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New Algorithm on Linearization-Discretization Solving Systems of Nonlinear Integro-Differential Fredholm Equations

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ABSTRACT: This article deals with a new strategy for solving a certain type of nonlinear integro-differential Fredholm equations with a weak singular kernel. We build our new algorithm starting with the linearization phase using Newton's iterative process, then for the discretization phase we apply Kantorovich's projection method. The discretized linear scheme will be approximated by the product integration method in the weak singular terms, and the other regular integrals will be approximated by the Nyström method. The convergence process of our new algorithm is carried out under certain predefined and necessary conditions. Finally, we give practical examples where, the results show the efficiency of our new algorithm for solving systems of weakly singular nonlinear integro-differential equations.

Key Words: Nonlinear Fredholm integrals equations, Newton-Kantorovich method, system of integro-differential equations, weak singular equation, product integration rule.

Contents

1. Introduction

In recent years, several problems in mathematics, engineering, physics, and science have been formulated in terms of integral equations, particularly singular integral equations [12]. As the analytical solution of these equations is challenging to find, we need to develop numerical methods to approach their solutions. However, constricting an approximate solution for these equations is not an easy task due to the singularity of the kernel. Projection methods (Kantorovich, Galerkin, Kulkarni,...) play an essential role in numerical analysis, particular for the numerical solution of these types of integral equations [3,10,13].

Many recent studies have dealt with nonlinear integral equations with a weak singular kernel. In [5], authors study the solution existence and uniqueness of a nonlinear mixed Volterra- Fredholm integro-differential equation with a weak singular kernel. In [14], authors develop an iterative classical scheme to approach the exact solution of nonlinear Fredholm integro-differential equation with a weak singular kernel using the product integration method. In [6], authors study the existence and the uniqueness of the solution of an integro-differential nonlinear Volterra equation with a weak singular kernel.

Product integration is one of the most common methods to estimate integrals with a singular kernel. Historically, note that Young introduces the origin of this approximation method [15]. De Hoog and Weiss Latter improved this result [8]. Other works were carried out in this direction by Atkinson [4].

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Solve systems of nonlinear integral equations can achieve by two strategies:

The classical strategy (**D.L**) : It is about starting with the discretization process to get a nonlinear algebraic system, and then the linearization phase using for example Newton iteration method.

The modern strategy (**L.D**) : The inverse vision of the (D.L) method, where we started with the linearization process applying the Newton method, then the discretization phase on the linear problem obtained.

In a recent papers [7,9,11], the authors examine the equivalent of the schemes (D.L) and (L.D), where they proved that the second method is more powerful than the first because the approximate solution of the modern strategy (L.D) converges under some conditions in the discretization process to the exact solution of the main problem, but the obtained estimate solution using the classical process (D.L) converges to an approximate vector.

In this paper, we will apply the modern strategy (L.D) to find the approximate solutions to a system of integro-differential nonlinear Fredholm equations of the second kind with a weak singular kernel. In the first step, we use the classical iterate Newton method for the linearization process and then apply the Kantorovich projection method for the discretization step to get a discretized linear scheme defined by integrals, that has a weak singular kernel that will be estimated using the product integration approach, and by the Nyström method for the remaining regular integrals.

The paper is organized as follows: In sec.2, we define the construction of our main problem and his notations. In sec.3, we describe the modern strategy (L.D), In sec.4, we prove the compactness of the approximates integral operators that have weakly singular kernels. sec.5 devoted to the convergence analysis of the (L.D) new process, what is more, we define the approximate methods to estimate integrals in the obtained new scheme. In sec.6, we give numerical examples, which confirm our theoretical framework and prove the effectiveness of the (L.D) modern strategy.

2. Main Problem

We consider the regular function κ as:

$$\begin{aligned} \kappa : [0,1]^2 \times \mathbb{R}^2 &\longmapsto \mathbb{R}, \\ (t,s,\varphi(s),\varphi'(s)) &\longmapsto \kappa(t,s,\varphi(s),\varphi'(s)), \end{aligned}$$

and the kernel g verify the same conditions defined in ([5], [14]) and others as follows

$$(\mathcal{H}_1) : \left\| \begin{array}{l} (i) \lim\limits_{s\to 0^+} g'(s) = +\infty, \\ (ii)\, g \in W^{1.1}(0,1), \\ (iii)\, g'(t) \geq 0 \text{ for all } t \in]0,1], \\ (iv)\, g' \text{ is a strictly decreasing function on }]0,1]. \end{array} \right.$$

These conditions indicate that the singularity in our problem produces from the derivative g' of the function g. We define the Banach space,

$$W^{1.1}(0,1) = \left\{ p \in L^1(0,1) : p' \in L^1(0,1), p' \text{ is the weak derivative of } p \right\},$$

under the following norm [1]:

$$\|p\|_{W^{1.1}(0,1)} = \|p\|_{L^1(0,1)} + \|p'\|_{L^1(0,1)} = \int_0^1 |p(s)|ds + \int_0^1 |p'(s)|ds.$$

In this work, we trait the following nonlinear type of equations:

$$\varphi(t) = \int_0^1 g(|t-s|)\kappa(t,s,\varphi(s),\varphi'(s))ds + f(t). \quad t \in [0,1], \tag{2.1}$$

with a given function $f \in C^1([0,1,],\mathbb{R})$ and φ is the exact solution of problem (2.1) in the same space.

To get more data concerning our exact solution φ, we derive both sides of equation (2.1) as follows:

$$\varphi'(t) = \int_0^1 S_{ts}\, g'(|t-s|)\kappa(t,s,\varphi(s),\varphi'(s))\,ds + \int_0^1 g(|t-s|)\frac{\partial\kappa}{\partial t}(t,s,\varphi(s),\varphi'(s))\,ds + f'(t), \tag{2.2}$$

where

$$S_{ts} = sign(t-s) = \begin{cases} 1 & t>s, \\ -1 & t<s, \\ 0 & t=s. \end{cases}$$

We set the ensuing notations for all $1 \le i \le 2$, $f_i = f^{(i-1)}, \varphi_i = \varphi^{(i-1)}$, to get the following system:

$$\begin{cases} \varphi_1(t) = \int_0^1 g(|t-s|)\kappa(t,s,\varphi_1(s),\varphi_2(s))ds + f_1(t), \\ \varphi_2(t) = \int_0^1 S_{ts}\, g'(|t-s|)\kappa(t,s,\varphi_1(s),\varphi_2(s))\,ds + \int_0^1 g(|t-s|)\frac{\partial\kappa}{\partial t}(t,s,\varphi_1(s),\varphi_2(s))\,ds + f_2(t). \end{cases} \tag{2.3}$$

We define the product Banach space $\mathcal{X} = C^1([0,1],\mathbb{R}) \times C^0([0,1],\mathbb{R})$ provided with the norm:

$$\forall \varPsi = (\psi_1,\psi_2) \in \mathcal{X}: \quad \|\varPsi\|_{\mathcal{X}} = \|\psi_1\|_\infty + \|\psi_2\|_\infty = \sup_{t\in[0,1]} |\psi_1(t)| + \sup_{t\in[0,1]} |\psi_2(t)|\,.$$

