Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) v. ???? (??) : 1–9.
ISSN-0037-8712 in press
doi:10.5269/bspm.64370

Orbits of Random Dynamical Systems

Ahmed Zaou, Otmane Benchiheb and Mohamed Amouch

Abstract: In this paper, we introduce and study the notions of hypercyclicity and transitivity for random dynamical systems and we establish the relation between them in a topological space. We also introduce the notions of mixing and weakly mixing for random dynamical systems and give some of their properties.

Key Words: Hypercyclicity, topological transitivity, Orbit, random dynamical system.

Contents

1. Introduction

Throughout the paper, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ will denote the set of positive integers while $\mathbb{N} = \{1, 2, 3, \dots\}$ will be the set of nonzero positive integers.

Let X be an F-space that is a complete and metrizable topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let T be a continuous linear operator (operator for short) acting on X. If x is vector of X, then the orbit of x under T is the set denoted by $\text{Orb}(T, x)$ and defined by

$$\text{Orb}(T, x) := \{T^n x : n \in \mathbb{N}_0\}.$$

We say that T is hypercyclic if there exists a vector $x \in X$ whose orbit under T is dense in X. In this case, the vector x is called a hypercyclic vector for T. We denote by $HC(T)$ the set of all hypercylic vectors for T. The first example of a hypercyclic operator in the Banach space setting was given by Rolewicz [20], who proved that if $\lambda \in \mathbb{C}$; $|\lambda| > 1$, then λB is hypercyclic, where B is the unilateral backward shift with weights constantly equal to 1. Rolewicz also proved that there are no hypercyclic operators on finite-dimensional space. Thus hypercyclicity is an infinite-dimensional phenomenon. If the space X is a separable space, then the hypercyclicity is equivalent to the notion of topological transitivity, that is; for any pair (U, V) of nonempty and open sets of X, there exists a positive integer n such that

$$T^n(U) \cap V \neq \emptyset.$$

In this case, the set $HC(T)$ is a dense G_δ subset of X, see [12].

A useful general criterion for hypercyclicity was isolated by C. Kitai in a restricted form [14] and then by R. Gethner and J. H. Shapiro in a form close to that given below [16]. The version used here appeared in the Ph.D. thesis of J. Bes [10]: we say that T satisfies the hypercyclicity criterion if there exist an increasing sequence of integers (n_k), two dense sets X_0, $Y_0 \subset X$ and a sequence of maps $S_{n_k} : Y_0 \longrightarrow X$ such that:

(1) $T^{n_k} x \longrightarrow 0$ for any $x \in X_0$;

(2) $S_{n_k} y \longrightarrow 0$ for any $y \in Y_0$;

(3) $T^{n_k} S_{n_k} y \longrightarrow y$ for any $y \in Y_0$.

2010 *Mathematics Subject Classification:* Primary: 47A16, 37A25 Secondary: 37A50, 60J10. .
Submitted July 15, 2022. Published July 28, 2022

Such an operator satisfying the hypercyclicity criterion is hypercyclic, see [10].

It is known that if $T \oplus S$ is hypercyclic on $X \oplus Y$, then T is hypercyclic on X and S is hypercyclic on Y. The converse is not true even in the case $T = S$ see [13]. From [17] if $T \oplus T$ is topologically transitive, then the operator T is called weakly mixing, i.e, $T \oplus T$ is hypercyclic. Clearly a weakly mixing operator is hypercyclic. Moreover, the following are equivalent:

(1) T satisfies the hypercyclicity criterion;

(2) T is hereditarily hypercyclic with respect to an increasing sequence of positive integers (n_k), that is, for any subsequence (m_k) of (n_k), the sequence $(T^{m_k})_{k\in\mathbb{N}_0}$ is hypercyclic;

(3) T is weakly mixing,

see [11]. The notions of hypercyclicity and supercyclicity are well studied in the last few years, see for example K.G. Grosse-Erdmann and A. Peris's book [17] and F. Bayart and E. Matheron's book [9], and the survey article [18] by K.G. Grosse-Erdmann, and the book [15] by Kostić. In [1,2,3,4,5,6,7,8] the authors have studied the dynamics of a set of operators instead of a single operator. In this paper, we introduce the notions of hyperciclycity, topological transitivity, and topological mixing of random dynamical systems and we study some of their properties.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{T} = \{T_\omega : X \longrightarrow X, \omega \in \Omega\}$ is a collection of measurable maps on a Polish space X. We will refer to $(\Omega, \mathcal{F}, \mathcal{T})$ as random dynamical system and we denote it in the following by $\mathcal{T}$.
By taking, $T^n_{\underline{\omega}} = T_{\omega_n} \circ \cdots \circ T_{\omega_1}$ for any $\underline{\omega} = (\omega_1, \omega_2, \dots) \in \Omega^{\mathbb{N}_0}$, we can relate this random dynamical system to a deterministic dynamical system obtained by defining the following skew-product transformation:

