^*_\mu - g$-closed sets but $A \cap B = \{d\}$ is not a $\Delta^*_\mu - g$-closed set.

Theorem 2.12. Let $(X, \mu)$ be a GTS. If $A$ is $\Delta^*_\mu$-open and $\Delta^*_\mu - g$-closed, then $A$ is $\mu$-closed.

Proof: Since $A$ is $\Delta^*_\mu$-open and $\Delta^*_\mu - g$-closed, $c_\mu(A) \subseteq A$ and hence $A$ is $\mu$-closed.
3. \( \Delta^*_\mu - g\mu - closed \) sets

In this section, we introduce the concept of \( \Delta^*_\mu - g\mu - closed \) sets and study its properties.

**Definition 3.1.** Let \((X, \mu)\) be a GTS. A subset \(A\) of \(X\) is called a \( \Delta^*_\mu - g\mu - closed \) set if \( c_{\lambda\mu}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a \( \Delta^*_\mu - open \) set in \(X\). The complement of a \( \Delta^*_\mu - g\mu - closed \) set is called a \( \Delta^*_\mu - g\mu - open \) set.

**Theorem 3.2.** For a GTS \((X, \mu)\), every \( \lambda\mu - closed \) set is \( \Delta^*_\mu - g\mu - closed \).

**Proof:** Let \(A\) be a \( \lambda\mu - closed \) set and \(U\) be any \( \Delta^*_\mu - open \) set containing \(A\). Since \(A\) is \( \lambda\mu - closed\), we have \( c_{\lambda\mu}(A) = A \). Therefore \( c_{\lambda\mu}(A) \subseteq U \) and hence \(A\) is \( \Delta^*_\mu - g\mu - closed \).

**Corollary 3.3.** For a GTS \((X, \mu)\), the following hold:

(i) Every \( \mu - closed \) set is \( \Delta^*_\mu - g\mu - closed \).

(ii) Every \( \mu - open \) set is \( \Delta^*_\mu - g\mu - closed \).

Example 3.4. Let \(X = \{a, b, c, d\}\) and \(\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}\). If \(A = \{c\}\), then \(A\) is \( \Delta^*_\mu - \mu - closed \) but not \( \lambda\mu - closed \) \( (\mu - closed, \mu - open) \).

**Theorem 3.5.** Let \((X, \mu)\) be a GTS and \(A \subseteq X\). If \(A\) is a \( \Delta^*_\mu - g\mu - closed \) set, then \(A\) is a \( g - \lambda\mu - closed \) set.

**Proof:** Let \(U\) be a \( \mu - open \) set containing \(A\) in \((X, \mu)\). Since every \( \mu - open \) set is \( \Delta^*_\mu - open \) and \(A\) is \( \Delta^*_\mu - g\mu - closed\), \( c_{\lambda\mu}(A) \subseteq U \). Therefore \(A\) is \( g - \lambda\mu - closed \).

Theorem 3.6 shows that the relation between \( \Delta^*_\mu - g - closed \) set and \( \Delta^*_\mu - g\mu - closed \) set.

**Theorem 3.6.** In a GTS \((X, \mu)\), every \( \Delta^*_\mu - g - closed \) set is \( \Delta^*_\mu - g\mu - closed \).

**Proof:** Let \(A\) be a \( \Delta^*_\mu - g - closed \) set and \(U\) be a \( \Delta^*_\mu - open \) set containing \(A\) in \((X, \mu)\). Then \( c_{\mu\mu}(A) \subseteq U \). Since \( c_{\mu\mu}(A) \subseteq c_{\mu}(A)\), we have \( c_{\mu\mu}(A) \subseteq U \). Therefore \(A\) is \( \Delta^*_\mu - g\mu - closed \).

**Remark 3.7.** \( \Delta^*_\mu - g\mu - closed \) sets and \( g\mu - closed \) (resp. \( \lambda\mu - g - closed \)) sets are independent of each other.

**Example 3.8.** Let \(X = \{a, b, c, d\}\) and \(\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}\). Then \( \{a, b, d\} \) is \( g\mu - closed \) but not \( \Delta^*_\mu - g\mu - closed \) and \( \{c\} \) is \( \Delta^*_\mu - g\mu - closed \) but not \( g\mu - closed \).