The system (2.3) can be rewritten as:

$$\begin{cases} \varphi_1(t) = K_1(\widetilde{\varphi}) + f_1(t), \\ \varphi_2(t) = K_2(\widetilde{\varphi}) + f_2(t) = K_2^s(\widetilde{\varphi}) + K_2^r(\widetilde{\varphi}) + f_2(t), \end{cases} \tag{2.4}$$

where $\widetilde{\varphi} = (\varphi_1,\varphi_2) \in \mathcal{X}$, and K_1, K_2^s, K_2^r are the operators presented by:

$$\begin{cases} K_1(\widetilde{\varphi})(t) = \int_0^1 g(|t-s|)\kappa(t,s,\varphi_1(s),\varphi_2(s))ds, \\ K_2^s(\widetilde{\varphi})(t) = \int_0^1 S_{ts}\, g'(|t-s|)\kappa(t,s,\varphi_1(s),\varphi_2(s))\,ds, \\ K_2^r(\widetilde{\varphi})(t) = \int_0^1 g(|t-s|)\frac{\partial\kappa}{\partial t}(t,s,\varphi_1(s),\varphi_2(s))\,ds. \end{cases} \tag{2.5}$$

So, the main problem we trait is:

$$\text{Find } \widetilde{\varphi} \in \mathcal{X}: \quad \widetilde{\varphi} = K(\widetilde{\varphi}) + F, \tag{2.6}$$

with : $\quad K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2^s + K_2^r \end{pmatrix}, F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$

For all $1 \le j \le 2$, let $T_{1j}(\cdot) = \frac{\partial K_1}{\partial \varphi_j}(\cdot), T_{2j}^s(\cdot) = \frac{\partial K_2^s}{\partial \varphi_j}(\cdot)$, and $T_{2j}^r(\cdot) = \frac{\partial K_2^r}{\partial \varphi_j}(\cdot)$ be the Fréchet derivative linear operators of the operators K_1, K_2^s, and K_2^r (respectively) associated to φ_j, such as:

$$\begin{cases} \left[T_{1j}(\widetilde{\varphi}).v\right](t) = \int_0^1 g(|t-s|)\frac{\partial\kappa}{\partial\varphi_j}(t,s,\varphi_1(s),\varphi_2(s))\,v(s)ds, \\ \left[T_{2j}^s(\widetilde{\varphi}).v\right](t) = \int_0^1 S_{ts}\, g'(|t-s|)\frac{\partial\kappa}{\partial\varphi_j}(t,s,\varphi_1(s),\varphi_2(s))\,v(s)ds, \\ \left[T_{2j}^r(\widetilde{\varphi}).v\right](t) = \int_0^1 g(|t-s|)\frac{\partial^2\kappa}{\partial t\partial\varphi_j}(t,s,\varphi_1(s),\varphi_2(s))\,v(s)ds, \end{cases} \tag{2.7}$$

for all $v \in C([0,1],\mathbb{R})$ and $t \in [0,1]$.

Let I_2 be the identity operator of the space $\mathcal{L}(\mathcal{X})$ of all linear bounded operators defined from $\mathcal{X}$ to $\mathcal{X}$, and $D(\cdot) : \mathcal{X} \longmapsto \mathcal{X}$ is the Fréchet derivative operator of the operator K defined for all $x \in \mathcal{X}$ as:

$$D(x)h = \begin{pmatrix} T_{11}(x) & T_{12}(x) \\ T_{21}^s(x) + T_{21}^r(x) & T_{22}^s(x) + T_{22}^r(x) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h \in \mathcal{X}. \tag{2.8}$$

We set $B_\nu(\widetilde{\varphi})$ as the ball of center $\widetilde{\varphi}$ and radius $\nu > 0$ and we assume that

$$(\mathcal{H}_2) : \left\| \begin{array}{l} (i)\ \text{Problem (2.6) has a unique solution } \widetilde{\varphi} \in \mathcal{X}, \\ (ii)\ I_2 - D(\widetilde{\varphi}) \text{ is invertible }, \exists \mu > 0, \left\|(I_2 - D(\widetilde{\varphi}))^{-1}\right\| \leq \mu < +\infty, \\ (iii) \exists \ell > 0, \text{ such that } D(\widetilde{\varphi}) : \mathcal{X} \to \mathcal{L}(\mathcal{X}) \text{ is } \ell - \text{Lipschitz over } B_\nu(\widetilde{\varphi}). \end{array} \right.$$

3. The modern strategy (L.D)

The first step to solve problem (2.6) is the linearization process using Newton's method to get the following linear scheme:

$$\left(I_2 - D(\widetilde{\varphi}^{(k)})\right)\left(\widetilde{\varphi}^{(k+1)} - \widetilde{\varphi}^{(k)}\right) = -\widetilde{\varphi}^{(k)} + K(\widetilde{\varphi}^{(k)}) + F, \quad \widetilde{\varphi}^{(0)} \in \mathcal{X},\ k = 1, 2, ... \quad . \tag{3.1}$$

The difficulty to find the operator $\left(I_2 - D(\widetilde{\varphi}^{(k)})\right)^{-1}$ in each iteration k, forces us to apply a discretization phase to our scheme (3.1). The discrete approximation is obtained operating a family of bounds projections of finis rank in $\mathcal{X}$ defined for $n \in \mathbb{N}^*$ as follows:

$$\pi_n X = \sum_{p=1}^{n} <X, \beta_{n,p}^*> \beta_{n,p}, \quad X \in \mathcal{X}, \tag{3.2}$$

where $(\beta_{n,p})_{p=1}^n$ is an ordered basis of the image space of π_n and $(\beta_{n,p}^*)_{p=1}^n$ is an adjoint basis of the previous one. For all $1 \leq i, j \leq 2$, for all $x \in \mathcal{X}$, we construct the approach operators

$$T_{n,ij}^c(x)v = (\pi_n T_{ij}^c(x))v = \sum_{p=1}^{n} <T_{ij}^c(x), \beta_{n,p}^*> \beta_{n,p}, \quad v \in C([0,1],\mathbb{R}), \quad c \in \{\ , s, r\}. \tag{3.3}$$

Based on the previous definition (2.8), For n large enough and $x \in \mathcal{X}$, we write the approximate operator $D_n(\cdot)$ of the operator $D(\cdot)$ as follows:

$$D_n(x)h = \pi_n(D(x)h) = \begin{pmatrix} T_{n,11}(x) & T_{n,12}(x) \\ T_{n,21}^s(x) + T_{n,21}^r(x) & T_{n,22}^s(x) + T_{n,22}^r(x) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h \in \mathcal{X}.$$

At this point, we can write our discretized scheme of the linear problem (3.1) as follows:

$$\left(I_2 - D_n(\widetilde{\varphi}_n^{(k)})\right)\left(\widetilde{\varphi}_n^{(k+1)} - \widetilde{\varphi}_n^{(k)}\right) = -\widetilde{\varphi}_n^{(k)} + K(\widetilde{\varphi}_n^{(k)}) + F, \quad \widetilde{\varphi}_n^{(0)} \in \mathcal{X},\ k = 1, 2, ... \quad , \tag{3.4}$$

where $\widetilde{\varphi}_n^{(k)} = (\varphi_{n,1}^{(k)}, \varphi_{n,2}^{(k)}) \in \mathcal{X}$ is the discretized approximate solution of the exact solution $\widetilde{\varphi} = (\varphi_1, \varphi_2) \in \mathcal{X}$ of our main problem (2.6). So, the discretized linear problem that we treat is written in the form: Find $\varphi_n^{(k+1)} \in \mathcal{X}$

$$\begin{cases} \widetilde{\varphi}_n^{(k+1)} - D_n(\widetilde{\varphi}_n^{(k)})\widetilde{\varphi}_n^{(k+1)} = b_n^{(k)}, \\ b_n^{(k)} = K(\widetilde{\varphi}_n^{(k)}) - D_n(\widetilde{\varphi}_n^{(k)})\widetilde{\varphi}_n^{(k)} + F. \end{cases} \tag{3.5}$$