$$\begin{array}{cccc} S & : & \Omega^{\mathbb{N}_0} \oplus X & \longrightarrow & \Omega^{\mathbb{N}_0} \oplus X \\ & & (\underline{\omega}, x) & \longmapsto & (\sigma\underline{\omega}, T_{\omega_1} x), \end{array}$$

where $\sigma : \Omega^{\mathbb{N}_0} \mapsto \Omega^{\mathbb{N}_0}$ is the unilateral shift. It is clear that $S^n(\underline{\omega}, x) = (\sigma^n \underline{\omega}, T^n_{\underline{\omega}} x)$, for any $n \in \mathbb{N}_0$. A probability measure μ on X is stationary if and only if the measure $\mathbb{P}^{\oplus\mathbb{N}_0} \oplus \mu$ is invariant under S that is $S^*(\mathbb{P}^{\oplus\mathbb{N}_0} \oplus \mu) = \mathbb{P}^{\oplus\mathbb{N}_0} \oplus \mu$, see [19].

Hereinafter, X will be a topological space and $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a collection of continuous functions that map X into itself. In this case, the orbit of a point $x \in X$ at some $\underline{\omega} \in \Omega^{\mathbb{N}}_0$ of this random dynamical system is defined by

$$\operatorname{Orb}(x, \mathcal{T}) = \{T^n_{\underline{\omega}} x : n \in \mathbb{N}_0\},$$

where $T^0_{\underline{\omega}} x = x$.

2. Hypercyclic and Topologically Transitive Random Dynamical Systems

In the following, we define the notion of hypercyclicity for a random dynamical system.

Definition 2.1. *Let X be a toplogical space. We say that a random dynamical system $\mathcal{T}$ is hypercyclic on X if there exists $x \in X$ and $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ such that*

$$\overline{\operatorname{Orb}(x, \mathcal{T})} = X.$$

In such a case, x is called a hypercyclic point for $\mathcal{T}$, and the set of hypercyclic points for $\mathcal{T}$ is denoted by HC($\mathcal{T}$).

Remark 2.2. *Let X be a topological space, and $T : X \to X$ be a continuous map on X. If we take $T_\omega = T$ for any $\omega \in \Omega$, then $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ is hypercyclic if only if T is hypercyclic.*

Example 2.3. *We pose $X = [0, 1]$ and $\Omega = \{1, 2\}$, and we consider the maps:*

$$\begin{array}{cccc} T_1 & : & X & \longrightarrow & X \\ & & x & \longmapsto & \begin{cases} 2x & if\ x \in [0, \frac{1}{2}] \\ 2 - 2x & if\ x \in]\frac{1}{2}, 1] \end{cases} \end{array},$$

and

$$\begin{array}{rcccl} T_2 & : & X & \longrightarrow & X \\ & & x & \longmapsto & x+\alpha (mod\ 1), \end{array}$$

where $\alpha \in [0,1[$. *There exists* $x \in X$ *such that* $\overline{\{T_1^n x, n \in \mathbb{N}_0\}} = X$. *Let* $\underline{\omega} = (1,1,1,\dots)$, *then*

$$\overline{\{T_{\underline{\omega}}^n x, n \in \mathbb{N}_0\}} = X.$$

Hence, $\mathcal{T} = \{T_1, T_2\}$ *is hypercyclic on* X.

In the following definition, we introduce the notion of quasi-conjugate for a random dynamical system.

Definition 2.4. *Let* X *and* Y *be topological spaces,* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be random dynamical systems on* X *and* Y *respectivly.* $\mathcal{T}$ *is called quasiconjugate to* $\mathcal{S}$ *if there exists a continuous map* $\phi : Y \to X$ *with dense range such that for all* $\omega \in \Omega$, $T_\omega \circ \phi = \phi \circ S_\omega$. *If* ϕ *can be chosen to be a homeomorphism then* $\mathcal{S}$ *and* $\mathcal{T}$ *are called conjugate.*

The property of hypercylicity of a dynamical system is preserved under quasiconjugacy, see [10, Proposition 1.19]. The following proposition proves that the same result holds for a random dynamical system.