**Example 3.9.** Let \(X = \{a, b, c\}\) and \(\mu = \{\emptyset, \{a\}\}\). Then \( \{b\} \) is \( \lambda\mu - g - closed \) but not \( \Delta^*_\mu - g\mu - closed \) and \( \{a\} \) is \( \Delta^*_\mu - g\mu - closed \) but not \( \lambda\mu - g - closed \).

**Remark 3.10.** By Theorems 3.5 and 3.6, the following diagram holds:

\[
\begin{align*}
\Delta^*_\mu - g - closed & \Rightarrow \Delta^*_\mu - g\mu - closed & \Rightarrow & g - \lambda\mu - closed
\end{align*}
\]

The converses of all implications in DIAGRAM II are not true.
Theorem 3.11 gives a characterization of $\Delta^*_\mu - g_\mu$-closed sets.

**Theorem 3.11.** Let $(X, \mu)$ be a GTS. A subset $A$ of $X$ is a $\Delta^*_\mu - g_\mu$-closed set if and only if $F \subseteq c_{\lambda_\mu}(A) \setminus A$ and $F$ is $\Delta^*_\mu$-closed implies that $F$ is empty.

**Proof:** The proof is similar to Theorem 2.6.

**Theorem 3.12.** If $A$ is a $\Delta^*_\mu - g_\mu$-closed set in a GTS $(X, \mu)$, then $c_{\lambda_\mu}(A) \setminus A$ does not contain any non-empty $\lambda_\mu$-closed ($\mu$-open / $\mu$-closed) subset of $X$.

**Proof:** The proof is similar to Theorem 2.7.

Example 3.13 shows that the converse of Theorem 3.12 is not true.

**Example 3.13.** Let $X = \{a, b, c\}$ and $\mu = \emptyset, \{a, b\}, X$. If $A = \{b, c\}$, $c_{\lambda_\mu}(A) \setminus A = \{a\}$, which does not contain any non-empty $\lambda_\mu$-closed (resp. $\mu$-open, $\mu$-closed) sets but $A$ is not $\Delta^*_\mu - g_\mu$-closed.

**Theorem 3.14.** Let $(X, \mu)$ be a GTS and $A$ and $B$ be subsets of $X$. If $A \subseteq B \subseteq c_{\lambda_\mu}(A)$ and $A$ is a $\Delta^*_\mu - g_\mu$-closed set, then $B$ is $\Delta^*_\mu - g_\mu$-closed.

**Proof:** If $F$ is a $\Delta^*_\mu$-closed set such that $F \subseteq c_{\lambda_\mu}(B) \setminus B$, then $F \subseteq c_{\lambda_\mu}(A) \setminus A$. By Theorem 3.11, $F = \emptyset$ and so $B$ is $\Delta^*_\mu - g_\mu$-closed.

**Theorem 3.15.** Let $A$ be a $\Delta^*_\mu - g_\mu$-closed set in a quasi-topological space $(X, \mu)$. Then the following hold:

(i) If $A$ is a $\pi$-regular set, then $i_\pi(A)$ and $c_\sigma(A)$ are $\Delta^*_\mu - g_\mu$-closed sets.

(ii) If $A$ is a $\sigma$-regular set, then $c_\sigma(A)$ and $i_\sigma(A)$ are $\Delta^*_\mu - g_\mu$-closed sets.

**Proof:**

(i) Since $A$ is a $\pi$-regular set, $c_\sigma(A) = A \cup i_\mu c_\mu(A) = A$ and $i_\pi(A) = A \cap i_\mu c_\mu(A) = A$. Thus $i_\pi(A)$ and $c_\sigma(A)$ are $\Delta^*_\mu - g_\mu$-closed sets.

(ii) Since $A$ is a $\sigma$-regular set, $c_\sigma(A) = A$ and $i_\pi(A) = A$. Thus $c_\sigma(A)$ and $i_\pi(A)$ are $\Delta^*_\mu - g_\mu$-closed sets.

**Remark 3.16.** The union (resp. intersection) of two $\Delta^*_\mu - g_\mu$-closed sets need not be a $\Delta^*_\mu - g_\mu$-closed set.