We define the formula of some notations that we use in determining the structure of matrices and linear forms represented in the last statement of our problem as follows:

$$\beta_n U = \sum_{p=1}^{n} U(p)\beta_{n,p}, \quad \forall U \in \mathbb{C}^n, \beta_n = (\beta_{n,1}, \beta_{n,2}, ..., \beta_{n,n}),$$

$$\ll W, \beta_n^* \gg (i,j) = \langle w_j, \beta_{n,i}^* \rangle, \quad \forall W = (w_1, w_2, ..., w_m) \in \mathcal{X}^{1\times m},$$

and we use these notations to simplify writing the representation of our projection operator (3.2) to be presented as:

$$\pi_n X = \beta_n \ll X, \beta_n^* \gg, \quad X \in \mathcal{X}.$$

According to scheme (3.5), we writing the following equalities as:

$$(I-\pi_n)\,\widetilde{\varphi}_n^{(k+1)} = (I-\pi_n)\,b_n^{(k)} = (I-\pi_n)\left(K(\widetilde{\varphi}_n^{(k)}) + F\right).$$

Finally, we can give the expression of our approximate solution as follows:

$$\widetilde{\varphi}_n^{(k+1)} = (I-\pi_n)\left(K(\widetilde{\varphi}_n^{(k)}) + F\right) + \beta_n Y_n^{(k+1)}, \tag{3.6}$$

where $Y_n^{(k+1)} \in \mathbb{C}^n$ is the column vectors represent the solution of the following linear system:

$$\left(I_2 - A_n^{(k)}\right) Y_n^{(k+1)} = B_n^{(k)}, \tag{3.7}$$

where, for $1 \leq i, j \leq 2$, we set

$$A_n^{(k)}(i,j) = \ll D(\widetilde{\varphi}_n^{(k)})\beta_n, \beta_n^* \gg (i,j) = \langle D(\widetilde{\varphi}_n^{(k)})\beta_{n,j}, \beta_{n,i}^* \rangle, \tag{3.8}$$

$$B_n^{(k)}(i) = \langle K(\widetilde{\varphi}_n^{(k)}), \beta_{n,i}^* \rangle - \langle D(\widetilde{\varphi}_n^{(k)})\widetilde{\varphi}_n^{(k)}, \beta_{n,i}^* \rangle + \langle F, \beta_{n,i}^* \rangle + \tag{3.9}$$

$$\langle D(\widetilde{\varphi}_n^{(k)})(I-\pi_n)(K(\widetilde{\varphi}_n^{(k)}) + F), \beta_{n,i}^* \rangle, \tag{3.10}$$

and $\langle \cdot, \cdot \rangle$ represent the duality brackets between the space $\mathcal{X}$ and its dual space $\mathcal{X}^*$.

4. The compactness of the operators $(T_{2j}^s(\cdot))_{(1 \leq j \leq 2)}$

For all $1 \leq j \leq 2$, for all $\theta = (\theta_1, \theta_2) \in \mathcal{X}$, we consider a function N_j taking the same form as the weak singular kernel of operator $T_{2j}^s(\cdot)$ as following:

$$N_j : \left\{[0,1]^2 \times \mathbb{R}^2, t \neq s\right\} \longmapsto \mathbb{R} \tag{4.1}$$

$$(t, s, \theta(s)) \longmapsto S_{ts}\, g'(|t-s|) \frac{\partial \kappa}{\partial \theta_j}(t, s, \theta(s)). \tag{4.2}$$

We denote

$$\left\| \frac{\partial \kappa}{\partial \theta_j} \right\|_\infty = \sup_{(t,s,\theta(s)) \in [0,1]^2 \times \mathbb{R}^2} \left| \frac{\partial \kappa}{\partial \theta_j}(t, s, \theta(s)) \right|.$$

Before showing the compactness of $T_{2j}^s(\cdot)$ on the Banach space $C([0,1], \mathbb{R})$, we will show some results which will be useful in the sequel. We first define the function $G : [0,1] \longmapsto \mathbb{R}$ below which will play an important technical role throughout along this section

$$G(t) = \int_0^t p'(s)\, ds. \tag{4.3}$$

We further define for all $\eta > 0$ and for all $1 \leq j \leq 2$,

$$\varepsilon(N_j, \eta) := \sup\left\{ \int_0^1 |N_j(t, s, \theta(s)) - N_j(\tau, s, \theta(s))|\, ds, (t, \tau) \in [0,1]^2,\ |t-\tau| \leq \eta \right\}.$$

The oscillation of a function $h \in C([0,1], \mathbb{R})$ with respect to $\eta > 0$ is defined by

$$\omega_\infty(h, \eta) := \sup\left\{|h(t) - h(\tau)|,\ (t, \tau) \in [0,1]^2,\ |t-\tau| \leq \eta\right\}.$$

This definition can be generalized in a partial way to any continuous function $Q \in C([0,1]^2 \times \mathbb{R}^2, \mathbb{R})$, $\forall s \in [0,1], \forall \theta = (\theta_1, \theta_2) \in \mathcal{X}$ as follows:

$$\omega_\infty(Q, \eta)(s, \theta(s)) := \sup\left\{|Q(t, s, \theta(s)) - Q(\tau, s, \theta(s))|, (t, \tau) \in [0,1]^2, |t-\tau| \leq \eta\right\}.$$

Lemma 4.1. *Let $\eta > 0$ and $Q \in C([0,1]^2 \times \mathbb{R}^2, \mathbb{R})$. Then the function: $(s, \theta(s)) \in [0,1] \times \mathbb{R}^2 \to \omega_\infty(Q,\eta)(s,\theta(s)) \in \mathbb{R}$ is continues on $[0,1] \times \mathbb{R}^2$ and*

$$\lim_{s \to 0^+} \|\omega_\infty(Q,\eta)\|_\infty = 0.$$

Proof: This follows properties of uniform continuity of Q on $[0,1]^2 \times \mathbb{R}^2$. □

Lemma 4.2. *Let g' verified $(\mathcal{H}_1)(ii) - (iii)$. Then*

$$\sup_{t \in [0,1]} \int_0^1 g'(|t-s|)\, ds = 2 \int_0^{1/2} g'(s)\, ds < +\infty.$$

Proof: Let $y : [0,1] \longmapsto \mathbb{R}$ defined by

$$y(t) = \int_0^1 g'(|t-s|)\, ds = G(t) + G(1-t),$$

where G is the function defined by (4.3). The function y has an axial symmetry with respect to $\frac{1}{2}$ and

$$y'(t) = g'(t) - g'(1-t) \begin{cases} > 0 & \text{if } 0 < t < \frac{1}{2}, \\ < 0 & \text{if } \frac{1}{2} < t < 1, \end{cases}$$

for all $t \in [0,1]$. So

$$\sup_{t \in [0,1]} \int_0^1 g'(|t-s|)\, ds = \max_{t \in [0,1]} y(t) = y(1/2) = 2 \int_0^{1/2} g'(s)\, ds,$$

and according to $(\mathcal{H}_1)(ii)$ this last integral is finite.

