Some Common Fixed Point Theorems for Four Self-Mappings Satisfying a General Contractive Condition

Manoj Kumar, Rashmi Sharma, Serkan Araci

ABSTRACT: In the paper, we derive a general case for four weakly compatible self maps satisfying a general contractive condition due to the same method introduced by Altun et al. [2]. We make use of such a study to prove common fixed point theorems for weakly compatible maps along with E.A. and (CLR) properties.

Key Words: Common fixed point, Weakly compatible, E.A. property, (CLR) property.

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1. Introduction

The study of common fixed point of mappings satisfying contractive conditions has been a very active field of research during recent years. The most general of the common fixed point theorems pertaining to four mappings $A, B, S$ and $T$ of a metric space $(X, d)$, uses either a Banach-type contractive condition [3] of the form

$$d(Ax, By) \leq km(x, y) \ (0 \leq k < 1),$$

where

$$m(x, y) = \max \{d(Ax, By), d(Sx, Ax), d(Ty, By) \text{ and } \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\},$$

or a Meir - Keeler - type $(\varepsilon, \delta)$ - contractive condition [6], that is, given $\varepsilon > 0$, there exists a $\delta > 0$ such that or a $\varphi$ - contractive condition [7] of the form

$$d(Ax, By) \leq \varphi(m(x, y)),$$

involving a contractive gauge function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(t) < t$ for each $t > 0$. Note that Banach-type contractive condition is a special case of both conditions Meir - Keeler - type $(\varepsilon, \delta)$ - contractive and $\varphi$ - contractive. A $\varphi$ - contractive condition does not guarantee the existence of a fixed point unless some additional condition is assumed. Moreover, a $\varphi$ - contractive condition, in general, does not imply the Meir - Keeler - type $(\varepsilon, \delta)$ - contractive condition. In the paper,
we aim to prove a common fixed point theorem for four weakly compatible self-maps satisfying a general contractive condition and also prove common fixed point theorems for weakly compatible maps along with E.A. and (CLR) properties.

We are now in a position to state the following three definitions which is an important to derive our main results.

**Definition 1.1.** [4] Two self maps \( f \) and \( g \) are said to be weakly compatible if they commute at coincidence points.

**Definition 1.2.** [1] Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) are said to satisfy E.A. property if there exists a sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t
\]

for some \( t \) in \( X \).

**Definition 1.3.** [8] Two self-mappings \( f \) and \( g \) of a metric space \((X,d)\) are said to satisfy (CLR) property if there exists a sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = fx
\]

for some \( x \) in \( X \).

### 2. Main Results

Now, we give the following theorems.

**Theorem 2.1.** Let \( A, B, S \) and \( T \) be self maps of a metric space \((X,d)\) satisfying the followings:

\[
SX \subseteq BX, \quad TX \subseteq AX,
\]

for all \( x \in X \), there exists right continuous functions \( \psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+ \), with

\[
\psi(0) = 0 = \phi(0) \quad \text{and} \quad \psi(s) < s \quad \text{for} \quad s > 0 \quad \text{such that}
\]

\[
\psi(d(Sx,Ty)) \leq \psi(m(x,y)) - \phi(m(x,y)),
\]

where

\[
m(x,y) = \max\{d(Ax,By), d(Sx, Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\}.
\]

If one of \( AX, BX, SX \) or \( TX \) is complete subspace of \( X \), then the pair \((A,S)\) or \((B,T)\) have a coincidence point. Moreover, if pairs \((A,S)\) and \((B,T)\) are weakly compatible, then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof:** Let \( x_0 \in X \) be an arbitrary point of \( X \). from (2.1), we can construct a sequence \( \{y_n\} \) in \( X \) as follows:

\[
y_{(2n+1)} = Sx_{2n} = Bx_{(2n+1)}, \quad y_{(2n+2)} = Tx_{(2n+1)} = Ax_{(2n+2)},
\]

for all \( n = 0, 1, 2, \ldots \). Define \( d_n = d(y_n, y_{(n+1)}) \). Suppose that \( d_{2n} = 0 \) for some \( n \). Then \( y_{2n} = y_{(2n+1)} \), that is, \( Tx_{(2n-1)} = Ax_{2n}, Sx_{2n} = Bx_{(2n+1)} \), and \( A \) and \( S \) have a coincidence point. Similarly, if \( d_{(2n+1)} = 0 \), then \( B \) and \( T \) have a coincidence point. Assume that \( d_n \neq 0 \) for each \( n \).
from (2.2), we have

