



Approximation and Analysis Regarding the Structure of a Multiple Variable Mapping

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ABSTRACT: The article introduces a several variables mapping as the multimixed quadratic-cubic mapping in order to characterize such mappings. It reduces a system of equations defining the multimixed quadratic-cubic mappings to obtain a single functional equation. It is shown that under some mild conditions, every multimixed quadratic-cubic mapping can be multi-quadratic, multi-cubic and multiquadratic-cubic. Further, the generalized Hyers-Ulam stability and hyperstability for multimixed quadratic-cubic functional equations in quasi- β -normed spaces have been investigated.

Key Words: Hyers-Ulam stability, multi-quadratic mapping, multi-cubic mapping, multiquadratic-cubic mapping, multimixed quadratic-cubic mapping.

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1. Introduction

The stability problem for functional equations, which Ulam [30] proposed for group homomorphisms, has been answered and explored for multiple variable mappings in recent decades. We recall that a functional equation Γ is said to be *stable* if any function f satisfying the equation Γ approximately must be near to an exact solution. Moreover, Γ is called *hyperstable* if any function f satisfying the equation Γ approximately (in some senses) is actually a solution for it; for some stability results in one variable mappings and functional equations see for instance the papers and books [13], [19], [24], [26], [29] and references therein.

We now state some basic notions and developments about the structure and the stability of several variables mappings. Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. A mapping $f : V^n \rightarrow W$ is called

- *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x + y) = A(x) + A(y)$) in each variable.
- *multi-quadratic* if it fulfills the quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \quad (1.1)$$

in each variable [11]. A lot of information about the structure of multi-additive mappings and their Ulam stabilities are available in [10], [12] and [20, Sections 13.4 and 17.2]. C.-G. Park was the first author who studied the stability of multi-quadratic in the setting of Banach algebras in [22]. After that, Ciepliński [11] studied the generalized Hyers-Ulam stability of multi-quadratic mappings in Banach spaces. Zhao et al. [32] described the structure of multi-quadratic mappings and in fact showed that a mapping $f : V^n \rightarrow W$ is multi-quadratic if and only if the equation

$$\sum_{s \in \{-1, 1\}^n} f(x_1 + sx_2) = 2^n \sum_{j_1, j_2, \dots, j_n \in \{1, 2\}} f(x_{1j_1}, x_{2j_2}, \dots, x_{nj_n}) \quad (1.2)$$

holds, where $x_j = (x_{1j}, x_{2j}, \dots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. Various versions of multi-quadratic mappings and their stability can be found in [7] and [28]. For the structure of multi-additive-quadratic, we refer to [1].

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Ghaemi et al. [15] introduced the multi-cubic mappings and then for a special case of such mappings have been studied in [8]. In fact, a mapping $f : V^n \rightarrow W$ is called *multi-cubic* if it is cubic in each variable, i.e., satisfies the equation

$$C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x) \quad (1.3)$$

in each variable [17]. In [8], the authors unified the system of functional equations defining a multi-cubic mapping to a single equation, namely, the multi-cubic functional equation. Furthermore, the general system of cubic functional equations which is defined in [15], characterized as a single equation in [14]. Other forms of cubic functional equations for instance are available in [3] and [23]. In [8], it is shown that every multi-cubic functional equation is stable and moreover such functional equations under some conditions can be hyperstable; for the miscellaneous versions of multi-cubic mappings and their stabilities in non-Archimedean normed and modular spaces, we refer to [14] and [21], respectively.

Chang and Jung [9] introduced the following mixed type quadratic and cubic functional equation

$$6f(x + y) - 6f(x - y) + 4f(3y) = 3f(x + 2y) - 3f(x - 2y) + 9f(2y). \quad (1.4)$$

They established the general solution of the functional equation (1.4) and investigated the Hyers-Ulam stability of this equation; for a different form of mixed type quadratic-cubic functional equation, one can see [18].

The following mixed type quadratic-cubic functional was considered in [27] which is somewhat different from (1.4) as follows:

$$f(x + 2y) - f(x - 2y) = 2[f(x + y) - f(x - y)] + 3f(2y) - 12f(y). \quad (1.5)$$

It is easily verified that the function $f(x) = ax^2 + bx^3$ is a solution of equations (1.4) and (1.5). Recently, the first author and Mitrović [6] have studied the structure of multimixed quadratic-cubic mappings and established ε -stability (Hyers' stability) of such mappings in Banach spaces setting by applying an alternative fixed point method.

Motivated by equation (1.5), in this paper, we define multimixed quadratic-cubic mappings and present a characterization of such mappings. In other words, we reduce the system of n equations defining the multimixed quadratic-cubic mappings to obtain a single functional equation. We also show that under some mild conditions, every multimixed quadratic-cubic mapping can be multi-quadratic, multi-cubic and multiquadratic-cubic. We also prove the generalized Hyers-Ulam stability and hyperstability for multimixed quadratic-cubic functional equations in quasi- β -normed spaces.