□

Then we define

$$\alpha = 2 \int_0^{1/2} g'(s)\, ds. \tag{4.4}$$

Lemma 4.3. *For all $\eta > 0$ small enough*

$$\varepsilon(g', \eta) \le 4 \int_0^\eta g'(s)\, ds \xrightarrow{s \longrightarrow 0^+} 0. \tag{4.5}$$

Proof: For $\eta > 0$, and $(t,\tau) \in [0,1]^2$ such as $0 < |t-\tau| \le \eta$, let's define ξ as

$$\begin{aligned} \xi : [0,1] &\longrightarrow \mathbb{R} \\ s &\longrightarrow |g'(|t-s|) - g'(|\tau - s|)|. \end{aligned}$$

Let's put $\zeta = \frac{t+\tau}{2}$, and we can write

$$\int_0^1 \xi(s)\, ds = \int_0^t \xi(s)\, ds + 2 \int_t^\zeta \xi(s)\, ds + \int_\tau^1 \xi(s)\, ds.$$

We can easily show that the restriction of ξ on $[0,1]$ has an axial symmetric with respect to ζ. So, with more simplification we find the following equalities

$$\begin{aligned} \int_0^t \xi(s)\, ds &= \int_0^t (g'(t-s) - g'(\tau - s))\, ds = G(t) + G(\tau - t) - G(\tau), \\ \int_t^\zeta \xi(s)\, ds &= \int_t^\zeta (g'(s-t) - g'(\tau - s))\, ds = G((\tau-t)/2) + G((\tau-t)/2) - G(\tau - t), \\ \int_\tau^1 \xi(s)\, ds &= \int_t^1 (g'(s-\tau) - g'(s-t))\, ds = G(1-\tau) - G(1-t) + G(\tau - t), \end{aligned}$$

where G is the function defined by (4.3). Thereby

$$\begin{aligned}\varepsilon(g',\eta) &= 4\int_0^{(\tau-t)/2} g'(s)\,ds - \int_t^{\tau} g'(s)\,ds - \int_{1-\tau}^{1-t} g'(s)\,ds \leq 4\int_0^{(\tau-t)/2} g'(s)\,ds \\ &\leq 4\int_0^{\eta/2} g'(s)\,ds \leq 4\int_0^{\eta} g'(s)\,ds,\end{aligned}$$

and that what we wanted to demonstrate. The assertion (4.5) follows from the integrability properties of g'. □

Lemma 4.4. *Let g' verified $(\mathcal{H}_1)$. Then for all $1 \leq j \leq 2$ we have*

$$\lim_{s\to 0^+} \varepsilon(N_j,\eta) = 0.$$

Proof: For $\eta > 0$ quite small and $(t,\tau,\theta) \in [0,1]^2 \times \mathcal{X}$ such that $|t-\tau| \leq \eta$,

$$\begin{aligned}\int_0^1 |N_j(t,s,\theta(s)) - N_j(\tau,s,\theta(s))|ds &= \int_0^1 |S_{ts}\,g'(|t-s|)\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s)) - S_{\tau s}\,g'(|\tau-s|)\frac{\partial\kappa}{\partial\theta_j}(\tau,s,\theta(s))|ds \\ &\leq \int_0^1 \left|\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\right| |g'(|t-s|) - g'(|\tau-s|)|\,ds \\ &\quad + \int_0^1 \left|S_{ts}\,\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s)) - S_{\tau s}\,\frac{\partial\kappa}{\partial\theta_j}(\tau,s,\theta(s))\right| |g'(|\tau-s|)|\,ds \\ &\leq \sup_{s\in[0,1]} \left|\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\right| \int_0^1 |g'(|t-s|) - g'(|\tau-s|)|\,ds \\ &\quad + \sup_{s\in[0,1]} \left|S_{ts}\,\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s)) - S_{\tau s}\,\frac{\partial\kappa}{\partial\theta_j}(\tau,s,\theta(s))\right| \int_0^1 g'(|\tau-s|)ds.\end{aligned}$$

Thus, according to the lemma (4.2)

$$\varepsilon(N_j,\eta) \leq \left\|\frac{\partial\kappa}{\partial\theta_j}\right\|_\infty \varepsilon(g',\eta) + \alpha\left\|\omega_\infty\left(S_{..}\,\frac{\partial\kappa}{\partial\theta_j},\eta\right)\right\|_\infty,$$

where α is defined by (4.4). Finlay, using lemmas (4.1) – (4.3) to get that

$$\lim_{s\to 0^+} \varepsilon(N_j,\eta) = 0.$$

□

Theorem 4.5. *Let $T_{2j}^s(\cdot)$ defined by (2.7) – (4.1) and (4.2), for all $1 \leq j \leq 2$, and let $(\mathcal{H}_1)$ holds. Then $T_{2j}^s(\cdot)$ is a linear compact operator on $C([0,1],\mathbb{R})$.*

Proof: Let $v \in C([0,1],\mathbb{R})$ and $\epsilon > 0$. So, according to lemma (4.4), there exists $\eta > 0$ such that $\varepsilon(N_j,\eta) < \frac{\epsilon}{\|v\|_\infty}$. Let $t \in [0,1]$. Then, for all $\tau \in [0,1]$ such as $|t-\tau| < \eta$, for all $1 \leq j \leq 2$, and for all $\theta \in \mathcal{X}$,

$$\begin{aligned}\left|[T_{2j}^s(\theta).v](t) - [T_{2j}^s(\theta).v](\tau)\right| &\leq \int_0^1 \left|S_{ts}\,g'(|t-s|)\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s)) - S_{\tau s}\,g'(|\tau-s|)\frac{\partial\kappa}{\partial\theta_j}(\tau,s,\theta(s))\right| |v(s)|ds \\ &\leq \|v\|_\infty\,\varepsilon(N_j,\eta) < \epsilon.\end{aligned}$$

Also, for all $v \in C([0,1],\mathbb{R})$ and for all $\theta \in \mathcal{X}, T_{2j}^s(\theta).v$ is continuous at all points of $[0,1]$, which proves that $C([0,1],\mathbb{R})$ is invariant by $T_{2j}^s(\theta)$.

Let $v \in C([0,1],\mathbb{R})$ such that $\|v\|_\infty \leq 1$. Then for all $t \in [0,1]$,

$$\left|[T^s_{2j}(\theta).v](t)\right| \leq \int_0^1 \left|\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\right| g'(|t-s|)|v(s)|ds$$
$$\leq \|v\|_\infty \sup_{s\in[0,1]} \left|\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\right| \int_0^1 g'(|t-s|)ds.$$

So. according the lemma (4.2),

$$\left\|T^s_{2j}(\theta)\right\|_\infty \leq 2\left|\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\right| \int_0^{1/2} g'(|t-s|)ds = \alpha \left\|\frac{\partial\kappa}{\partial\theta_j}\right\|_\infty.$$

Thereby, for all $1 \leq j \leq 2$, for all $\theta \in \mathcal{X}$, $T^s_{2j}(\theta)$ is bounded from $C([0,1],\mathbb{R})$ into itself.