Proposition 2.5. *Let* X *and* Y *be topological spaces,* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be random dynamical systems on* X *and* Y *respectively, such that* $\mathcal{T}$ *is quasiconjugate to* S *with respect to* ϕ. *If* $\mathcal{S}$ *is hypercyclic on* Y, *then* $\mathcal{T}$ *is hypercyclic on* X. *Furthermore,*

$$\phi(HC(\mathcal{S})) \subset HC(\mathcal{T}).$$

Proof. Suppose that $\mathcal{S}$ is hypercyclic, then there exists some $x \in Y$ and $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ such that $\{S_{\underline{\omega}}^n x, n \in \mathbb{N}_0\}$ visits every nonempty open subset of Y. Let U be a nonempty open subset of X, then $\phi^{-1}(U)$ is a nonempty open subset of Y, implies that there exists some $n \in \mathbb{N}_0$ such that $S_{\underline{\omega}}^n x \in \phi^{-1}(U)$. This implies that $T_{\underline{\omega}}^n(\phi(x)) \in U$. Thus,

$$\overline{\{T_{\underline{\omega}}^n \phi(x) : n \in \mathbb{N}_0\}} = X.$$

Hence, $\mathcal{T}$ is hypercyclic and $\phi(x) \in HC(\mathcal{T})$. □

Corollary 2.6. *Let* X *and* Y *be topological spaces,* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be random dynamical systems on* X *and* Y *respectively, such that* $\mathcal{T}$ *is conjugate to* S *with respect to* ϕ. *Then* $\mathcal{T}$ *is hypercyclic on* X *if only if* $\mathcal{S}$ *is hypercyclic on* Y. *Furthermore,*

$$\phi(HC(\mathcal{S})) = HC(\mathcal{T})$$

Let $\{X\}_{i=1}^p$ be a family of topological spaces and let $\mathcal{T}_i = \{T_{i,\omega} : \omega \in \Omega\}$ be a random dynamical system on X_i, for all $i = 1,2,\dots,p$. Let

$$\oplus_{i=1}^p X_i = X_1 \oplus X_2 \oplus \cdots \oplus X_p = \{(x_1, x_2, \dots, x_p) : x_i \in X_i,\ i = 1,2,\dots,p\}$$

and define the random dynamical system $\oplus_{i=1}^p \mathcal{T}_i = \{(\oplus_{i=1}^p T_i)_\omega, \omega \in \Omega\}$ on $\oplus_{i=1}^p X_i$ by,

$$(\oplus_{i=1}^p T_i)_\omega : \oplus_{i=1}^p X_i \to \oplus_{i=1}^p X_i,\ (x_1, x_2, \dots, x_p) \mapsto (T_{1,\omega}x_1, T_{2,\omega}x_2, \dots, T_{p,\omega}x_p).\ (\forall \omega \in \Omega)$$

Remark 2.7. *For all* $\underline{\omega} \in \Omega^{\mathbb{N}_0}$, *for all* $n \in \mathbb{N}_0$, *and for all* $(x_1, x_2, \dots, x_p)$,

$$(\oplus_{i=1}^n T_i)_{\underline{\omega}}^n (x_1, x_2, \dots, x_p) = (T_{1,\underline{\omega}}^n x_1, T_{2,\underline{\omega}}^n x_2, \dots, T_{p,\underline{\omega}}^n x_p).$$

Proposition 2.8. *Let* $\{X\}_{i=1}^p$ *be a family of topological spaces and let* $\mathcal{T}_i = \{T_{i,\omega} : \omega \in \Omega\}$ *be a random dynamical system on* X_i *for all* $i = 1,2,\dots,p$. *If* $\oplus_{i=1}^p \mathcal{T}_i$ *is hypercyclic on* $\oplus_{i=1}^p X_i$, *then* $\mathcal{T}_i$ *is hypercyclic in* X_i *for all* $i = 1,2,\dots,p$. *Moreover if* $(x_1, x_2, \dots, x_p) \in HC(\oplus_{i=1}^p \mathcal{T}_i)$, *then* $x_i \in HC(\mathcal{T}_i)$ *for all* $i = 1,2,\dots,p$.

Proof. Suppose that $\oplus_{i=1}^{p}\mathcal{T}_i$ is hypercyclic on $\oplus_{i=1}^{p}X_i$. Let $(x_1,x_2,\dots,x_p)\in HC(\oplus_{i=1}^{p}\mathcal{T}_i)$, then there exists $\underline{\omega}\in\Omega^{\mathbb{N}_0}$ such that

$$\overline{\{(\oplus_{i=1}^{p}T_i)^n_{\underline{\omega}}(x_1,x_2,\dots,x_p), n\in\mathbb{N}_0\}}=\oplus_{i=1}^{p}X_i,$$