2. Characterization of the multimixed quadratic-cubic mappings

Throughout this paper, \mathbb{N} , \mathbb{Z} and \mathbb{Q} are the set of all positive integers, integers and rational numbers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$. For any $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $t = (t_1, \dots, t_n) \in \{-1, 1\}^n$ and $x = (x_1, \dots, x_n) \in V^n$ we write $lx := (lx_1, \dots, lx_n)$ and $tx := (t_1x_1, \dots, t_nx_n)$, where lx stands, as usual, for the scalar product of l on x in the commutative group $(V, +)$.

Let V and W be linear spaces, $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$. Put

$$\mathbf{n} := \{1, \dots, n\}. \quad (2.1)$$

Each subset of \mathbf{n} with m elements is denoted by $(m)_{\mathbf{n}}$. Recall from [5] that a mapping $f : V^n \rightarrow W$ is called k -quadratic and $n - k$ -cubic (briefly, multiquadratic-cubic) if f satisfies the following functional equations system.

$$\left\{ \begin{array}{l} f(v_1, \dots, v_{i-1}, v_i + v'_i, v_{i+1}, \dots, v_n) + f(v_1, \dots, v_{i-1}, v_i - v'_i, v_{i+1}, \dots, v_n) \\ \quad = 2f(v_1, \dots, v_n) + 2f(v_1, \dots, v'_i, \dots, v_n), \quad i \in (k)_{\mathbf{n}}, \\ \\ f(v_1, \dots, v_{i-1}, 2v_i + v'_i, v_{i+1}, \dots, v_n) + f(v_1, \dots, v_{i-1}, 2v_i - v'_i, v_{i+1}, \dots, v_n) \\ \quad = 2f(v_1, \dots, v_{i-1}, v_i + v'_i, v_{i+1}, \dots, v_n) + 2f(v_1, \dots, v_{i-1}, v_i - v'_i, v_{i+1}, \dots, v_n) \\ \quad \quad + 12f(v_1, \dots, v_n), \quad i \in (n - k)_{\mathbf{n}}. \end{array} \right.$$

Note that we can suppose for simplicity that f is quadratic in each of the first k variables, but one can obtain analogous results without this assumption. Let us note that for $k = n$ ($k = 0$), the above definition leads to the so-called multi-quadratic (multi-cubic) mappings; some basic facts on such mappings can be found for instance in [8] and [32].

Definition 2.1. *Let V and W be vector spaces over \mathbb{Q} , $n \in \mathbb{N}$. A several variables mapping $f : V^n \rightarrow W$ is called n -mixed quadratic-cubic or briefly multimixed quadratic-cubic if f fulfills (1.5) in each of its n arguments, that is*

$$\begin{aligned} & f(v_1, \dots, v_{i-1}, v_i + 2v'_i, v_{i+1}, \dots, v_n) - f(v_1, \dots, v_{i-1}, v_i - 2v'_i, v_{i+1}, \dots, v_n) \\ & \quad - 3f(v_1, \dots, v_{i-1}, 2v'_i, v_{i+1}, \dots, v_n) \\ & = 2[f(v_1, \dots, v_{i-1}, v_i + v'_i, v_{i+1}, \dots, v_n) - f(v_1, \dots, v_{i-1}, v_i - v'_i, v_{i+1}, \dots, v_n)] \\ & \quad - 12f(v_1, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_n). \end{aligned}$$

Let $n \in \mathbb{N}$ with $n \geq 2$ and $x_i^n = (x_{i1}, x_{i2}, \dots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We will write x_i^n simply x_i when no confusion can arise. For $x_1, x_2 \in V^n$, set

$$\mathcal{M}^n = \{\mathfrak{M}_n = (M_1, \dots, M_n) \mid M_j \in \{x_{1j} \pm 2x_{2j}, 2x_{2j}\}\},$$

and

$$\mathcal{N}^n = \{\mathfrak{N}_n = (N_1, \dots, N_n) \mid N_j \in \{x_{1j} \pm x_{2j}, x_{2j}\}\},$$

for all $j \in \{1, \dots, n\}$. For $p_i, q_i \in \mathbb{N}_0$ with $0 \leq p_i, q_i \leq n$, consider the subsets $\mathcal{M}_{(q_1, q_2)}^n$ and $\mathcal{N}_{(p_1, p_2)}^n$ of \mathcal{M}^n and \mathcal{N}^n , respectively, as follows:

$$\mathcal{M}_{(q_1, q_2)}^n := \{\mathfrak{M}_n \in \mathcal{M}^n \mid \text{Card}\{M_j : M_j = x_{1j} - 2x_{2j}\} = q_1, \text{Card}\{M_j : M_j = x_{2j}\} = q_2\},$$

$$\mathcal{N}_{(p_1, p_2)}^n := \{\mathfrak{N}_n \in \mathcal{N}^n \mid \text{Card}\{N_j : N_j = x_{1j} - x_{2j}\} = p_1, \text{Card}\{N_j : N_j = x_{2j}\} = p_2\}.$$