The compactness is deduced by considering the truncated function g'_n defined by

$$g'_n(s) = \begin{cases} g'(1/n) & if \quad 0 \leq s < 1/n, \\ g'(s) & else. \end{cases}$$

For all $1 \leq j \leq 2$, for all $\theta \in \mathcal{X}$, we define the integral operator $T^s_{n,2j}(\cdot)$ as

$$[T^s_{n,2j}(\theta).v](t) = \int_0^1 S_{ts}\, g'_n(|t-s|)\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\, v(s)ds,$$

and it is compact from $C([0,1],\mathbb{R})$ into itself, because its kernel is a continuous function on $[0,1]^2 \times \mathcal{X}$. If $v \in C([0,1],\mathbb{R})$ and $\theta \in \mathcal{X}$, then

$$\left[\left(T^s_{2j}(\theta) - T^s_{n,2j}(\theta)\right).v\right](t) = \int_{t-1/n}^t \left[g'(|t-s|) - g'(1/n)\right] S_{ts}\,\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))v(s)\,ds$$
$$+ \int_t^{t+1/n} \left[g'(|t-s|) - g'(1/n)\right] S_{ts}\,\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))v(s)\,ds$$
$$= \int_0^{1/n} \left[g'(s) - g'(1/n)\right] S_{t(t-s)}\,\frac{\partial\kappa}{\partial\theta_j}(t,t-s,\theta(s))v(t-s)\,ds$$
$$+ \int_0^{1/n} \left[g'(s) - g'(1/n)\right] S_{t(t+s)}\,\frac{\partial\kappa}{\partial\theta_j}(t,t+s,\theta(s))v(t+s)\,ds.$$

So. for all $v \in C([0,1],\mathbb{R})$ and $\theta \in \mathcal{X}$

$$\left\|T^s_{2j}(\theta) - T^s_{n,2j}(\theta)\right\|_\infty \leq 4\left\|\frac{\partial\kappa}{\partial\theta_j}\right\|_\infty \int_0^{1/n} g'(s)\,ds \xrightarrow{n\to+\infty} 0.$$

Finally, for all $1 \leq j \leq 2$, the operators $T^s_{2j}(\cdot)$ are the uniform limits of a sequences of compacts operators. □

5. Convergence analysis

Now, we can consider that all the operators of the matrix $D(\cdot)$ are a compact operators. In this work, we suppose that $(\pi_n)_{n\geq 1}$ is punctually convergent to the identity operator in the Banach space $\mathcal{X}$ on which $D(\cdot)$ is defined. So, the approximate operators matrix $D_n(\cdot)$ converge uniformly to the operators matrix $D(\cdot)$ (see ([2]) Theorem 4.1, pp.186); ie, For n large enough

$$(\mathcal{H}_3): \Big\| \; \exists\, \gamma_n > 0, \forall\theta \in \mathcal{X}, \quad \|D_n(\theta) - D(\theta)\| \leq \gamma_n \xrightarrow{n\to\infty} 0.$$

Lemma 5.1. *Assume that* $(\mathcal{H}_2)-(\mathcal{H}_3)$ *holds, and let* $r=\min\left\{\nu,\frac{1}{2\ell\mu}\right\}$. *Then for all* $\theta\in B_r(\widetilde{\varphi})$ *the operator* $(I_2-D(\theta))$ *is invertible such that*

$$\left\|(I_2-D(\theta))^{-1}\right\|\leq 2\mu.$$

Proof: Let be $\theta\in B_r(\widetilde{\varphi})$, we have

$$\begin{aligned}I_2-D(\theta)=&I_2-D(\widetilde{\varphi})-D(\theta)+D(\widetilde{\varphi})\\ =&(I_2-D(\widetilde{\varphi}))\left(I_2-(I_2-D(\widetilde{\varphi}))^{-1}\left[D(\theta)-D(\widetilde{\varphi})\right]\right),\end{aligned}$$

and according $(\mathcal{H}_2)(iii)$, we have

$$\|D(\theta)-D(\widetilde{\varphi})\|\leq\ell\,\|\theta-\widetilde{\varphi}\|_{\mathcal{X}}\leq\ell r.$$

Then

$$\left\|(I_2-D(\widetilde{\varphi}))^{-1}\left[D(\theta)-D(\widetilde{\varphi})\right]\right\|\leq\mu\ell r\leq\frac{1}{2},$$

and using the Geometric Series Theorem (see [3] pp.516), we conclude that the operator $(I_2-D(\theta))$ is invertible and uniformly bounded such that, for all $\theta\in B_r(\widetilde{\varphi})$,

$$(I_2-D(\theta))^{-1}=\left(I_2-(I_2-D(\widetilde{\varphi}))^{-1}\left[D(\theta)-D(\widetilde{\varphi})\right]\right)^{-1}(I_2-D(\widetilde{\varphi}))^{-1},$$

and

$$\left\|(I_2-D(\theta))^{-1}\right\|\leq\mu\sum_{k=0}^{\infty}\left\|(I_2-D(\widetilde{\varphi}))^{-1}\left[D(\theta)-D(\widetilde{\varphi})\right]\right\|^k\leq 2\mu.$$

□

Lemma 5.2. *We assume that the aforementioned conditions in* $(\mathcal{H}_2)$ *are fulfilled. Then for* n *big enough, and all* $\theta\in B_r(\widetilde{\varphi})$, *the approximate operator* $(I_2-D_n(\theta))$ *is invertible, and there exists* $\epsilon_n\in]0,1]$ *such that,*

$$\sup_{\theta\in B_r(\widetilde{\varphi})}\left\|I_2-(I_2-D_n(\theta))^{-1}(I_2-D(\theta))\right\|\leq\epsilon_n,$$

$$\sup_{\theta\in B_r(\widetilde{\varphi})}\left\|(I_2-D_n(\theta))^{-1}\right\|\leq 2\mu(1+\epsilon_n).$$

Proof: For all $\theta\in B_r(\widetilde{\varphi})$, we have

$$\begin{aligned}I_2-D_n(\theta)=&I_2-D(\theta)+D(\theta)-D_n(\theta)\\ =&(I_2-D(\theta))\left(I_2-(I_2-D(\theta))^{-1}\left[D_n(\theta)-D(\theta)\right]\right),\end{aligned}$$

and according $(\mathcal{H}_3)$, and by taking n big enough to achieve $\gamma_n<\frac{1}{2\mu}$. Then

$$\left\|(I_2-D(\theta))^{-1}\left[D_n(\theta)-D(\theta)\right]\right\|\leq 2\mu\gamma_n<1,$$

and by the Geometric Series Theorem (see [3] pp.516), we have for all $\theta\in B_r(\widetilde{\varphi})$, the operator $(I_2-D_n(\theta))$ is invertible and its bounded such that,

$$\left\|(I_2-D_n(\theta))^{-1}\right\|\leq\frac{2\mu}{1-2\mu\gamma_n}.$$

As

$$I_2-(I_2-D_n(\theta))^{-1}(I_2-D(\theta))=(I_2-D_n(\theta))^{-1}(D(\theta)-D_n(\theta)),$$

we set $\epsilon_n = \frac{2\mu\gamma_n}{1-2\mu\gamma_n}$ and for n large enough, $\gamma_n < 1$ we find

$$\sup_{\theta\in B_r(\widetilde{\varphi})} \left\| I_2 - (I_2 - D_n(\theta))^{-1}(I_2 - D(\theta)) \right\| \leq \epsilon_n,$$

and our inverse operator $(I_2 - D_n(\theta))^{-1}$ can be writing as

$$(I_2 - D_n(\theta))^{-1} = (I_2 - D(\theta))^{-1} - \left[I_2 - (I_2 - D_n(\theta))^{-1}(I_2 - D_n(\theta)) \right](I_2 - D(\theta))^{-1}.$$

In this way, we close at last that

$$\left\| (I_2 - D_n(\theta))^{-1} \right\| \leq 2\mu(1+\epsilon_n).$$

□

Theorem 5.3. *(Convergence Theorem)*
If the initial function $\widetilde{\varphi}_n^{(0)} \in B_{\varrho_n}(\widetilde{\varphi})$, for $n \in \mathbb{N}$. Then the sequence $(\widetilde{\varphi}_n^{(k)})_{k\geq 0}$ defined by the scheme (3.5), converges to $\widetilde{\varphi}$ the exact solution of problem (2.1), such that

$$\left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_\chi \leq \varrho_n \left(\frac{1+\epsilon_n}{2} \right)^k \xrightarrow{k\to\infty} 0,$$

where

$$\varrho_n := \min \left\{ r, \left(\frac{1-\epsilon_n}{2\ell\mu(1+\epsilon_n)} \right) \right\}.$$