$$\psi(d(Sx_{2n}, Tx_{2(n+1)}) \leq \psi(m(x_{2n}, x_{2(n+1)})) - \phi(m(x_{2n}, x_{2(n+1)})), \quad (2.4)$$

where

$$m(x_{2n}, x_{2(n+1)}) = \max\{d(Ax_{2n}, Bx_{2(n+1)}), d(Sx_{2n}, Ax_{2n}),$$

$$d(Sx_{2n}, Bx_{2(n+1)}) + d(Tx_{2(n+1)}, Ax_{2n}) / 2$$

$$d(Tx_{2(n+1)}, Bx_{2(n+1)})\}$$

$$= \max\{d_{2n}, d_{(2n+1)}\} \quad (2.5)$$

Thus, from (2.4), we have

$$\psi(d(Sx_{2n}, Tx_{2(n+1)}) \leq \psi(\max\{d_{2n}, d_{2(n+1)}\}) - \phi(\max\{d_{2n}, d_{2(n+1)}\}). \quad (2.6)$$

Now, if $d_{(2n+1)} \geq d_{2n}$, for some $n$, then from (2.6), we have

$$\psi(d_{(2n+1)}) \leq \psi(d_{(2n+1)}) - \phi(d_{(2n+1)})$$

$$< \psi(d_{(2n+1)}), \quad (2.7)$$

which is a contradiction. Thus, $d_{2n} > d_{(2n+1)}$ for all $n$, and so, from (2.6), we have

$$\psi(d_{(2n+1)}) \leq \psi(d_{2n}) - \phi(d_{2n}), \quad \text{for all } n \in \mathbb{N}. \quad (2.8)$$

Similarly,

$$\psi(d_{2n}) \leq \psi(d_{(2n-1)}) - \psi(d_{(2n-1)}),$$

$$\psi(d_{(2n-1)}) \leq \psi(d_{(2n-2)}) - \phi(d_{(2n-2)}).$$

In general, we have for all $n = 1, 2, \ldots$

$$\psi(d_{n}) \leq \psi(d_{(n-1)}) - \phi(d_{(n-1)})$$

$$< \psi(d_{(n-1)}). \quad (2.9)$$

Hence the sequence $\{\psi(d_n)\}$ is monotonically decreasing and bounded below. Thus, there exists, $r \geq 0$, such that

$$\lim_{n \to \infty} \psi(d_n) = r. \quad (2.10)$$

From (9), we deduce that

$$0 \leq \phi(d_{(n-1)}) \leq \psi(d_{(n-1)}) - \psi(d_n).$$

Letting limit as $n \to \infty$ and using (10), we get $\lim_{n \to \infty} \phi(d_{(n-1)}) = 0$ implies that

$$\lim_{n \to \infty} \phi(d_{(n-1)}) = \lim_{n \to \infty} (d(y_{(n-1)}, y_n)) = 0, \quad (2.11)$$
or

\[ \lim_{n \to \infty} d_n = \lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \] (2.12)

Now, we show that \( \{y_n\} \) is a Cauchy sequence. For this, it is sufficient to show that \( \{y_{2n}\} \) is a Cauchy sequence. Let, if possible, \( \{y_{2n}\} \) is not a Cauchy sequence. Then there exists an \( \varepsilon > 0 \) such that for each even integer \( 2k \) there exists even integers \( 2m(k) > 2n(k) > 2k \) such that

\[ d(y_{2(n(k))}, y_{2(m(k))}) \geq \varepsilon. \] (2.13)

For every even integer \( 2k \), suppose that \( 2m(k) \) be the least positive integer exceeding \( 2n(k) \) satisfying (13) such that

\[ d(y_{2(n(k))}, y_{2(m(k))}) < \varepsilon. \] (2.14)

from (2.13), we have

\[
\varepsilon \leq d(y_{2(n(k))}, y_{2(m(k))}) \\
\leq d(y_{2(n(k))}, y_{2(m(k)) - 2}) + d(y_{2(m(k)) - 2}, y_{2(m(k)) - 1}) + d(y_{2(m(k)) - 1}, y_{2m(k)}).
\]