For all $i=1,2,\dots,n$, let U_i be a nonempty open subset of X_i, then $U_1\oplus U_2\oplus\cdots\oplus U_p$ is a nonempty open subset of $\oplus_{i=1}^{p}X_i$, implies that there exists some $p\in\mathbb{N}_0$ such that

$$(\oplus_{i=1}^{p}T_i)^n_{\underline{\omega}}(x_1,x_2,\dots,x_p)=(T^n_{1,\underline{\omega}}x_1,T^n_{2,\underline{\omega}}x_2,\dots,T^n_{p,\underline{\omega}}x_p)\in U_1\oplus U_2\oplus\cdots\oplus U_p,$$

that is $T^p_{i,\underline{\omega}}x_i\in U_i$ for all $i=1,2,\dots,p$, it follows that

$$\overline{\{T^n_{i,\underline{\omega}}x_i, n\in\mathbb{N}_0\}}=X_i,$$

Hence $\mathcal{T}_i$ is hypercyclic in X_i and $x_i\in HC(\mathcal{T}_i)$, for all $i=1,2,\dots,p$.

□

Remark 2.9. *The converse of Proposition 2.8 is not true in general. Indeed, let $X=\{z\in\mathbb{C}:|z|=1\}$ and $\Omega=\{0,1\}$. We consider the maps $T_0:X\to X$, $z\mapsto e^{i\alpha}z$, where α is irrational in $[0,2\pi[$, and $T_1=Id_X$. There exists $x\in X$, such that $\overline{\{T_0^n x, n\in\mathbb{N}_0\}}=X$, see [17]. Take $\underline{\omega}=(0,0,0,\dots)$, then $\overline{\{T^n_{\underline{\omega}}x, n\in\mathbb{N}_0\}}=X$, implies that, the random dynamical system $\mathcal{T}=\{T_\omega\}_{\omega\in\Omega}$ is hypercyclic. But $\mathcal{T}\oplus\mathcal{T}$ is not hypercyclic.*

In the following definition, we introduce the notion of topological transitivity for a random dynamical system.

Definition 2.10. *Let X be a topological space, and $\mathcal{T}=\{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X. We say that $\mathcal{T}$ is topologically transitive on X if: for any U and V nonempty open subsets of X, there exists $\underline{\omega}\in\Omega^{\mathbb{N}_0}$ and $n\in\mathbb{N}_0$, such that*

$$T^n_{\underline{\omega}}(U)\cap V\neq\varnothing.$$

Remark 2.11. *Let X be a topological space, and $T:X\to X$ be a continuous map on X. Take $T_\omega=T$ for any $\omega\in\Omega$. Then $\{T_\omega\}_{\omega\in\Omega}$ is topologically transitive on X if only if T is a topologically transitive operator on X.*

Example 2.12. *Let $X=\{x\in\mathbb{C}:|x|=1\}$ and $\Omega=\{0,1\}$. Consider the maps: $T_0:X\to X$, $x\mapsto e^{i\alpha}x$, where $\alpha\in\mathbb{R}-\mathbb{Q}$ and $T_1:X\to X$, $x\mapsto T_1(x)=x^2$. For any U and V nonempty open subsets of X, there exists some $n\in\mathbb{N}_0$ such that $T_1^n(U)\cap V\neq\varnothing$. Take $\underline{\omega}=(1,1,1,\dots)$, then for any pair (U,V) of nonempty open subsets of X there exists some $n\in\mathbb{N}_0$, such that*

$$T^n_{\underline{\omega}}(U)\cap V\neq\varnothing.$$

Thus, the random dynamical $\mathcal{T}=\{T_0,T_1\}$ is topologically transitive on X.

The topological transitivity of a dynamical system is preserved under quasiconjugacy, see [10]. The following proposition proves that the same result holds for a random dynamical system.

Proposition 2.13. *Let X and Y be topological spaces. Let $\mathcal{T}=\{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S}=\{S_\omega\}_{\omega\in\Omega}$ be random dynamical systems on X and Y respectivly, such that $\mathcal{T}$ is quasiconjugate to S. If $\mathcal{S}$ is topologically transitive on Y, then $\mathcal{T}$ is topologically transitive on X.*

Proof. Suppose that $\mathcal{S}$ is topologically transitive. Let U and V be nonempty open subsets of X, then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are nonempty and open of Y. Hence there exists $\underline{\omega}\in\Omega^{\mathbb{N}_0}$ and $n\in\mathbb{N}_0$, such that

$$S^n_{\underline{\omega}}(\phi^{-1}(U))\cap\phi^{-1}(V)\neq\varnothing.$$

This implies that

$$T^n_{\underline{\omega}}(U)\cap V\neq\varnothing.$$

Thus $\mathcal{T}$ is toplogically transitive. □

Corollary 2.14. *Let X and Y be two topological spaces. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ be two random dynamical systems on X and Y respectivly, such that $\mathcal{T}$ is conjugate to S. Then $\mathcal{S}$ is topologically transitive on Y if only if $\mathcal{T}$ is topologically transitive on X.*