Hereafter, for a multimixed quadratic-cubic mappings f , we use the following notations:

$$f\left(\mathcal{M}_{(q_1, q_2)}^n\right) := \sum_{\mathfrak{M}_n \in \mathcal{M}_{(q_1, q_2)}^n} f(\mathfrak{M}_n), \quad (2.2)$$

$$f\left(\mathcal{M}_{(q_1, q_2)}^n, z\right) := \sum_{\mathfrak{M}_n \in \mathcal{M}_{(q_1, q_2)}^n} f(\mathfrak{M}_n, z) \quad (z \in V),$$

$$f\left(\mathcal{N}_{(p_1, p_2)}^n\right) := \sum_{\mathfrak{N}_n \in \mathcal{N}_{(p_1, p_2)}^n} f(\mathfrak{N}_n), \quad (2.3)$$

and

$$f\left(\mathcal{N}_{(p_1, p_2)}^n, z\right) := \sum_{\mathfrak{N}_n \in \mathcal{N}_{(p_1, p_2)}^n} f(\mathfrak{N}_n, z) \quad (z \in V).$$

For each $x_1, x_2 \in V^n$, we consider the equation

$$\begin{aligned} & \sum_{q_1=0}^n \sum_{q_2=0}^{n-q_1} (-1)^{q_1} (-3)^{q_2} f\left(\mathcal{M}_{(q_1, q_2)}^n\right) \\ & = \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f\left(\mathcal{N}_{(p_1, p_2)}^n\right), \end{aligned} \quad (2.4)$$

where $f\left(\mathcal{M}_{(q_1, q_2)}^n\right)$ and $f\left(\mathcal{N}_{(p_1, p_2)}^n\right)$ are defined in (2.2) and (2.3), respectively.

We recall that the binomial coefficient for all $n, r \in \mathbb{N}_0$ with $n \geq r$ is defined and denoted by $\binom{n}{r} := \frac{n!}{r!(n-r)!}$.

Definition 2.2. Let $r \in \mathbb{N}$. We say the mapping $f : V^n \rightarrow W$

(i) satisfies (has) the r -power condition in the j th variable if

$$f(z_1, \dots, z_{j-1}, 2z_j, z_{j+1}, \dots, z_n) = 2^r f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n),$$

for all $z_1, \dots, z_n \in V^n$. In particular, 2-power and 3-power conditions are called quadratic and cubic condition, respectively.

(ii) has zero condition if $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.

(iii) is odd in the j th variable if

$$f(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = -f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n).$$

(iv) is even in the j th variable if

$$f(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n).$$

Here, we bring an elementary lemma from [4].

Lemma 2.3. Let $n, k, p_l \in \mathbb{N}_0$, such that $k + \sum_{l=1}^m p_l \leq n$, where $l \in \{1, \dots, m\}$. Then

$$\begin{aligned} \binom{n-k}{n-k-\sum_{l=1}^m p_l} \binom{\sum_{l=1}^m p_l}{\sum_{l=1}^{m-1} p_l} \cdots \binom{p_1+p_2}{p_1} \\ = \binom{n-k}{p_1} \binom{n-k-p_1}{p_2} \cdots \binom{n-k-\sum_{l=1}^{m-1} p_l}{p_m}. \end{aligned}$$

Consider \mathbf{n} as in (2.1). For a subset $T = \{j_1, \dots, j_i\}$ of \mathbf{n} with $1 \leq j_1 < \dots < j_i \leq n$ and $x = (x_1, \dots, x_n) \in V^n$,

$$_T x := (0, \dots, 0, x_{j_1}, 0, \dots, 0, x_{j_i}, 0, \dots, 0) \in V^n$$

denotes the vector which coincides with x in exactly those components, which are indexed by the elements of T and whose other components are set equal zero. Note that ${}_0 x = 0$, $_{\mathbf{n}} x = x$. We use these notations in the proof of upcoming lemma.

Next, we reduce the system of n equations defining the multimixed quadratic-cubic mapping in obtaining the single functional equation (2.4). For doing this, we need the next lemma.

Lemma 2.4. If a mapping $f : V^n \rightarrow W$ satisfies equation (2.4), then it has zero condition.

Proof. We argue by induction on k that $f({}_k x) = 0$, when $0 \leq k \leq n-1$. Putting $x_1 = x_2 = {}_0 x$ in (2.4), we have

$$\begin{aligned} \left[\sum_{q_1=0}^n \sum_{q_2=0}^{n-q_1} \binom{n}{n-q_1-q_2} \binom{q_1+q_2}{q_2} (-1)^{q_1} (-3)^{q_2} \right] f({}_0 x) \\ = \left[\sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} \binom{n}{n-p_1-p_2} \binom{p_1+p_2}{p_2} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} \right] f({}_0 x). \end{aligned} \quad (2.5)$$

Here we compute the the left side of (2.5). Using Lemma 2.3 for $k = 0$, we have

$$\begin{aligned} \sum_{q_1=0}^n \sum_{q_2=0}^{n-q_1} \binom{n}{n-q_1-q_2} \binom{q_1+q_2}{q_2} (-1)^{q_1} (-3)^{q_2} \\ = \sum_{q_1=0}^n \binom{n}{q_1} (-1)^{q_1} \sum_{q_2=0}^{n-q_1} \binom{n-q_1}{q_2} 1^{n-q_1-q_2} (-3)^{q_2} \\ = \sum_{q_1=0}^n \binom{n}{q_1} (-1)^{q_1} (-2)^{n-q_1} = (-1-2)^n = (-3)^n. \end{aligned} \quad (2.6)$$