Using (12) and (14) in the above inequality, we get

\[ \lim_{k \to \infty} d(y_{2(n(k))}, y_{2(m(k))}) = \varepsilon. \] (2.15)

Also, by the triangular inequality,

\[
|d(y_{2(n(k))}, y_{2(m(k)) - 1}) + d(y_{2(n(k))}, y_{2(m(k))})| \leq d(2(m(k)) - 1),
\]

\[
|d(y_{2(n(k)) + 1}, y_{2(m(k)) - 1}) + d(y_{2(n(k))}, y_{2(m(k))})| \leq d(2(m(k)) - 1) + d_{2m(k)}.
\] (2.16)

Using (12), we get

\[ \lim_{k \to \infty} d(y_{2(n(k))}, y_{2(m(k)) - 1}) = \lim_{k \to \infty} d(y_{2(n(k)) + 1}, y_{2(m(k)) - 1}) = \varepsilon. \] (2.17)

from (2.2), we have

\[
\psi(d(Sx_{2n(k)}, Tx_{2m(k) - 1})) \leq \psi(m(x_{2(n(k))}, x_{2(m(k)) - 1})) \\
- \phi(m(x_{2(n(k))}, x_{2(m(k)) - 1})),
\] (2.18)

where

\[
m(x_{2(n(k))}, x_{2(m(k)) - 1}) = \max\{d(Ax_{2n(k)}, Bx_{2n(k)}) + d(Sx_{2n(k)}, Ax_{2n(k)}), \}
\]

\[
(d(Sx_{2n(k)}, Bx_{2(m(k)) - 1}) + d(Tx_{2n(k)}, Ax_{2(m(k)) - 1)})^2,
\]

\[
d(Tx_{2(m(k) - 1)}, Bx_{2(m(k) - 1)})^2
\]

\[
= \max\{d(y_{2(n(k))}, y_{2(m(k) - 1)}), d(y_{2(n(k))}, y_{2(n(k) + 1)}) \}
\]

\[
(d(y_{2(n(k)) + 1}, y_{2(m(k) - 1)}) + d(y_{2(n(k))}, y_{2(m(k)) - 1}))^2,
\]

\[
d(y_{2(m(k) - 1)}); y_{2m(k)})}.\]
Letting limit as $k \to \infty$ and using (17), we get
\[
\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon),
\]
which is a contradiction, since $\varepsilon > 0$. Thus, $\{y_{2n}\}$ is a Cauchy sequence and so $\{y_n\}$ is a Cauchy sequence. Now, suppose that $A(X)$ is complete. Note that $\{y_{2n}\}$ is contained in $A(X)$ and has a limit in $A(X)$, say $v$, that is, $\lim_{n \to \infty} y_{2n} = v$. Let $v \in A^{(-1)} u$. Then $Av = u$. Now, we shall prove that $Sv = u$. Let, if possible, $Sv \neq u$, that is, $d(Sv, u) = p > 0$.