Proposition 2.15. *Let $\{X_i\}_{i=1}^n$ be a family of topological spaces and let $\mathcal{T}_i = \{T_{i,\omega} : \omega \in \Omega\}$ be a random dynamical system on X_i, for all $i = 1, 2, \dots, n$. If $\oplus_{i=1}^n \mathcal{T}_i$ is topologically transitive in $\oplus_{i=1}^n X_i$, then $\mathcal{T}_i$ is topologically transitive in X_i, for all $i = 1, 2, \dots, n$.*

Proof. Suppose that $\oplus_{i=1}^n \mathcal{T}_i$ is topologically transitive. Let U_i and V_i be nonempty open subsets of X_i; $1 \le i \le n$. Then, $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ and $V_1 \oplus V_2 \oplus \cdots \oplus V_n$ are nonempty open subsets of $\oplus_{i=1}^n X_i$, which implies that there exist $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $p \in \mathbb{N}_0$ such that

$$(\oplus_{i=1}^n T_i)^p_{\underline{\omega}}(U_1 \oplus U_2 \oplus \cdots \oplus U_n) \cap (V_1 \oplus V_2 \oplus \cdots \oplus V_n) \neq \varnothing$$

then

$$(T^p_{1,\underline{\omega}}(U_1) \oplus T^p_{2,\underline{\omega}}(U_2) \oplus \cdots \oplus T^p_{n,\underline{\omega}}(U_n)) \cap (V_1 \oplus V_2 \oplus \cdots \oplus V_n) \neq \varnothing,$$

it follows that

$$T^p_{i,\underline{\omega}}(U_i) \cap V_i \neq \varnothing \text{ for any } i = 1, 2, \dots, n.$$

Thus, $\mathcal{T}_i$ is topologically transitive on X_i, for all $i = 1, 2, \dots, n$. □

Remark 2.16. *The converse is not true. Let $X = \{z \in \mathbb{C} : |z| = 1\}$ and $\Omega = \{0, 1\}$. We consider the maps $T_0 : X \to X$, $z \mapsto e^{i\alpha} z$, where α is irrational in $[0, 2\pi[$, and $T_1 = Id_X$. There exists $x \in X$, such that*

$$\overline{\{T_0^n x, n \in \mathbb{N}_0\}} = X.$$

Then the random dynamical system $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ is hypercyclic on X, but $\mathcal{T} \oplus \mathcal{T}$ is not hypercyclic in $X \oplus X$.

By the Birkhoff's transitivity theorem [12], if X is a separable F-space, then a continuous map on X is hypercyclic if and only if it is topologically transitive. For $\mathcal{T}$ a random dynamical system we have the following remark. Recall that

$$\begin{array}{cccc} \sigma : & \Omega^{\mathbb{N}_0} & \longrightarrow & \Omega^{\mathbb{N}_0} \\ & (\omega_1, \omega_2, \dots) & \longmapsto & \sigma\underline{\omega} = (\omega_2, \omega_3, \dots). \end{array}$$

the full shift in $\Omega^{\mathbb{N}_0}$.

Remark 2.17. *Let X be a topological space without isolated points and $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X. It is easy to see that if $x \in HC(\mathcal{T})$ with $\underline{\omega}$, then so is every $T_{\sigma^p \underline{w}} x$ $(p \ge 1)$. As a result, we have*

$$\mathrm{Orb}(x, \mathcal{T}) \subset HC(\mathcal{T})$$

and this shows that $HC(\mathcal{T})$ is dense in X.

In the following proposition, we prove that if $\mathcal{T}$ is hypercyclic then it is topologically transitive.

Proposition 2.18. *Let X be a topological space and $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X, such that for any $\omega \in \Omega$, T_ω is a continuous map on X. If $\mathcal{T}$ is hypercyclic on X, then it is topologically transitive on X.*

Proof. Suppose that $\mathcal{T}$ is hypercyclic. Then there exists some $x \in X$ and $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ such that

$$\overline{\{T^n_{\underline{\omega}} x, n \in \mathbb{N}_0\}} = X.$$

Let U and V be two nonempty open subsets of X, then there exists some $p, n \in \mathbb{N}_0$, such that $T^p_{\underline{\omega}}x \in U$ and $T^n_{\underline{\omega}}x \in V$. Suppose that $n \geqslant p$, then $T^n_{\underline{\omega}}x = T^{n-p}_{\sigma^p\underline{\omega}} \circ T^p_{\underline{\omega}}x$, which implies that,

$$T^{n-p}_{\sigma^p\underline{\omega}}(U) \cap V \neq \varnothing.$$

Hence, $\mathcal{T}$ is topologically transitive.