Similarly, one can show from Lemma 2.3 that

$$\sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} \binom{n}{n-p_1-p_2} \binom{p_1+p_2}{p_2} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} = (-12)^n. \quad (2.7)$$

It follows from relations (2.5), (2.6) and (2.7) that $f_{(0)x} = 0$. Assume that $f_{(k-1)x} = 0$ for any $k \in \{1, \dots, n-1\}$. We show that $f_{(k)x} = 0$. Without loss of generality, we assume that the first k variables are non-zero. By our assumption, replacing (x_1, x_2) by $({}_kx_1, 0)$ in equation (2.4), we have

$$\begin{aligned} & \left[\sum_{q_1=0}^{n-k} \sum_{q_2=0}^{n-k-q_1} \binom{n-k}{n-k-q_1-q_2} \binom{q_1+q_2}{q_2} (-1)^{q_1} (-3)^{q_2} \right] f_{(k)x} \\ &= \left[\sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \binom{n-k}{n-k-p_1-p_2} \binom{p_1+p_2}{p_2} 2^{n-k-p_1-p_2} (-2)^{p_1} (-12)^{p_2} \right] f_{(k)x}. \end{aligned}$$

Similar the above and by using lemma 2.3, we can obtain $(-3)^{n-k} f_{(k)x} = (-12)^{n-k} f_{(k)x}$, and this implies that $f_{(k)x} = 0$. This finishes the proof. \square

In the upcoming results which are our aim in this section, we unify the general system of quadratic-cubic functional equations defining a multimixed quadratic-cubic mapping to an equation and indeed this functional equation describe a multimixed quadratic-cubic mapping.

Proposition 2.5. *If a mapping $f : V^n \rightarrow W$ is multimixed quadratic-cubic, then it satisfies equation (2.4).*

Proof. We proceed the proof by induction on n , and in fact we show that equation (2.4) is valid for f . Clearly, f satisfies equation (1.5) and this guarantees the assertion for $n = 1$. If (2.4) holds for some

positive integer $n > 1$, then

$$\begin{aligned}
& \sum_{q_1=0}^{n+1} \sum_{q_2=0}^{n+1-q_1} (-1)^{q_1} (-3)^{q_2} f \left(\mathcal{M}_{(q_1, q_2)}^{n+1} \right) = \sum_{q_1=0}^n \sum_{q_2=0}^{n-q_1} (-1)^{q_1} (-3)^{q_2} f \left(\mathcal{M}_{(q_1, q_2)}^n, x_{1, n+1} + 2x_{2, n+1} \right) \\
& \quad - \sum_{q_1=0}^n \sum_{q_2=0}^{n-q_1} (-1)^{q_1} (-3)^{q_2} f \left(\mathcal{M}_{(q_1, q_2)}^n, x_{1, n+1} - 2x_{2, n+1} \right) - 3 \sum_{q_1=0}^n \sum_{q_2=0}^{n-q_1} (-1)^{q_1} (-3)^{q_2} f \left(\mathcal{M}_{(q_1, q_2)}^n, 2x_{2, n+1} \right) \\
& = \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f \left(\mathcal{N}_{(p_1, p_2)}^n, x_{1, n+1} + 2x_{2, n+1} \right) \\
& \quad - \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f \left(\mathcal{N}_{(p_1, p_2)}^n, x_{1, n+1} - 2x_{2, n+1} \right) \\
& \quad - 3 \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f \left(\mathcal{N}_{(p_1, p_2)}^n, 2x_{2, n+1} \right) \\
& = 2 \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f \left(\mathcal{N}_{(p_1, p_2)}^n, x_{1, n+1} + x_{2, n+1} \right) \\
& \quad - 2 \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f \left(\mathcal{N}_{(p_1, p_2)}^n, x_{1, n+1} - x_{2, n+1} \right) \\
& \quad - 12 \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f \left(\mathcal{N}_{(p_1, p_2)}^n, x_{2, n+1} \right) \\
& = \sum_{p_1=0}^{n+1} \sum_{p_2=0}^{n+1-p_1} 2^{n+1-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f \left(\mathcal{N}_{(p_1, p_2)}^{n+1} \right).
\end{aligned}$$

This means that (2.4) holds for $n + 1$. □

It follows from Proposition 2.5 and by a mathematical computation that the mapping $f(z_1, \dots, z_n) = \prod_{j=1}^n (\alpha_j z_j^2 + \beta_j z_j^3)$ satisfies (2.4) and so this equation is said to be *multimixed quadratic-cubic* functional equation.