Putting $x = v$ and $y = x_{(2n-1)}$ in (1.2), we have
\[
\psi(d(Sv, Tx_{(2n-1)})) \leq \psi(m(v, x_{(2n-1)})) - \phi(m(v, x_{(2n-1)})).
\]
Letting limit as $n \to \infty$, we have
\[
\lim_{n \to \infty} \psi(d(Sv, Tx_{(2n-1)})) \leq \lim_{n \to \infty} \psi(m(v, x_{(2n-1)})) - \lim_{n \to \infty} \phi(m(v, x_{(2n-1)})), \quad (2.19)
\]
where,
\[
\lim_{n \to \infty} m(v, x_{(2n-1)}) = \lim_{n \to \infty} \left[ \max\{d(u, y_{(2n-1)}), d(Sv, u), d(y_{2n}, y_{(2n-1)})\}, \right. \frac{(d(Sv, y_{(2n-1)}) + d(y_{2n}, u))}{2} \left. \right]
\]
\[
= \max\{d(u, u), d(Sv, u), d(u, u), \frac{1}{2}(d(Sv, u) + d(u, u))\}
\]
\[
= d(Sv, u) = p.
\]
Thus, from (2.19), we have
\[
\psi(d(Sv, u)) \leq \psi(p) - \phi(p),
\]
that is
\[
\psi(p) \leq \psi(p) - \phi(p),
\]
which is a contradiction, since $p > 0$. Thus, $Sv = u = Av$. Hence $u$ is the coincidence point of the pair $(A, S)$. Since $SX \subseteq BX$, $Sv = u$, implies that, $u \in BX$. Let $w \in B^{(-1)} u$. Then $Bw = u$. By using the same arguments as above, one can easily verify that, $Tw = u = Bw$, that is, $u$ is the coincidence point of the pair $(B, T)$. The same result holds, if we assume that $BX$ is complete instead of $AX$. Now, if $TX$ is complete, then by (1), $u \in TX \subseteq AX$. Similarly, if $SX$ is complete, then $u \in SX \in BX$. Now, since the pairs $(A, S)$ and $(B, T)$ are weakly compatible, so
\[
u = Sv = Av = Tw = Bw,
\]
then
\[
Au = ASv = Sav = Su,
Bu = BTw = TBw = Tu. \quad (2.20)
\]
Now, we claim that \( Tu = u \). Let, if possible, \( Tu \neq u \).

From (2.2), we have

\[
\psi(d(u,Tu)) = \psi(d(Sv,Tu)) \\
\leq \psi(m(v,u)) - \phi(m(v,u)),
\]

where

\[
m(v,u) = \max\{d(Av,Bu), d(Sv,Av), d(Tu,Bu), \frac{1}{2}(d(Sv,Bu) + d(Tu,Av))\}
\]

\[
= \max\{d(u,Tu), d(u,u), 0, \frac{1}{2}(d(u,u) + d(Tu,u))\}
\]

\[
= d(u,Tu).
\]

Thus, we have

\[
\psi(d(u,Tu)) \leq \psi(d(u,Tu)) - \phi(d(u,Tu))
\]

\[
< \psi(d(u,Tu)),
\]

which is a contradiction. So, \( Tu = u \). Similarly, \( Su = u \). Thus, we get \( Au = Su = Bu = Tu = u \). Hence \( u \) is the common fixed point of \( A, B, S \) and \( T \). For the uniqueness, let \( z \) be another common fixed point of \( A, B, S \) and \( T \).

Now, we claim that \( u = z \). Let, if possible, \( u \neq z \).

From (2.2), we have

\[
\psi(d(u,z)) = \psi(d(Su,Tz)) \\
\leq \psi(m(u,z)) - \phi(m(u,z)) \\
= \psi(d(u,z)) - \phi(d(u,z)),
\]

since

\[
m(u,z) = d(u,z) \\
< \psi(d(u,z)),
\]

a contradiction. Thus, \( u = z \), and the uniqueness follows.

**Theorem 2.2.** Let \( A, B, S \) and \( T \) be self mappings of a metric space \((X,d)\) satisfying (1), (2) and the followings:

- pairs \((A,S)\) and \((B,T)\) are weakly compatible, \((2.21)\)
- pair \((A,S)\) or \((B,T)\) satisfy the E.A. property. \((2.22)\)

If any one of \( AX, BX, SX \) and \( TX \) is a complete subspace of \( X \), then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof:** Suppose that \((A,S)\) satisfies the E.A. property. Then there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z \), for some \( z \) in
X. Since $SX \subseteq BX$, there exists a sequence \( \{y_n\} \) in \( X \) such that \( Sx_n = By_n \). Hence \( \lim_{n \to \infty} By_n = z \). We shall show that \( \lim_{n \to \infty} Ty_n = z \). Let, if possible, \( \lim_{n \to \infty} Ty_n = t = z \).