Proof: If we choose $\widetilde{\varphi}_n^{(k)} \in B_r(\widetilde{\varphi})$, so according lemma (5.2), the operator $I_2 - D_n(\widetilde{\varphi}_n^{(k)})$ is invertible. Then $\widetilde{\varphi}_n^{(k+1)}$ defined in scheme (3.5) can given by

$$\widetilde{\varphi}_n^{(k+1)} - \widetilde{\varphi} = \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} - \left(I_2 - D_n(\widetilde{\varphi}_n^{(k)}) \right)^{-1} \left(\widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} - K(\widetilde{\varphi}_n^{(k)}) + K(\widetilde{\varphi}) \right).$$

Since

$$K(\widetilde{\varphi}) - K(\widetilde{\varphi}_n^{(k)}) = -\int_0^1 D\left((1-x)\widetilde{\varphi}_n^{(k)} + x\widetilde{\varphi} \right) \left(\widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right) dx,$$

then, we write

$$\widetilde{\varphi}_n^{(k+1)} - \widetilde{\varphi} = \int_0^1 \left[I_2 - \left(I_2 - D_n(\widetilde{\varphi}_n^{(k)}) \right)^{-1} \left[I_2 - D\left((1-x)\widetilde{\varphi}_n^{(k)} + x\widetilde{\varphi} \right) \right] \right] \left(\widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right) dx,$$

by added $I_2 - D(\widetilde{\varphi}_n^{(k)})$ to and subtracted from $I_2 - D\left((1-x)\widetilde{\varphi}_n^{(k)} + x\widetilde{\varphi} \right)$, we get

$$\begin{aligned} \widetilde{\varphi}_n^{(k+1)} - \widetilde{\varphi} &= \int_0^1 \left[I_2 - \left(I_2 - D_n(\widetilde{\varphi}_n^{(k)}) \right)^{-1} \left(I_2 - D(\widetilde{\varphi}_n^{(k)}) \right) \right] \left(\widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right) dx \\ &+ \int_0^1 \left(I_2 - D_n(\widetilde{\varphi}_n^{(k)}) \right)^{-1} \left[D\left((1-x)\widetilde{\varphi}_n^{(k)} + x\widetilde{\varphi} \right) - D(\widetilde{\varphi}_n^{(k)}) \right] \left(\widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right) dx, \end{aligned}$$

and

$$\begin{aligned} \left\| \widetilde{\varphi}_n^{(k+1)} - \widetilde{\varphi} \right\|_\chi &\leq \left\| I_2 - \left(I_2 - D_n(\widetilde{\varphi}_n^{(k)}) \right)^{-1} \left(I_2 - D(\widetilde{\varphi}_n^{(k)}) \right) \right\| \left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_\chi \\ &+ \left\| \left(I_2 - D_n(\widetilde{\varphi}_n^{(k)}) \right)^{-1} \right\| \left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_\chi \int_0^1 \left\| D\left((1-x)\widetilde{\varphi}_n^{(k)} + x\widetilde{\varphi} \right) - D(\widetilde{\varphi}_n^{(k)}) \right\| dx. \end{aligned}$$

As we are taking $\widetilde{\varphi}_n^{(k)} \in B_r(\widetilde{\varphi})$, so according to lemma (5.2)

$$\left\| I_2 - \left(I_2 - D_n(\widetilde{\varphi}_n^{(k)}) \right)^{-1} \left(I_2 - D(\widetilde{\varphi}_n^{(k)}) \right) \right\| \leq \epsilon_n,$$

and since $B_r(\widetilde{\varphi})$ is a convex set, for all $x \in [0,1], (1-x)\widetilde{\varphi}_n^{(k)} + x\widetilde{\varphi} \in B_r(\widetilde{\varphi})$, and according to $(\mathcal{H}_2)(iii)$, we have

$$\left\| D\left((1-x)\widetilde{\varphi}_n^{(k)} + x\widetilde{\varphi} \right) - D(\widetilde{\varphi}_n^{(k)}) \right\| \leq \ell x \left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_{\mathcal{X}}.$$

Hence

$$\int_0^1 \left\| D\left((1-x)\widetilde{\varphi}_n^{(k)} + x\widetilde{\varphi} \right) - D(\widetilde{\varphi}_n^{(k)}) \right\| dx \leq \frac{\ell}{2} \left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_{\mathcal{X}}.$$

Now, we take the second result of lemma (5.2), we get

$$\left\| \widetilde{\varphi}_n^{(k+1)} - \widetilde{\varphi} \right\|_{\mathcal{X}} \leq \epsilon_n \left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_{\mathcal{X}} + \left(2\mu(1+\epsilon_n) \left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_{\mathcal{X}} \right) \frac{\ell}{2} \left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_{\mathcal{X}}.$$

Then if $\widetilde{\varphi}_n^{(k)} \in B_{\varrho_n}(\widetilde{\varphi})$, we get $\frac{\ell}{2} \left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_{\mathcal{X}} \leq \frac{1-\epsilon_n}{4\mu(1+\epsilon_n)}$. Hence

$$\left\| \widetilde{\varphi}_n^{(k+1)} - \widetilde{\varphi} \right\|_{\mathcal{X}} \leq \left(\frac{1+\epsilon_n}{2} \right) \left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_{\mathcal{X}},$$

since $1+\epsilon_n < 2$ the previous inequality implies that $\widetilde{\varphi}_n^{(k+1)} \in B_{\varrho_n}(\widetilde{\varphi})$ and finally we find

$$\left\| \widetilde{\varphi}_n^{(k)} - \widetilde{\varphi} \right\|_{\mathcal{X}} \leq \varrho_n \left(\frac{1+\epsilon_n}{2} \right)^k \to 0, \quad as \quad k \to \infty.$$

□

5.1. Integrals approximation

In this part, we will define wish approximate methods we are using to estimate integrals represented in the interpolation formula of our approximate iterate solution in (3.6) and in the linear system characterized by (3.7). For integrals that have a weak singular kernel, we must use the product integration rule to remove the singularity (*see* [3], [14]). For integrals with a regular kernel, we use the Nyström method (*see* [4], [16]).

First, we define for $N >> n$ in $\mathbb{N}$ an equidistance subdivision Ω_N as follows:

$$\Omega_N = \left\{ t_r = rh, h = \frac{1}{N}, r = 0, 1, ..., N \right\}, \tag{5.1}$$

and the Nyström approximate integration formula on the previous subdivision Ω_N is given by:

$$\int_0^1 G(t, s, \theta(s))ds = \sum_{r=0}^{N} q_r G(t, t_r, \theta(t_r)),$$

where q_r are reals and there exists $M < +\infty$, such that $\max_{N \geq 1} \sum_{r=0}^{N} |q_r| < M$.