It is shown in [5, Proposition 2.1] that if a mapping $f : V^n \rightarrow W$ is k -quadratic and $n - k$ -cubic (multiquadratic-cubic) mapping, then f satisfies equation

$$\sum_{s \in \{-1, 1\}^k} \sum_{t \in \{-1, 1\}^{n-k}} f(x_1^k + sx_2^k, 2x_1^{n-k} + tx_2^{n-k}) = 2^k \sum_{m=0}^{n-k} 2^{n-k-m} 12^m \sum_{i \in \{1, 2\}} f(x_i^k, \mathbb{M}_m^{n-k}), \quad (2.8)$$

for all $x_i^k = (x_{i1}, \dots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i, k+1}, \dots, x_{in}) \in V^{n-k}$ where $i \in \{1, 2\}$ in which

$$f(x_i^k, \mathbb{M}_m^{n-k}) := \sum_{\mathfrak{N}_n \in \mathcal{M}_m^{n-k}} f(x_i^k, \mathfrak{N}_n),$$

whereas

$$\mathbb{M}^{n-k} = \{\mathfrak{N}_n = (N_{k+1}, \dots, N_n) \mid N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\}$$

and

$$\mathbb{M}_m^{n-k} := \{\mathfrak{N}_n = (N_{k+1}, \dots, N_n) \in \mathbb{M}^{n-k} \mid \text{Card}\{N_j : N_j = x_{1j}\} = m\}.$$

Note that in the the case $k = n$ and $k = 0$, equation (2.8) converts to (1.2) and

$$\sum_{t \in \{-1, 1\}^n} f(2x_1^n + tx_2^n) = \sum_{m=0}^n 2^{n-m} 12^m f(\mathbb{M}_m^n), \quad (2.9)$$

respectively. In addition, it is proved in [32, Theorem 2] (resp., [8, Proposition 2.2]) that if the mapping $f : V^n \rightarrow W$ is multi-quadratic (resp. multi-cubic), then it satisfies the equation (1.2) (resp., (2.9)).

Proposition 2.6. *Suppose that a mapping $f : V^n \rightarrow W$ satisfies equation (2.4). Under one of the following conditions, it is multimixed quadratic-cubic.*

- (i) f is even in each variable and satisfies the quadratic condition for all variables;
- (ii) f is odd in each variable and satisfies the cubic condition for all variables.

Proof. (i) Let $j \in \{1, \dots, n\}$ be arbitrary and fixed. Set

$$f_j^*(z) := f(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n).$$

Putting $x_{1k} = 0$ for all $k \in \{1, \dots, n\} \setminus \{j\}$, $x_{1j} = z$ and $x_2 = (z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_n)$ in (2.4), using Lemma 2.4, we get

$$\begin{aligned} & (-3)^{n-1} [f(2z_1, \dots, 2z_{j-1}, z + 2w, 2z_{j+1}, \dots, 2z_n) \\ & \quad - f(2z_1, \dots, 2z_{j-1}, z - 2w, 2z_{j+1}, \dots, 2z_n) - 3f(2z_1, \dots, 2z_{j-1}, 2w, 2z_{j+1}, \dots, 2z_n)] \\ & = (-12)^{n-1} [2f_j^*(z + w) - 2f_j^*(z - w) - 12f_j^*(w)]. \end{aligned} \quad (2.10)$$

Our assumption (2.10) converts to

$$\begin{aligned} & 2^{2(n-1)} (-3)^{n-1} [f_j^*(z + 2w) - f_j^*(z - 2w) - 3f_j^*(2w)] \\ & = (-12)^{n-1} [2f_j^*(z + w) - 2f_j^*(z - w) - 12f_j^*(w)], \end{aligned}$$

and so

$$f_j^*(z + 2w) - f_j^*(z - 2w) - 3f_j^*(2w) = 2f_j^*(z + w) - 2f_j^*(z - w) - 12f_j^*(w). \quad (2.11)$$

This finishes the proof of part (i).

(ii) Similar to the proof of part (i), Putting $x_{1k} = 0$ for all $k \in \{1, \dots, n\} \setminus \{j\}$, $x_{1j} = z$ and $x_2 = (z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_n)$ in (2.4), using our assumptions and Lemma 2.4, we obtain the left side of (2.4) as follows:

$$\begin{aligned} & \left[\sum_{q_2=0}^{n-1} \binom{n-1}{q_2} 2^{3(n-1-q_2)} (-3)^{q_2} 2^{n-1-q_2} 2^{3q_2} \right] (f_j^*(z + 2w) - f_j^*(z - 2w)) \\ & + \left[\sum_{q_2=1}^n \binom{n-1}{q_2-1} 2^{3(n-1)} (-3)^{q_2} 2^{n-q_2} \right] f_j^*(w) \\ & = 2^{3(n-1)} (-1)^{n-1} (f_j^*(z + 2w) - f_j^*(z - 2w)) - 3 \times 2^{3(n-1)} (-1)^{n-1} f_j^*(2w). \end{aligned} \quad (2.12)$$

On the other hand, the right side of (2.4) will be

$$\begin{aligned} & \left[\sum_{p_2=0}^{n-1} \binom{n-1}{p_2} 2^{n-1-p_2} (-12)^{p_2} 2^{n-p_2} \right] (f_j^*(z + w) - f_j^*(z - w)) \\ & + \left[\sum_{p_2=1}^n \binom{n-1}{p_2-1} 2^{n-p_2} (-12)^{p_2} 2^{n-p_2} \right] f_j^*(w) \\ & = 2 \left[\sum_{p_2=0}^{n-1} \binom{n-1}{p_2} 4^{n-1-p_2} (-12)^{p_2} \right] (f_j^*(z + w) - f_j^*(z - w)) \\ & \quad - 12 \left[\sum_{p_2=0}^n \binom{n-1}{p_2} 4^{n-1-p_2} (-12)^{p_2} \right] f_j^*(w) \\ & = 2(-8)^{n-1} [f_j^*(z + w) - f_j^*(z - w)] - 12(-8)^{n-1} f_j^*(w). \end{aligned} \quad (2.13)$$