From (2.2), we have
\[
\psi(d(Sx, Ty)) \leq \psi(m(x, y)) - \phi(m(x, y)).
\]
Letting limit as \( n \to \infty \), we have
\[
\lim_{n \to \infty} \psi(d(Sx, Ty)) \leq \lim_{n \to \infty} \psi(m(x, y)) - \lim_{n \to \infty} \phi(m(x, y)),
\]
where,
\[
\lim_{n \to \infty} m(x, y_n) = \lim_{n \to \infty} \max \{d(Ax_n, By_n), d(Sx_n, Ax_n), d(Ty_n, By_n), \frac{1}{2}(d(Sx_n, By_n) + d(Ty_n, Ax_n))\}
\]
\[
= \max \{d(z, z), d(z, z), d(t, z), \frac{1}{2}(d(z, z) + d(t, z))\}
\]
\[
= d(t, z).
\]
Thus, from (2.23), we get
\[
\psi(d(z, t)) \leq \psi(d(z, t)) - \phi(d(z, t))
\]
\[
< \psi(d(z, t)),
\]
which is a contradiction. Therefore, \( t = z \), that is, \( \lim_{n \to \infty} Ty_n = z \). Suppose that \( BX \) is a complete subspace of \( X \). Then \( z = Bu \) for some \( u \) in \( X \). Subsequently, we have
\[
\lim_{n \to \infty} Ty_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} By_n = z = Bu.
\]
Now, we shall show that \( Tu = Bu \). Let, if possible, \( Tu \neq Bu \).

From (2.2), we have
\[
\psi(d(Sx, Tu)) \leq \psi(m(x, u)) - \phi(m(x, u)).
\]
Letting limit as \( n \to \infty \), we have
\[
\lim_{n \to \infty} \psi(d(Sx, Tu)) \leq \lim_{n \to \infty} \psi(m(x, u)) - \lim_{n \to \infty} \phi(m(x, u)),
\]
where
\[
\lim_{n \to \infty} m(x, u) = \lim_{n \to \infty} \max \{d(Ax_n, Bu), d(Sx_n, Ax_n), d(Tu, Bu), \frac{1}{2}(d(Sx_n, Bu) + d(Tu, Ax_n))\}
\]
\[
= \max \{d(z, z), d(z, z), d(Tu, z), \frac{1}{2}(d(z, z) + d(Tu, z))\}
\]
\[
= d(Tu, z).
\]
Thus, from (2.24), we have
\[
\psi(d(z, Tu) \leq \psi(d(z, Tu)) - \phi(d(z, Tu))
\]
\[
<\psi(d(z, Tu)),
\]
which is a contradiction. Therefore, \( Tu = z = Bu \). Since \( B \) and \( T \) are weakly compatible, therefore, \( BTu = TBu \), implies that, \( TTu = TBu = BTu = BBu \).

Since \( TX \subseteq AX \), there exists \( v \in X \), such that, \( Tu = Av \).

Now, we claim that \( Av = Sv \). Let, if possible, \( Av \neq Sv \).

from (2.2), we have
\[
\psi(d(Sv, Tu) \leq \psi(m(v, u)) - \phi(m(v, u)),
\]
(2.25)

where
\[
m(v, u) = \max \{d(Av, Bu), d(Sv, Av), d(Tu, Bu), \frac{1}{2}(d(Sv, Bu) + d(Tu, Av))\}
\]
\[
=d(Sv, Av) = d(Sv, Tu).
\]

Thus, from (2.25), we have
\[
\psi(d(Sv, Tu) \leq \psi(d(Sv, Tu)) - \phi(d(Sv, Tu))
\]
\[
<\psi(d(Sv, Tu)),
\]
which is a contradiction. Therefore, \( Sv = Tu = Av \). The weak compatibility of \( A \) and \( S \) implies that \( ASv = SAv = SSv = AAu \).

Now, we claim that \( Tu \) is the common fixed point of \( A, B, S \) and \( T \). Suppose that, \( TTu \neq Tu \).

from (2.2), we have
\[
\psi(d(Tu, TTu) = \psi(d(Tu, TTu))
\]
\[
\leq \psi(m(v, T u)) - \psi(m(v, T u)),
\]
(2.26)

where
\[
m(v, Tu) = \max \{d(Av, BTu), d(Sv, Av), d(BTu, TTu),
\]
\[
\frac{1}{2}(d(Sv, BTu) + d(TTu, Av))\}
\]
\[
= \max \{d(Tu, TTu), 0, 0, d(Tu, TTu)\}
\]
\[
=d(Tu, TTu).
\]