What is more, the integral product method consists of interpolating the regular terms of the kernel κ and his derivative $\frac{\partial \kappa}{\partial t}$ on the subdivision Ω_N using the piecewise linear functions in every subinterval $[t_r, t_{r+1}]$. So, for all $1 \leq i \leq n$ or $1 \leq i \leq N$ depend wish situation we have in our equations (3.8) and for all $\theta \in \mathcal{X}$, we have

$$P_{N,1}\left[\kappa\right](t_i,s,\theta(s)) \simeq \left(\frac{s-t_r}{h}\right)\kappa(t_i,t_{r+1},\theta(t_{r+1})) + \left(\frac{t_{r+1}-s}{h}\right)\kappa(t_i,t_r,\theta(t_r)), \quad s\in[t_r,t_{r+1}],$$

$$P_{N,1}\left[\frac{\partial\kappa}{\partial t}\right](t_i,s,\theta(s)) \simeq \left(\frac{s-t_r}{h}\right)\frac{\partial\kappa}{\partial t}(t_i,t_{r+1},\theta(t_{r+1})) + \left(\frac{t_{r+1}-s}{h}\right)\frac{\partial\kappa}{\partial t}(t_i,t_r,\theta(t_r)), \quad s\in[t_r,t_{r+1}].$$

Then, for all $1\le i\le N$ or $1\le i\le n$, we approximate integrals with weak singular kernel $p(\cdot)G(\cdot,\cdot,\cdot)$ as follows

$$\int_0^1 p(|t_i-s|)G(t_i,s,\theta(s))ds = \sum_{r=0}^{N}\vartheta_r G(t_i,t_r,\theta(t_r)),$$

where ϑ_r are given by:

$$\vartheta_0 = \frac{1}{h}\int_0^{t_1}(t_1-s)sign(t_i-s)p(|t_i-s|)ds,$$
$$\vartheta_r = \frac{1}{h}\left(\int_{t_{r-1}}^{t_r}(s-t_{r-1})sign(t_i-s)p(|t_i-s|)ds + \int_{t_r}^{t_{r+1}}(t_{r+1}-s)sign(t_i-s)p(|t_i-s|)ds\right),$$
$$\vartheta_N = \frac{1}{h}\int_{t_{N-1}}^{1}(s-t_{N-1})sign(t_i-s)p(|t_i-s|)ds.$$

6. Numerical examples

In this section, we will show the efficacy of our (L.D) new process to solve systems of nonlinear Fredholm integro-differential weak singular equations by applying it to solve two examples. All results got are represented in the tables below.

Let $\widetilde{\varphi}_n^{(k)} = (\widetilde{\varphi}_{n,1}^{(k)}, \widetilde{\varphi}_{n,2}^{(k)}) \in \mathcal{X}$ be the k order approximative solution obtained by solving the discretized linear problem (3.5) applying our new process and $\widetilde{\varphi}_{ext} = (\widetilde{\varphi}_{1,ext}, \widetilde{\varphi}_{2,ext}) \in \mathcal{X}$ our exact solution of the main problem (2.6). The stopping condition on the parameter k is defined as:

$$E_{n,N}^{k} = \max_{1\le p\le N}\left(\left|\widetilde{\varphi}_{n,1}^{(k+1)}(t_p)-\widetilde{\varphi}_{n,1}^{(k)}(t_p)\right| + \left|\widetilde{\varphi}_{n,2}^{(k+1)}(t_p)-\widetilde{\varphi}_{n,2}^{(k)}(t_p)\right|\right) \le 10^{-12}.$$

The estimate error obtained by applying our process is denoted by:

$$E_{n,N} = \max_{1\le p\le N}\left(\left|\widetilde{\varphi}_{1,ext}(t_p)-\widetilde{\varphi}_{n,1}^{(k)}(t_p)\right| + \left|\widetilde{\varphi}_{2,ext}(t_p)-\widetilde{\varphi}_{n,2}^{(k)}(t_p)\right|\right),$$

where for all $1\le p\le N$, the points t_p are taking from the subdivision Ω_N presented in (5.1).

Algorithm 1. *(L.D) Algorithm*

Data: n, N, g, φ_{ext}
Result: $E_{n,N}$, *plot* $(\varphi_n^{(k+1)}, \varphi_{ext}, \log(E_{n,N}^k))$
Initialization: $\varphi_n^{(0)} \leftarrow 0, M_n^{(0)} \leftarrow 0_{n\times n}, B_n^{(0)} \leftarrow 0_n, k \leftarrow 1, Tol \leftarrow 10^{-12}, E_{n,N}^k \leftarrow 1$
$h_n \leftarrow 1/(n-1)$
$T_n \leftarrow (0 : h_n : 1)^t$
while $E_{n,N}^k > Tol$ **do**
  for $i \leftarrow 1$ **to** n **do**
    for $j \leftarrow 1$ **to** n **do**
      Calculate and save $M_n^{(k)}(i,j)$
      if i=j **then**
        $A_n^{(k)} \leftarrow 1 - M_n^{(k)}(i,j)$
      else
        $A_n^{(k)} \leftarrow -M_n^{(k)}(i,j)$
      end
    end
    Calculate and save $B_n^{(k)}(i)$
  end
  $X_n^{(k+1)} \leftarrow (A_n^{(k)})^{-1}.B_n^{(k)}$ *(Calculate and save the vector solution of the linear system)*
  $\pi_n K(t) \leftarrow 0_{1\times 2}, \pi_n g \leftarrow 0_{1\times 2}, e_n X_n^{(k+1)}(t) \leftarrow 0_{1\times 2}$
  for $i \leftarrow 1$ **to** 2 **do**
    for $p \leftarrow 1$ **to** n **do**
      Calculate the vector basis $e_p(t)$
      $\pi_n K_i(t) \leftarrow \pi_n K_i(t) + K_i(T_n(p)) \,.\, e_p(t)$
      $\pi_n g_i(t) \leftarrow \pi_n g_i(t) + g_i(T_n(p)) \,.\, e_p(t)$
      $e_n X_{n,i}^{(k+1)}(t) \leftarrow e_n X_{n,i}^{(k+1)}(t) + X_{n,i}^{(k+1)}(p) \,.\, e_p(t)$
    end
  end
  for $i \leftarrow 1$ **to** 2 **do**
    $\varphi_{n,i}^{(k+1)}(t) \leftarrow K_i(t) + g_i(t) - \pi_n K_i(t) - \pi_n g_i(t) + e_n X_{n,i}^{(k+1)}(t)$
    $\varphi_{n,i}^{(k+1)}(t) \leftarrow \varphi_{n,i}^{(k)}(t)$ *(Save the previous iterate solution, k = 1,2...)*
  end
  $E_{n,N}^k(t) \leftarrow \max_{t\in[0,1]} \left(\left\| \varphi_{n,1}^{(k+1)}(t) - \varphi_{n,1}^{(k)}(t)) \right\| + \left\| \varphi_{n,2}^{(k+1)}(t) - \varphi_{n,2}^{(k)}(t)) \right\| \right)$ *(Calculate the iterate error)*
  $k \leftarrow k+1$ *(Increment the number of iterations k by 1)*
end
$E_{n,N} \leftarrow \max_{t\in[0,1]} \left(\left\| \varphi_{n,1}^{(k+1)}(t) - \varphi_{1,ext}(t) \right\| + \left\| \varphi_{n,2}^{(k+1)}(t) - \varphi_{2,ext}(t) \right\| \right)$ *(Calculate the error between the exact solution and the last iterate solution).*

Example 1: We take the following nonlinear integro-differential equation:

$$\varphi(t) = \frac{1}{20} \int_0^1 \sqrt{|t-s|} \sin\left(e^s + \arcsin\left(\frac{s+t}{3} \right) + \varphi(s) - \varphi'(s) \right) ds + f(t), \quad t \in [0,1], \tag{6.1}$$

and the function f and the exact solution φ_{ext} are given by:

$$f(t) = te^t - \frac{(7t+3)(1-t)^{\frac{3}{2}} + 7t^{\frac{5}{2}}}{450}, \qquad \varphi_{ext}(t) = te^t.$$

To get more information about our solution, we derive equation (6.1) and according the same notation defined before in the construction of our main system (2.3), we get the following system of nonlinear integro-differential equations:

$$\begin{cases} \varphi_1(t) = & \frac{1}{20}\int_0^1 \sqrt{|t-s|}\sin\left(e^s + \arcsin\left(\frac{s+t}{3}\right) + \varphi_1(s) - \varphi_2(s)\right)ds + f_1(t) \\ \varphi_2(t) = & \frac{1}{20}\int_0^1 \frac{sign(t-s)}{2\sqrt{|t-s|}}\sin\left(e^s + \arcsin\left(\frac{s+t}{3}\right) + \varphi_1(s) - \varphi_2(s)\right)ds \\ & +\frac{1}{60}\int_0^1 \frac{\sqrt{|t-s|}}{\sqrt{1-(\frac{s+t}{3})^2}}\cos\left(e^s + \arcsin\left(\frac{s+t}{3}\right) + \varphi_1(s) - \varphi_2(s)\right)ds + f_2(t). \end{cases}$$

We solve this system with our (L.D) new strategy and the results obtained are presented in Table 1.