Comparing (2.12) and (2.13), we achieve (2.11). \square

Corollary 2.7. *Suppose that a mapping $f : V^n \rightarrow W$ satisfies equation (2.4).*

- (i) *If f is even in each variable and satisfies the quadratic condition in all variables, then it is multi-quadratic. Moreover, f satisfies equation (1.2);*
- (ii) *If f is odd in each variable and satisfies the cubic condition in all variables, then it is multi-cubic. In addition, equation (2.9) is valid for f ;*
- (iii) *If f is even in each of some k variables with the quadratic condition and is odd in each of the other variables with the cubic condition, then it is multiquadratic-cubic. In particular, f satisfies equation (2.8).*

Proof. (i) It is shown in Proposition 2.6 that for each j , f_j^* satisfies (1.5). Putting $z = w = 0$ in (2.11), we have $f_j^*(0) = 0$. Letting $z = 0$ in (2.11), we get by the evenness of f_j^* that $f_j^*(2w) = 4f_j^*(w)$ for all $w \in V$. The last equality converts (2.11) to

$$f_j^*(z + 2w) - f_j^*(z - 2w) = 2[f_j^*(z + w) - f_j^*(z - w)], \quad (2.14)$$

for all $z, w \in V$. It is seen that (2.14) is the same relation (2.2) from [9]. Repeating the proof of Lemma 2.1 of [9], one can find (1.1) for f_j^* .

(ii) Putting $z = 0$ in (2.11) and using the oddness of f_j^* , we have $f_j^*(2w) = 8f_j^*(w)$ for all $w \in V$. Applying the last equality in (2.11), we arrive at

$$f_j^*(z + 2w) - f_j^*(z - 2w) = 2[f_j^*(z + w) - f_j^*(z - w)] + 12f_j^*(w), \quad (2.15)$$

for all $z, w \in V$. Replacing (z, w) by (w, z) in (2.15), we obtain

$$f_j^*(2z + w) + f_j^*(2z - w) = 2[f_j^*(z + w) + f_j^*(z - w)] + 12f_j^*(z),$$

for all $z, w \in V$. This completes the proof.

(iii) The result follows from the previous parts. □

3. Stability of the multimixed quadratic-cubic functional equations

We first recall some basic facts concerning quasi- β -normed space.

Definition 3.1. *Let β be a fixed real number with $0 < \beta < 1$, and \mathbb{K} denote either \mathbb{R} or \mathbb{C} . Let X be a linear space over \mathbb{K} . A quasi- β -norm is a real valued function on X fulfilling the following conditions*

- (1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
- (2) $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and $t \in \mathbb{K}$;
- (3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

When $\beta = 1$, the norm above is a quasinorm. Recall that K is the *modulus of concavity* of the norm $\|\cdot\|$. Moreover, if $\|\cdot\|$ is a quasi- β -norm on X , the pair $(X, \|\cdot\|)$ is said to be a quasi- β -normed space. A quasi- β -Banach space is a complete quasi- β -normed space. A quasi- β -norm $\|\cdot\|$ is called a (β, p) -norm ($0 < p \leq 1$) if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$, for all $x, y \in X$. In this case, a quasi- β -Banach space is called a (β, p) -Banach space.

Given a p -norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X . By the Aoki-Rolewicz Theorem [25], each quasi-norm is equivalent to some p -norm; see also [2]. Since it is much easier to work with p -norms, here and subsequently, we restrict our attention mainly to p -norms. In this section, by using an idea of Găvruta [16], we prove the stability of (2.4) in quasi- β -normed spaces. Here, we need the following fundamental lemma which is a main tool to achieve our goal in this section taken from [31, Lemma 3.1].

Lemma 3.2. *Let $j \in \{-1, 1\}$ be fixed, $a, s \in \mathbb{N}$ with $a \geq 2$. Suppose that X is a linear space, Y is a (β, p) -Banach space with (β, p) -norm $\|\cdot\|_Y$. If $\psi : X \rightarrow [0, \infty)$ is a function such that there exists an $L < 1$ with $\psi(a^j x) < L a^{js\beta} \psi(x)$ for all $x \in X$ and $f : X \rightarrow Y$ is a mapping satisfying*

$$\|f(ax) - a^s f(x)\|_Y \leq \psi(x),$$

for all $x \in X$, then there exists a uniquely determined mapping $F : X \rightarrow Y$ such that $F(ax) = a^s F(x)$ and

$$\|f(x) - F(x)\|_Y \leq \frac{1}{a^{s\beta}|1 - L^j|} \psi(x),$$

for all $x \in X$. Moreover, $F(x) = \lim_{l \rightarrow \infty} \frac{f(a^{jl} x)}{a^{jls}}$ for all $x \in X$.