Thus, from (2.26), we have
\[
\psi(d(Tu, TTu) \leq \psi(d(Tu, TTu)) - \phi(d(Tu, TTu))
\]
\[
<\psi(d(Tu, TTu)),
\]
which is a contradiction. Therefore, \( Tu = TTu = BTu \). Hence \( Tu \) is the common fixed point of \( B \) and \( T \). Similarly, we prove that \( Sv \) is the common fixed point of
A and S. Since \( Tu = Sv \), \( Tu \) is the common fixed point of \( A, B, S \) and \( T \). The proof is similar when \( AX \) is assumed to be a complete subspace of \( X \). The cases in which or \( SX \) is a complete subspace of \( X \) are similar to the cases in which \( AX \) or \( BX \), respectively is complete subspace of \( X \), since \( TX \subseteq AX \) and \( SX \subseteq BX \).

Now, we shall prove that the common fixed point is unique. If possible, let \( p \) and \( q \) be two common fixed points of \( A, B, S \) and \( T \), such that, \( p \neq q \).

from (2.2), we have
\[
\psi(d(p, q)) = \psi(d(Sp, Tq)) 
\leq \psi(m(p, q)) - \phi(m(p, q)),
\]
(2.27)
where
\[
m(p, q) = \max\{d(Ap, Bq), d(Sp, Aq), d(Bq, Tq), \frac{1}{2}(d(Sp, Bq) + d(Tq, Ap))\}
\]
\[
= \max\{d(p, q), 0, 0, d(p, q)\}
= d(p, q).
\]
Thus, from (2.27), we have
\[
\psi(d(p, q)) \leq \psi(d(p, q)) - \phi(d(p, q))
< \psi(d(p, q)),
\]
which is a contradiction. Therefore, \( p = q \), and the uniqueness follows.

**Theorem 2.3.** Let \( A, B, S \) and \( T \) be self maps of a metric space \( (X, d) \) satisfying (2), (21) and the following:
\[
SX \subseteq BX \quad \text{and the pair } (A, S) \text{ satisfies } (CLR_A) \text{ property or } \quad TX \subseteq AX \quad \text{and the pair } (B, T) \text{ satisfies } (CLR_B) \text{ property.}
\]
(2.28)
Then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof:** Without loss of generality, assume that \( SX \subseteq BX \) and the pair \( (A, S) \) satisfies \( (CLR_A) \) property, then there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = Ax \), for some \( x \) in \( X \). Since \( SX \subseteq BX \), there exists a sequence \( \{y_n\} \) in \( X \) such that \( Sx_n = By_n \). Hence \( \lim_{n \to \infty} By_n = Ax \). We shall show that \( \lim_{n \to \infty} Ty_n = Ax \). Let, if possible, \( \lim_{n \to \infty} Ty_n = z \neq Ax \).

from (2.2), we have
\[
\psi(d(Sx_n, Ty_n)) \leq \psi(m(x_n, y_n)) - \phi(m(x_n, y_n)).
\]
Letting limit as \( n \to \infty \), we have
\[
\lim_{n \to \infty} \psi(d(Sx_n, Ty_n)) \leq \lim_{n \to \infty} \psi(m(x_n, y_n)) - \lim_{n \to \infty} \phi(m(x_n, y_n)),
\]
(2.29)
where
\[
\lim_{n \to \infty} m(x_n, y_n) = \lim_{n \to \infty} \left[ \max\{d(Ax_n, By_n), d(Sx_n, Ax_n), d(Ty_n, By_n), \frac{1}{2}(d(Sx_n, By_n) + d(Ty_n, Ax_n))\} \right] \\
= \max\{d(Ax, Ax), d(Ax, Ax), d(z, Ax), \frac{1}{2}(d(z, z) + d(z, Ax))\} \\
= d(z, Ax).
\]
Thus, from (2.29), we get
\[
\psi(d(Ax, z) \leq \psi(d(Ax, z)) - \phi(d(Ax, z)) < \psi(d(Ax, z)),
\]
which is a contradiction. Therefore, \(Ax = z\), that is, \(\lim_{n \to \infty} Ty_n = Ax\). Subsequently, we have
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = Ax = z.
\]
Now, we shall show that \(Sx = z\). Let, if possible, \(Sx \neq z\). from (2.2), we have
\[
\psi(d(Sx, Ty_n) \neq \psi(m(x, y_n)) - \phi(m(x, y_n)).
\]
Letting limit as \(n \to \infty\), we have
\[
\lim_{n \to \infty} \psi(d(Sx, Ty_n) \leq \lim_{n \to \infty} \psi(m(x, y_n)) - \lim_{n \to \infty} \phi(m(x, y_n)),
\]
where
\[
\lim_{n \to \infty} m(x, y_n) = \lim_{n \to \infty} \left[ \max\{d(Ax, By_n), d(Sx, Ax), d(Ty_n, By_n), \frac{1}{2}(d(Sx, By_n) + d(Ty_n, Ax_n))\} \right] \\
= \max\{d(z, z), d(Sx, z), d(z, z), \frac{1}{2}(d(Sx, z) + d(z, z))\} \\
= d(Sx, z).
\]
Thus, from (2.30), we get
\[
\psi(d(Sx, z) \leq \psi(d(Sx, z)) - \phi(d(Sx, z)) < \psi(d(Sx, z)),
\]
which is a contradiction. Therefore, \(Sx = z = Ax\). Since, the pair \((A, S)\) is weakly compatible, it follows that \(Az = Sz\). Also, since \(SX \subseteq BX\), there exists some \(y\) in \(X\) such that \(Sx = By\), that is, \(By = z\). Now, we show that \(Ty = z\). Let, if possible, \(Ty \neq z\).
from (2.2), we have
\[ \psi(d(Sx, Ty) \leq \psi(m(x, y)) - \phi(m(x, y)). \]