□

With some additional assumptions on the topological space, the following theorem shows that we have the equivalence between the properties of hypercyclicity and topological transitivity.

Theorem 2.19. *Let X be a separable complete metric space X without isolated points. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X. Then $\mathcal{T}$ is topologically transitive on X if only if it is hypercyclic on X.*

Proof. Let $\{U_k\}_{k\geq 1}$ be a countable base for the topology of X. Then there is some $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ such that for any nonempty open set V in X and each fixed $k \geq 1$, there is some $n \geq 0$ such that

$$T^n_{\underline{\omega}}(V) \cap U_k \neq \varnothing$$

or equivalently

$$V \cap T^{-n}_{\underline{\omega}}(U_k) \neq \varnothing$$

This shows that $\bigcup_{n\geq 0} T^{-n}_{\underline{\omega}}(U_k)$ is dense in X and hence, since X is a Baire space, the set $\cap_{k\geq 1} \cup_{n\geq 0} T^{-n}_{\underline{\omega}}(U_k)$ is also dense in X. Now, if we define the set

$$D_{\underline{\omega}}(\mathcal{T}) = \{x \in X : \overline{\{T^n_{\underline{\omega}}x : n \in \mathbb{N}_0\}} = X\},$$

then it is easy to see that

$$D_{\underline{\omega}}(\mathcal{T}) = \bigcap_{k\geq 1} \bigcup_{n\geq 0} T^{-n}_{\underline{\omega}}(U_k)$$

Thus, is a dense G_δ set in X and in particular nonempty. So, $\mathcal{T}$ is hypercyclic and we are done. □

3. Topological mixing and Weakly Topological Mixing Random Dynamical Systems

In the following definition, we introduce the notion of topological mixing for a random dynamical system.

Definition 3.1. *Let X be a topological space, and $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X. We say that $\mathcal{T}$ is topologically mixing on X if, for any U and V nonempty open subsets of X, there exist $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $N \in \mathbb{N}_0$, such that*

$$T^n_{\underline{\omega}}(U) \cap V \neq \varnothing, \textit{ for all } n \geqslant N.$$

Remark 3.2. *Let X be a topological space, and $T : X \to X$ be a continuous map on X. Take $T_\omega = T$ for any $\omega \in \Omega$. Then $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ is topologically mixing on X if only if T is a topologically mixing operator on X.*

Example 3.3. *We pose $X = [0, 1]$ and $\Omega = \{0, 1\}$, and we consider the maps: $T_1 : X \to X$,*

$$x \mapsto \begin{cases} 2x & if\ x \in [0, \frac{1}{2}] \\ 2 - 2x & if\ x \in]\frac{1}{2}, 1] \end{cases}$$

and $T_2 : X \to X$,

$$x \mapsto T_2(x) = x + \alpha (mod\ 1),$$

with $\alpha \in [0, 1[$. For any U and V of nonempty open subsets of X there exists some $N \in \mathbb{N}_0$, such that

$$T_1^n(U) \cap V \neq \varnothing, \text{ for all } n \geqslant N,$$

see [17]. Take $\underline{\omega} = (1, 1, 1, \dots)$, *then for any pair* (U, V) *of nonempty open subsets of* X *there exists some* $N \in \mathbb{N}_0$, *such that* $T_{\underline{\omega}}^n(U) \cap V \neq \varnothing$, *for all* $n \geqslant N$, *hence* $\mathcal{T} = \{T_\omega\}_{\omega \in \Omega}$ *is topologically mixing on* X.

Proposition 3.4. *Let* X *and* Y *be two topological spaces. Let* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be two random dynamical systems on* X *and* Y *respectively, such that* $\mathcal{T}$ *is quasi-conjugate to* S *with respect to* ϕ. *If* $\mathcal{S}$ *is topologically mixing on* Y, *then* $\mathcal{T}$ *is topologically mixing on* X.

Proof. Suppose that $\mathcal{S}$ is topologically mixing on X. Let U and V be two nonempty open subsets of X, then $\phi^{-1}(V)$ and $\phi^{-1}(U)$ are nonempty and open in Y. Hence there exist $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $N \in \mathbb{N}_0$, such that

$$S_{\underline{\omega}}^n(\phi^{-1}(U)) \cap \phi^{-1}(V) \neq \varnothing \text{ for all } n \geqslant N,$$

which implies that

$$T_{\underline{\omega}}^n(U) \cap V \neq \varnothing \text{ for all } n \geqslant N.$$

Thus $\mathcal{T}$ is topologically mixing.