From now on, for a mapping $f : V^n \rightarrow W$, we consider the difference operator $\mathbf{D}_{qc} f : V^n \times V^n \rightarrow W$ by

$$\begin{aligned} \mathbf{D}_{qc} f(x_1, x_2) &:= \sum_{q_1=0}^n \sum_{q_2=0}^{n-q_1} (-1)^{q_1} (-3)^{q_2} f\left(\mathcal{M}_{(q_1, q_2)}^n\right) \\ &\quad - \sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f\left(\mathcal{N}_{(p_1, p_2)}^n\right), \end{aligned}$$

where $f\left(\mathcal{M}_{(q_1, q_2)}^n\right)$ and $f\left(\mathcal{N}_{(p_1, p_2)}^n\right)$ are defined in (2.2) and (2.3), respectively. In the sequel, we assume that all mappings $f : V^n \rightarrow W$ satisfy (have) zero condition.

Theorem 3.3. *Let $j \in \{-1, 1\}$ be fixed, V be a linear space and W be a (β, p) -Banach space with (β, p) -norm $\|\cdot\|_W$ and $\varphi : V^n \times V^n \rightarrow \mathbb{R}_+$ be a function such that there exists an $0 < L < 1$ with $\varphi(2^j x_1, 2^j x_2) \leq 2^{(3n-k)j\beta} L \varphi(x_1, x_2)$ for all $x_1, x_2 \in V^n$. Suppose that $f : V^n \rightarrow W$ is an even mapping in each of some k variables and is odd in each of the other variables and moreover fulfilling the inequality*

$$\|\mathbf{D}_{qc} f(x_1, x_2)\|_W \leq \varphi(x_1, x_2), \quad (3.1)$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{F} : V^n \rightarrow W$ of (2.4) such that

$$\|f(x) - \mathcal{F}(x)\|_W \leq \frac{1}{|1 - L^j|} \frac{1}{3^{k\beta} \times 2^{(3n-k)\beta}} \varphi(0, x), \quad (3.2)$$

for all $x \in V^n$.

Proof. Without loss of generality, we assume that f is even in the k first of variables. Replacing (x_1, x_2) by $(0, x_1)$ in (3.1) and using the assumptions, we have

$$\left\| (-3)^k T f(2x) - (-12)^k S f(x) \right\|_W \leq \varphi(0, x), \quad (3.3)$$

for all $x = x_1 \in V^n$, in which

$$T = \sum_{q_2=0}^{n-k} \binom{n-k}{n-k-q_2} (-3)^{q_2} 2^{n-k-q_2} = (-3+2)^{n-k} = (-1)^{n-k}$$

and

$$S = \sum_{p_2=0}^{n-k} \binom{n-k}{n-k-p_2} 2^{n-k-p_2} 2^{n-k-p_2} (-12)^{p_2} = (4-12)^{n-k} = (-8)^{n-k}.$$

A computational shows that inequality (3.3) is converted to

$$\left\| (-3)^k (-1)^{n-k} f(2x) - (-12)^k (-8)^{n-k} f(x) \right\|_W \leq \varphi(0, x),$$

for all $x \in V^n$, and so

$$\|f(2x) - 2^{3n-k}f(x)\|_W \leq \frac{1}{3^{k\beta}}\varphi(0, x),$$

for all $x \in V^n$. By Lemma 3.2, there exists a unique mapping $\mathcal{F} : V^n \rightarrow W$ such that $\mathcal{F}(2x) = 2^{3n-k}\mathcal{F}(x)$ and

$$\|f(x) - \mathcal{F}(x)\|_W \leq \frac{1}{|1 - L^j|} \frac{1}{3^{k\beta} \times 2^{(3n-k)\beta}} \varphi(0, x),$$

for all $x \in V^n$. It remains to show that \mathcal{F} satisfies (2.4). Here, we note from Lemma 3.2 that for all $x \in V^n$, $\mathcal{F}(x) = \lim_{l \rightarrow \infty} \frac{f(2^{jl}x)}{2^{(3n-k)jl}}$. Now, by (3.1), we have

$$\begin{aligned} \left\| \frac{\mathbf{D}_{qc}f(2^{jl}x_1, 2^{jl}x_2)}{2^{(3n-k)jl}} \right\|_W &\leq 2^{-(3n-k)jl\beta} \varphi(2^{jl}x_1, 2^{jl}x_2) \\ &\leq 2^{-(3n-k)jl\beta} (2^{(3n-k)j\beta} L)^l \varphi(x_1, x_2) = L^l \varphi(x_1, x_2), \end{aligned}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}$. Letting $l \rightarrow \infty$ in the above inequality, we observe that $\mathbf{D}_{qc}\mathcal{F}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that \mathcal{F} satisfies (2.4). \square

We now have the next stability result for functional equation (2.4) in the special case of Theorem 3.3 when f is either an even or odd mapping in each of variable.

Theorem 3.4. *Let $j \in \{-1, 1\}$ be fixed, V be a linear space and W be a (β, p) -Banach space with (β, p) -norm $\|\cdot\|_W$. Suppose that $f : V^n \rightarrow W$ is a mapping such that*

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \varphi(x_1, x_2),$$

for all $x_1, x_2 \in V^n$, where φ is as in Theorem 3.3.