Letting limit as \( n \to \infty \), we have
\[ \lim_{n \to \infty} \psi(d(Sx, Ty) \leq \lim_{n \to \infty} \psi(m(x, y)) - \lim_{n \to \infty} \phi(m(x, y)), \tag{2.31} \]

where
\[ \lim_{n \to \infty} m(x, y) = \lim_{n \to \infty} [\max\{d(Ax, By), d(Sx, Ax), d(Ty, By), \]
\[ \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\}]
\[ = \max\{d(z, z), d(z, z), d(z, Ty), \frac{1}{2}(d(z, z) + d(Ty, z))\}
\[ = d(z, Ty). \]

Thus, from (2.31), we get
\[ \psi(d(z, Ty) \leq \psi(d(z, Ty)) - \phi(d(z, Ty)) \]
\[ \leq \psi(d(z, Ty)), \]

which is a contradiction. Thus, \( z = Ty = By \). Since the pair \( (B, T) \) is weakly compatible, it follows that \( Tz = Bz \). Now, we claim that \( Sz = Tz \). Let, if possible, \( Sz \neq Tz \).

from (2.2), we have
\[ \psi(d(Sz, Tz) \leq \psi(m(z, z)) - \phi(m(z, z)), \tag{2.32} \]

where
\[ m(z, z) = \max\{d(Az, Bz), d(Sz, Az), d(Bz, Tz), \frac{1}{2}(d(Sz, Bz) + d(Tz, Az)\}
\[ = d(Sz, Tz). \]

Thus, from (2.32), we have
\[ \psi(d(Sz, Tz) \leq \psi(d(Sz, Tz)) - \phi(d(Sz, Tz)) \]
\[ < \psi(d(Sz, Tz)), \]

which is a contradiction. Therefore, \( Sz = Tz \), that is, \( Az = Sz = Tz = Bz \). Now, we shall show that \( z = Tz \). Let, if possible, \( z \neq Tz \).

from (2.2), we have
\[ \psi(d(Sx, Tz) \leq \psi(m(x, z)) - \phi(m(x, z)), \tag{2.33} \]
where
\[
m(x, z) = \max\{d(Ax, Bz), d(Sx, Ax), d(Bz, Tz), \frac{1}{2}(d(Sx, Bz) + d(Tz, Ax))\}
\]
\[
= d(Sx, Tz) = d(z, Tz).
\]

Thus, from (2.33), we have
\[
\psi(d(z, Tz)) \leq \psi(d(z, Tz)) - \phi(d(z, Tz))
\]
\[
< \psi(d(z, Tz)),
\]
which is a contradiction. Therefore, \( z = Tz = Bz = Az = Sz \). Hence \( z \) is the common fixed point of \( A, B, S \) and \( T \). Now, we shall prove that the common fixed point is unique. Let \( u \) be another common fixed point of \( A, B, S \) and \( T \). Let, if possible, \( z \neq u \).

from (2.2), we have
\[
\psi(d(u, z)) = \psi(d(Su, Tz))
\]
\[
\leq \psi(m(u, z)) - \phi(m(u, z))
\]
\[
= \psi(d(u, z)) - \phi(d(u, z)), \text{ since } m(u, z) = d(u, z)
\]
\[
< \psi(d(u, z)),
\]
which is a contradiction. Thus, \( u = z \), and hence the uniqueness follows.