□

Corollary 3.5. *Let* X *and* Y *be two topological spaces. Let* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be two random dynamical systems on* X *and* Y *respectively, such that* $\mathcal{T}$ *is conjugate to* S. *Then* $\mathcal{T}$ *is topologically mixing on* X *if only if* $\mathcal{S}$ *is topologically mixing on* Y.

Proposition 3.6. *Let* $\{X_i\}_{i=1}^n$ *be a family of topological spaces, and let* $\mathcal{T}_i = \{T_{i,\omega} : \omega \in \Omega\}$ *be a random dynamical system on* X_i, *for all* $i = 1, 2, \dots, n$. *If* $\oplus_{i=1}^n \mathcal{T}_i$ *is topologically mixing on* $\oplus_{i=1}^n X_i$, *then* $\mathcal{T}_i$ *is topologically mixing in* X_i *for all* $i = 1, 2, \dots, n$.

Proof. Suppose that $\oplus_{i=1}^n \mathcal{T}_i$ is topologically mixing. Let U_i and V_i be nonempty open subsets of X_i; $1 \leq i \leq n$. Then $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ and $V_1 \oplus V_2 \oplus \cdots \oplus V_n$ are nonempty open subsets of $\oplus_{i=1}^n X_i$, which implies that there exists $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $N \in \mathbb{N}_0$, such that

$$(\oplus_{i=1}^n T_i)_{\underline{\omega}}^p(U_1 \oplus U_2 \oplus \cdots \oplus U_n) \cap (V_1 \oplus V_2 \oplus \cdots \oplus V_n) \neq \varnothing, \text{ for all } p \geqslant N.$$

Then

$$(T_{1,\underline{\omega}}^p(U_1) \oplus T_{2,\underline{\omega}}^p(U_2) \oplus \cdots \oplus T_{n,\underline{\omega}}^p(U_n)) \cap (V_1 \oplus V_2 \oplus \cdots \oplus V_n) \neq \varnothing, \text{ for all } p \geqslant N.$$

It follows that

$$T_{i,\underline{\omega}}^p(U_i) \cap V_i \neq \varnothing,$$

for all $p \geqslant N$, for any $i = 1, 2, \dots, n$. Thus, $\mathcal{T}_i$ is topologically mixing on X_i for all $i = 1, 2, \dots, n$.

□

In the following definition, we introduce the notion of weakly topologically mixing for a random dynamical system.

Definition 3.7. *Let* X *be a topological space. A random dynamical system* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *is called weakly topologically mixing on* X, *if* $\mathcal{T} \oplus \mathcal{T}$ *is topologically transitive on* $X \oplus X$.

Proposition 3.8. *Let* X *be a topological space, and* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *be a random dynamical system on* X. *If* $\mathcal{T}$ *is weakly topologically mixing on* X, *then it is topologically transitive on* X.

Proof. This is a consequence of Proposition 2.15. □

Remark 3.9. *Let* X *be a topological space, and* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *be a random dynamical system on* X, *then*

topologically mixing $\Rightarrow$ *weak topologically mixing* $\Rightarrow$ *topologically transitive.*

Furthermore, if X is a separable complete metric space without isolated points, then

$$\text{topologically transitive} \Leftrightarrow \text{hypercyclic.}$$

Proposition 3.10. *Let X and Y be two topological spaces. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ be two random dynamical systems on X and Y respectivly such that $\mathcal{T}$ is quasiconjugate to $\mathcal{S}$. If $\mathcal{S}$ is weakly topologically mixing on Y then $\mathcal{T}$ is weakly topologically mixing on X.*

Proof. Suppose that $\mathcal{S}$ is weakly topologically mixing on Y, then $\mathcal{S} \oplus \mathcal{S}$ is topologically transitive in X. Let $\phi : Y \to X$ be a continuous map with dense range such that for all $\omega \in \Omega$, $T_\omega \circ \phi = \phi \circ S_\omega$. Take $\psi = \phi \oplus \phi$, then ψ defines a continuous map with dense range from $Y \oplus Y$ to $X \oplus X$. Furthermore, for all $\omega \in \Omega$ we have $\psi \circ (S \oplus S)_\omega = (T \oplus T)_\omega \circ \psi$. That is $\mathcal{S} \oplus \mathcal{S}$ is quasiconjugate to $\mathcal{T} \oplus \mathcal{T}$ via ψ. Hence by Proposition (2.13), we deduce that $\mathcal{T} \oplus \mathcal{T}$ is topologically transitive on $X \oplus X$. Thus $\mathcal{T}$ is weakly topologically mixing on X.