**Example 3.17.** Let $X = \{a, b, c, d\}$ and $\mu = \emptyset, \{c\}, \{a, b, c\}, \{b, c, d\}, X$. Then \{a\} and \{c\} are $\Delta^*_\mu - g_\mu$-closed sets but their union is not a $\Delta^*_\mu - g_\mu$-closed set. Further \{a, b, c\} and \{a, c, d\} are $\Delta^*_\mu - g_\mu$-closed sets but their intersection is not a $\Delta^*_\mu - g_\mu$-closed set.

4. $g\Delta^*_\mu$-closed sets

In this section, we introduce the notion of $g\Delta^*_\mu$-closed sets and discuss its properties.

**Definition 4.1.** Let $(X, \mu)$ be a GTS. A subset $A$ of $X$ is called a $g\Delta^*_\mu$-closed set if $c_{\lambda_\mu}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $\Delta^*_\mu$-open set in $X$. The complement of a $g\lambda_\mu$-closed set is called a $g\Delta^*_\mu$-open set.

**Theorem 4.2.** For a GTS $(X, \mu)$, every $\Delta^*_\mu$-closed set is $g\Delta^*_\mu$-closed.
Proof: Let $A$ be a $\Delta^*_\mu$-closed set and $U$ be any $\Delta^*_\mu$-open set containing $A$. Since $A$ is $\Delta^*_\mu$-closed, $c\Delta^*_\mu(A) = A$. Therefore $c\Delta^*_\mu(A) \subseteq U$ and hence $A$ is $g\Delta^*_\mu$-closed.

Corollary 4.3. For a GTS $(X, \mu)$, the following hold:
(i) Every $\lambda_\mu$-closed set is $g\Delta^*_\mu$-closed.
(ii) Every $\mu$-closed set is $g\Delta^*_\mu$-closed.
(iii) Every $\mu$-open set is $g\Delta^*_\mu$-closed.

Example 4.4 shows that the converses of Theorem 4.2 and Corollary 4.3 are not true.

Example 4.4. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{c\}, \{a, b, c\}, \{b, c, d\}, X\}$. Then $\{b, c\}$ is a $g\Delta^*_\mu$-closed set but it is not $\Delta^*_\mu$-closed (resp. $\lambda_\mu$-closed, $\mu$-closed, $\mu$-open).

Remark 4.5. $g\Delta^*_\mu$-closed sets and $\lambda_\mu$-g-closed (resp. $\mu$-closed) sets are independent of each other.

Example 4.6. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}\}$. Then $\{b, c\}$ is $\lambda_\mu$-g-closed but not $g\Delta^*_\mu$-closed and $\{a\}$ is $g\Delta^*_\mu$-closed but neither $\lambda_\mu$-g-closed nor $g_\mu$-closed.

Example 4.7. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b\}, \{c, d\}, \{b, c, d\}, X\}$. Then $\{a, c\}$ is $g_\mu$-closed but not $g\Delta^*_\mu$-closed.

Theorem 4.8 shows the relation between $g\Delta^*_\mu$-closed set and $\Delta^*_\mu - g_\mu$-closed set.

Theorem 4.8. For a GTS $(X, \mu)$, every $\Delta^*_\mu - g_\mu$-closed set is $g\Delta^*_\mu$-closed.

Proof: Let $A$ be a $\Delta^*_\mu - g_\mu$-closed set and $U$ be a $\Delta^*_\mu$-open set containing $A$ in $(X, \mu)$. Then $c\lambda_\mu(A) \subseteq U$. Since $c\Delta^*_\mu(A) \subseteq c\lambda_\mu(A)$, we have $c\Delta^*_\mu(A) \subseteq U$. Therefore $A$ is $g\Delta^*_\mu$-closed.

Example 4.9 shows that the converse of Theorem 4.8 is not true.

Example 4.9. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a, b\}, X\}$. Then $\{a\}$ is $g\Delta^*_\mu$-closed but not $\Delta^*_\mu - g_\mu$-closed.

Remark 4.10. By Theorems 3.6 and 4.8, the following diagram holds:
\[
\Delta^*_\mu - g_\mu \text{- closed} \Rightarrow \Delta^*_\mu - g_\mu \text{- closed} \Rightarrow g\Delta^*_\mu \text{- closed}
\]

References


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