Example 2.4. Let \( X = [0, 1] \) be endowed with the Euclidean metric \( d(x, y) = |x - y| \). Let the self maps \( A, B, S \) and \( T \) be defined by
\[
Sx = \frac{x}{8}, Bx = \frac{x}{4}, Tx = \frac{x}{2}, Ax = x.
\]

Clearly,
\[
SX = [0, \frac{1}{8}] \subseteq [0, \frac{4}{1}] = BX,
\]
\[
TX = [0, \frac{1}{2}] \subseteq [0, 1] = AX.
\]

Also \( AX \) is complete subspace of \( X \) and pairs \((A, S), (B, T)\) are weakly compatible. Now,
\[
d(Sx, Ty) = |\frac{x}{8} - \frac{y}{2}| = \frac{x}{8}|x - 4y|.
\]
\[
d(Ax, By) = |x - \frac{y}{4}| = \frac{1}{4}|4x - y|.
\]
\[
d(Sx, Ax) = |\frac{x}{8} - x| = \frac{7}{8}x.
\]
\[
d(By, Ty) = |\frac{y}{4} - \frac{y}{2}| = \frac{y}{4}.
\]
\[
\frac{(d(Sx, By) + d(Ty, Ax))}{2} = \frac{1}{2}[|\frac{x}{8} - \frac{y}{4}| + |\frac{y}{2} - x|]
\]
\[
= \frac{1}{16}|x - 2y| + 4|y - 2x|.
\]
Let $\psi(t) = \frac{t}{3}$ and $\phi(t) = \frac{t}{6}$. Thus, we have
\[
\psi(d(Sx, Ty)) = \frac{1}{24}|x - 4y|.
\]

\[
m(x, y) = \max\{d(Ax, By), d(Sx, Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\}
\]
\[= d(Sx, Ax).
\]
Therefore,
\[
\psi(d(Sx, Ax)) = \frac{1}{3}\left(\frac{7}{8}\right) = \frac{7}{24}x,
\]
\[
\phi(d(Sx, Ax)) = \frac{1}{6}\left(\frac{7}{8}\right) = \frac{7}{48}x.
\]
Thus, we have
\[
\psi(m(x, y)) - \phi(m(x, y)) = \frac{7}{24}x - \frac{7}{48}x = \frac{7}{48}x.
\]
Therefore,
\[
\psi(d(Sx, Ty)) \leq \psi(m(x, y)) - \phi(m(x, y)).
\]
Hence condition (2) is satisfied. If, we consider the sequence $\{x_n\} = \{\frac{1}{n}\}$, then
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{n} = 0.
\]
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} x_{\frac{n}{8}} = \lim_{n \to \infty} \frac{1}{8n} = 0.
\]
Therefore,
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = 0, \text{ where } 0 \in X.
\]
So the pair $(A, S)$ satisfies the $E.A.$ property. Also,
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Ax_n = 0 = A(0).
\]
So the pair $(A, S)$ satisfies the $(CLR_A)$ property. Hence all the conditions of above Theorems are satisfied. Here 0 is the unique common fixed point of $A, S, B$ and $T$.

References


Manoj Kumar (Corresponding Author),
Department of Mathematics,
School of Physical Sciences,
Starex University, Gurugram,
India.
E-mail address: manojantil18@gmail.com, manoj.19564@lpu.co.in

and

Rashmi Sharma,
Department of Mathematics,
Lovely Professional University, Phagwara, Punjab,
India.
E-mail address: rashmisharma.lpu@gmail.com

and

Serkan Araci,
Faculty of Economics,
Administrative and Social Sciences, Department of Economics,
Hasan Kalyoncu University, TR-27410 Gaziantep,
Turkey.
E-mail address: mtsrk@hotmail.com