- (i) *If f is even in each variable and there exists an $0 < L < 1$ with $\varphi(2^j x_1, 2^j x_2) \leq 4^{nj\beta} L \varphi(x_1, x_2)$ for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{Q} : V^n \rightarrow W$ of (2.4) such that*

$$\|f(x) - \mathcal{Q}(x)\|_W \leq \frac{1}{|1 - L^j|} \frac{1}{12^{n\beta}} \varphi(0, x)$$

for all $x \in V^n$. In particular, if \mathcal{Q} is even and has the quadratic condition in each variable, then it is multi-quadratic;

- (ii) *If f is odd in each variable and there exists an $0 < L < 1$ with $\varphi(2^j x_1, 2^j x_2) \leq 8^{nj\beta} L \varphi(x_1, x_2)$ for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{C} : V^n \rightarrow W$ of (2.4) such that*

$$\|f(x) - \mathcal{C}(x)\|_W \leq \frac{1}{|1 - L^j|} \frac{1}{8^{n\beta}} \varphi(0, x)$$

for all $x \in V^n$. In particular, if \mathcal{C} is odd and has the cubic condition in each variable, then it is multi-cubic.

Proof. The results follow from Theorem 3.3 and Corollary 2.7. \square

The following corollary is a direct consequence of Theorem 3.4 concerning the stability of (2.4) when the norm of $\mathbf{D}_{qc}f(x_1, x_2)$ is controlled by sum of variables norms of x_1 and x_2 with positive powers.

Corollary 3.5. *Given the positive numbers θ and λ . Let V be a quasi- α -normed space with quasi- α -norm $\|\cdot\|_V$, and W be a (β, p) -Banach space with (β, p) -norm $\|\cdot\|_W$.*

(i) If $\lambda \neq 2n\frac{\beta}{\alpha}$ and $f : V^n \rightarrow W$ is an even mapping in each variable fulfilling the inequality

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \theta \sum_{k=1}^2 \sum_{j=1}^n \|x_{kj}\|_V^\lambda,$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{Q} : V^n \rightarrow W$ of (2.4) such that

$$\|f(x) - \mathcal{Q}(x)\|_W \leq \begin{cases} \frac{\theta}{3^{n\beta}(4^{n\beta} - 2^{\alpha\lambda})} \sum_{j=1}^n \|x_{1j}\|_V^\lambda & \lambda \in \left(0, 2n\frac{\beta}{\alpha}\right), \\ \frac{2^{\alpha\lambda}\theta}{12^{n\beta}(2^{\alpha\lambda} - 4^{n\beta})} \sum_{j=1}^n \|x_{1j}\|_V^\lambda & \lambda \in \left(2n\frac{\beta}{\alpha}, \infty\right), \end{cases}$$

for all $x = x_1 \in V^n$. Moreover, if \mathcal{Q} is even and has the quadratic condition in each variable, then it is multi-quadratic;

(ii) If $\lambda \neq 3n\frac{\beta}{\alpha}$ and $f : V^n \rightarrow W$ is an odd mapping in each variable fulfilling the inequality

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \theta \sum_{k=1}^2 \sum_{j=1}^n \|x_{kj}\|_V^\lambda,$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{C} : V^n \rightarrow W$ of (2.4) such that

$$\|f(x) - \mathcal{C}(x)\|_W \leq \begin{cases} \frac{\theta}{(8^{n\beta} - 2^{\alpha\lambda})} \sum_{j=1}^n \|x_{1j}\|_V^\lambda & \lambda \in \left(0, 3n\frac{\beta}{\alpha}\right), \\ \frac{2^{\alpha\lambda}\theta}{8^{n\beta}(2^{\alpha\lambda} - 8^{n\beta})} \sum_{j=1}^n \|x_{1j}\|_V^\lambda & \lambda \in \left(3n\frac{\beta}{\alpha}, \infty\right), \end{cases}$$

for all $x = x_1 \in V^n$. In particular, if \mathcal{C} is odd and has the cubic condition in each variable, then it is multi-cubic.

Under some conditions the functional equation (2.4) can be hyperstable as follows.

Corollary 3.6. Given the positive number θ and $p_{ij} > 0$ for $i \in \{1, 2\}$, $j \in \{1, \dots, n\}$. Let V be a quasi- α -normed space with quasi- α -norm $\|\cdot\|_V$, and W be a (β, p) -Banach space with (β, p) -norm $\|\cdot\|_W$.

(i) If $\sum_{i=1}^2 \sum_{j=1}^n p_{ij} \neq 2n\frac{\beta}{\alpha}$ and $f : V^n \rightarrow W$ is an even mapping and has the quadratic condition in each variable fulfilling the inequality

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \theta \prod_{i=1}^2 \prod_{j=1}^n \|x_{ij}\|_V^{p_{ij}},$$

for all $x_1, x_2 \in V^n$, then it is multi-quadratic;

(ii) If $\sum_{i=1}^2 \sum_{j=1}^n p_{ij} \neq 3n\frac{\beta}{\alpha}$ and $f : V^n \rightarrow W$ is an odd mapping and has the cubic condition in each variable fulfilling the inequality

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \theta \prod_{i=1}^2 \prod_{j=1}^n \|x_{ij}\|_V^{p_{ij}},$$

for all $x_1, x_2 \in V^n$, then it is multi-cubic.

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