□

Corollary 3.11. *Let X and Y be two topological spaces. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ be two random dynamical systems on X and Y respectively such that $\mathcal{T}$ is conjugate to $\mathcal{S}$. Then $\mathcal{S}$ is weakly topologically mixing on X if only if $\mathcal{T}$ is weakly topologically mixing on Y.*

Proposition 3.12. *Let X and Y be two topological spaces. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ be two random dynamical systems on X and Y respectively. If $\mathcal{T} \oplus \mathcal{S}$ is weakly topologically mixing on $X \oplus Y$, then $\mathcal{T}$ and $\mathcal{S}$ are topologically weakly mixing on X and Y respectively.*

Proof. Suppose that $\mathcal{T} \oplus \mathcal{S}$ is weakly mixing. We consider the maps, $\phi : X \oplus Y \to X$, $(x, y) \mapsto x$ and $\psi : X \oplus Y \to X$, $(x, y) \mapsto y$. For all $\omega \in \Omega$ we have $\phi \circ (T \oplus S)_\omega = T_\omega \circ \phi$ and $\psi \circ (T \oplus S)_\omega = S_\omega \circ \psi$, then $\mathcal{T}$ is quasiconjugate to $\mathcal{T} \oplus \mathcal{S}$ via ϕ and $\mathcal{S}$ is quasiconjugate to $\mathcal{T} \oplus \mathcal{S}$ via ψ. Thus, by Proposition 3.10 we deduce that $\mathcal{T}$ and $\mathcal{S}$ are weakly topologically mixing on X and Y respectively.

□

References

1. M. Amouch, O. Benchiheb, On cyclic sets of operators, Rendiconti del Circolo Matematico di Palermo Series 2, 68, 521-529, 2019.
2. M. Amouch, and O. Benchiheb. "On linear dynamics of sets of operators." Turkish Journal of Mathematics 43.1 (2019): 402-411.
3. M. Amouch, O. Benchiheb, Diskcyclicity of sets of operators and applications, Acta Mathematica Sinica, English Series. 36, 1203-1220, 2020.
4. M. Amouch, O. Benchiheb, Some versions of supercyclicity for a set of operators, Filomat, 35, 1619-1627, 2021.
5. M. Amouch, O. Benchiheb. Codiskcyclic sets of operators on complex topological vector spaces, Proyecciones Journal (2022).
6. M. Amouch, O. Benchiheb. On recurrent sets of operators. arXiv:1907.05930v3 [math.FA] 7 May 2022
7. M. Ansari, B. Khani-Robati, K. Hedayatian. On the density and transitivity of sets of operators. Turkish Journal of Mathematics 42.1, 181-189, 2018.
8. M. Ansari, K. Hedayatian, B. Khani-Robati, A. Moradi. A note on topological and strict transitivity. Iranian Journal of Science and Technology, Transactions A: Science, 42(1), 59-64, 2018.
9. F. Bayart, E Matheron. Dynamics of linear operators. 2009; New York, NY, USA, Cambridge University Press, 2009.
10. J. P. Bés. Three problem's on hypercyclic operators. Ph.D. thesis, Bowling Green State University, Bowling Green, Ohio, 1998.
11. J. P. Bés, A. Peris. Hereditarily hypercyclic operators. Journal of Functional Analysis 167.1, 94-112, 1999.
12. G. D. Birkhoff. Surface transformations and their dynamical applications. Acta Mathematica, 43, 1-119, 1922.
13. M. De La Rosa, C. Read. A hypercyclic operator whose direct sum T⊕T is not hypercyclic. Journal of Operator Theory 369-380, 2009.
14. C. Kitai. Invariant closed sets for linear operators. 1148-1148, (984.
15. M. Kostić. Chaos for linear operators and abstract differential equations. Nova Science Publishers, 2020.

16. R. M. Gethner, J. H. Shapiro. Universal vectors for operators on spaces of holomorphic functions. Proceedings of the American Mathematical Society 100.2, 281-288, 1987.

17. K-G. Grosse-Erdmann, A. Peris Manguillot. Linear chaos. Springer Science and Business Media, 2011.

18. K-G. Grosse-Erdmann. Universal families and hypercyclic operators. Bulletin of the American Mathematical Society 36.3, 345-381, 1999.

19. A. Ludwig. Trends and open problems in the theory of random dynamical systems. Probability towards 2000. Springer, New York, NY, 34-46, 1998.

20. S. Rolewicz. On orbits of elements. Studia Mathematica, 32.1, 17-22, 1969.

Ahmed Zaou, Otmane Benchiheb and Mohamed Amouch,
Department of Mathematics,
University Chouaib Doukkali,
Faculty of science,
Eljadida, Morocco .
E-mail address: ahmed.zaou1991@gmail.com
E-mail address: otmane.benchiheb@gmail.com
E-mail address: amouch.m@ucd.ac.ma