Example 2: Consider the following nonlinear integro-differential equation:

$$\varphi(t) = \frac{1}{10}\int_0^1 \sqrt{|t-s|}\frac{ts^2\left(1+e^{-\sin(4s)}+e^{-4\cos(4s)}\right)}{1+e^{-\varphi(s)}+e^{-\varphi'(s)}}ds + f(t), \quad t\in[0,1], \tag{6.2}$$

where the function f and the exact solution φ_{ext} are given by:

$$f(t) = \sin(4t) - \frac{(1-t)^{3/2}\left(8t^3+12t^2+15t\right)+8t^{9/2}}{525}, \qquad \varphi_{ext}(t) = \sin(4t).$$

Using the same technique do before to construct the following system of equations:

$$\begin{cases} \varphi_1(t) = & \frac{1}{10}\int_0^1 \sqrt{|t-s|}\frac{ts^2\left(1+e^{-\sin(4s)}+e^{-4\cos(4s)}\right)}{1+e^{-\varphi_1(s)}+e^{-\varphi_2(s)}}ds + f_1(t) \\ \varphi_2(t) = & \frac{1}{10}\int_0^1 \frac{sign(t-s)}{2\sqrt{|t-s|}}\frac{s^2\left(1+e^{-\sin(4s)}+e^{-4\cos(4s)}\right)}{1+e^{-\varphi_1(s)}+e^{-\varphi_2(s)}}ds \\ & +\frac{1}{10}\int_0^1 \sqrt{|t-s|}\frac{s^2\left(1+e^{-\sin(4s)}+e^{-4\cos(4s)}\right)}{1+e^{-\varphi_1(s)}+e^{-\varphi_2(s)}}ds + f_2(t). \end{cases}$$

We solve this system with our (L.D) new strategy and the results obtained are presented in Table 2.

Results obtained on the both examples and represented in tables 1 and 2 are confirm the efficacy of our new strategy to solve this genre of weak singular problems. By choosing $n = 5$ (the number of vectors basis taken for the discretization phase using the Kantorovich projection) and by increasing N (the number of nodes in the subdivision defined for the approximation of integrals in our problem) we remark that the estimated error $E_{n,N}$ is rapidly declining towards 0. However, in Figures 1 and 2 we see clearly that our estimate solutions obtained using the (L.D) new process converge to the exact solutions, and the logarithm of the iterate stopping condition $Log(E^k_{n,N})$ prove that the (L.D) modern strategy has a linear convergence, which confirm that our method is reasonable. The Algorithm using for programming the resolution process of both numerical examples is given by the Algorithm 1.

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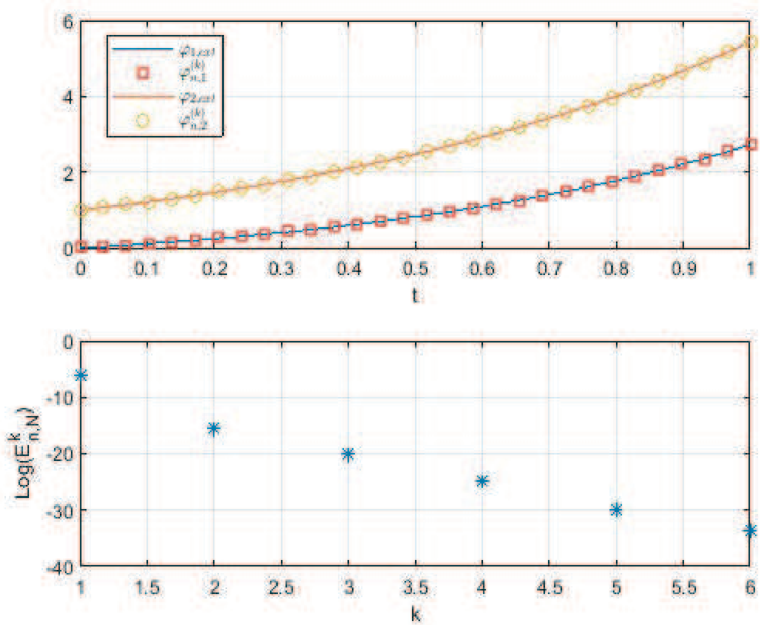

Figure 1: Approximate solutions of example 1, using the (L.D) method and the variation of $Log(E_{n,N}^k)$ during the iteration process.

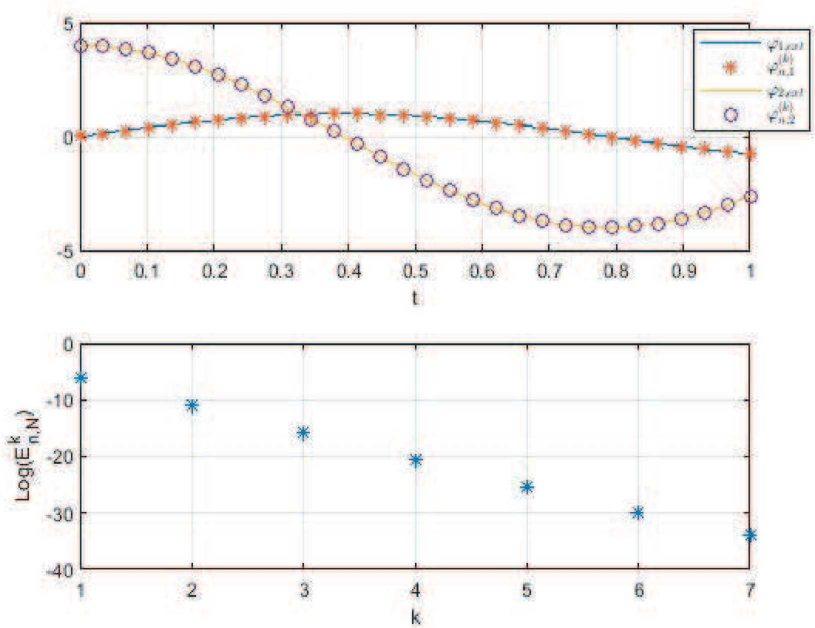

Figure 2: Approximate solutions of example 2, using the (L.D) method and the variation of $Log(E_{n,N}^k)$ during the iteration process.

N	The Error $E_{n,N}$ with $n = 5$
10	6.3633 E-04
30	1.1846 E-04
50	5.5029 E-05
100	1.9571 E-05
500	1.7862 E-06
1000	6.3642 E-07

Table 1: Numerical results of example 1.

N	The Error $E_{n,N}$ with $n = 5$
10	1.8124 E-03
30	4.2009 E-04
50	2.0258 E-04
100	7.3898 E-05
500	7.0205 E-06
1000	2.5260 E-06

Table 2: Numerical results of example 2.

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