Igusa-Todorov Function on Path Rings

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Abstract: The aim of this paper is to study the relation between the Igusa-Todorov functions for $A$, a finite dimensional algebra, and the algebra $AQ$. In particular it is proved that $\phi \dim(AQ) = \phi \dim(A) + 1$ when $A$ is a Gorenstein algebra. As a consequence of the previous result, it is exhibited an example of a family of algebras $\{A_n\}_{n \in \mathbb{N}}$ such that $\phi \dim(A_n) = n$ and each $A_n$ is of $\Omega^\infty$-infinite representation type.

Key Words: Igusa-Todorov Functions, Gorenstein Algebra, Path ring.

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1. Introduction

One of the most important conjectures in Representation Theory of Artin algebras is the finitistic conjecture. It states that $\sup\{\text{pd}(M) : M$ is a finitely generated module of finite projective dimension\} is finite. In an attempt to prove the conjecture Igusa and Todorov defined in [9] two functions from the objects of mod$A$ (the category of finitely generated right modules over an Artin algebra $A$) to the natural numbers, which generalizes the notion of projective dimension. Nowadays they are known as the Igusa-Todorov functions, $\phi$ and $\psi$. One of its nicest features is that they are finite for each module, and allow us to define the $\phi$-dimension and the $\psi$-dimension of an algebra. These are new homological measures in the module category. In particular it holds that

$$\text{findim}(A) \leq \phi \dim(A) \leq \psi \dim(A) \leq \text{gldim}(A)$$

and they all agree in the case of algebras with finite global dimension.

This article is organized as follows: after the introduction and the preliminary section devoted to fixing the notation and recalling the basic facts needed in this work, section 3 is devoted to Igusa-Todorov function for path rings. The main results in this section are the following

**Theorem A:** Let $A$ be a finite dimensional $k$-algebra. If $Q$ is a finite acyclic quiver with at least two vertices, then the inequality below holds:

$$\phi \dim AQ \geq \phi \dim A + 1.$$ 

In section 4, using results given in Cohen-Macaulay and Gorenstein Algebras [2] and the previous theorem, we obtain the following result

**Theorem B:** Let $A$ be a finite dimensional $k$-algebra and $Q$ be a finite acyclic quiver with at least two vertices. Then $A$ is $n$-Gorenstein if and only if $AQ$ is $(n + 1)$-Gorenstein.

In [11] it was proved that if $A$ is of $\Omega^n$-finite representation type for some $n$ then $\phi \dim(A)$ and $\psi \dim(A)$ are both finite. In this article, as a consequence of the above theorem, we give an example of a family of algebras of $\Omega^\infty$-infinite representation type with finite $\phi$-dimension and $\psi$-dimension.

2. Preliminaries

2.1. Some notation and definitions

In this article, $A$ denotes a finite dimensional algebra over a field $k$ and $\text{mod}A$ the category of finitely generated right $A$-modules. For $M$ in $\text{mod}A$, $\text{pd}M$ and $\text{id}M$ are the projective and injective dimension of $M$ respectively.

2.2. Gorenstein projective modules and Gorenstein algebras

The concept of Gorenstein projective module goes back to a work of Auslander and Bridger [1]. In this work it was introduced the $G$-dimension for finitely generated modules over a two-sided noetherian ring. Later was proved by Avramov, Martisinkovsky, and Rieten that if $M$ is a finitely generated module, $M$ is Gorenstein projective if and only if the $G$-dimension of $M$ is zero ([3, Theorem 4.2.6]).

**Definition 2.1.** [5]

A finitely generated $A$-module $G$ is **Gorenstein projective** if there exist a complex such that $G \cong \text{Ker} P_0$, $P_i$ is a projective module for $i \in \mathbb{Z}$ and the following complex is exact:

$$\cdots \longrightarrow P_{-2} \xrightarrow{P_{-2}} P_{-1} \xrightarrow{P_{-1}} P_0 \xrightarrow{P_0} P_1 \xrightarrow{P_1} P_2 \xrightarrow{P_2} \cdots$$
where $(\cdot)^* = \text{Hom}_A(\cdot, A)$.

**Definition 2.2.** [8]

An Artin algebra $A$ is called $n$-Gorenstein if $\text{id}(A) \leq n$ and $\text{pd}(DA^{op}) \leq n$ with $n \in \mathbb{N}$, where $D = \text{Hom}_k(\cdot, k)$. An Artin algebra $A$ is called Gorenstein if it is $n$-Gorenstein for some $n \in \mathbb{N}$.

**Remark 2.3.** An Artin algebra $A$ is 0-Gorenstein if and only if $A$ is selfinjective.

The following proposition can be seen in [13]:

**Proposition 2.4.** [13, Corollary 3.4]

Let $A$ be an $n$-Gorenstein algebra, and

\[
0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_0 \longrightarrow M \longrightarrow 0
\]

be an exact sequence with $P_i$ projective, then $K$ is a Gorenstein projective $A$-module.

### 2.3. Igusa-Todorov functions

In this section, we show some general facts about the Igusa-Todorov functions. Our objective is to introduce some properties that will be used in the following sections.

**Definition 2.5.** [9]

Let $K_0$ be the abelian group generated by all symbols $[M]$, where $M$ is a f.g. $A$-module, modulo the relations:

1. $[M] = [M'] - [M'']$ if $M \cong M' \oplus M''$.
2. $[P] = 0$ if $P$ is projective.

The group $K_0$ may also be described as the free abelian group with basis the set of symbols $[M]$, one for each isomorphism class of indecomposable non-projective module. Moreover every element in $K_0$ can be written in the form $[M] - [N]$, for some pair of, not necessarily indecomposable, $M, N$ $A$-modules.

For any finitely generated $A$-module $M$ let $\overline{\Omega}[M] = [\Omega M]$. Since $\Omega$ commutes with direct sums and takes projective modules to zero this gives a group homomorphism $\overline{\Omega} : K_0 \rightarrow K_0$.

For every finitely generated $A$-module $M$, let $\langle \text{add}M \rangle$ denotes the subgroup of $K_0$ generated by the classes of indecomposable summands of $M$.

**Definition 2.6.** The **Igusa-Todorov function** $\phi$ is defined for $M \in \text{mod}A$, as

\[
\phi(M) = \min \left\{ \overline{\Omega}^{s+1}, (\text{add}M) \text{ is a monomorphism for all } s \in \mathbb{N} \right\}.
\]

**Definition 2.7.** The **Igusa-Todorov function** $\psi$ is defined for $M \in \text{mod}A$, as

\[
\psi(M) = \phi(M) + \sup \left\{ \text{pd}(N) : \Omega^{\phi(M)}(M) = N \oplus N' \text{ and } \text{pd}(N) < \infty \right\}.
\]
The main properties of the Igusa-Todorov functions are summarised below. One can find the next propositions in [9] and [7].

**Proposition 2.8.** [9], [7]
Let $A$ be an Artin algebra and $M, N \in \text{mod}(A)$. Then, the following statements hold.

1. $\phi(M) = \text{pd}(M)$ if $M$ has finite projective dimension.
2. $\phi(M) = 0$ if $M$ is indecomposable and has infinite projective dimension.
3. $\phi(M) \leq \phi(M \oplus N)$.
4. $\phi(M^k) = \phi(M)$ for $k \in \mathbb{N}$.
5. $\phi(M) \leq \phi(\Omega(M)) + 1$.

**Proposition 2.9.** [9], [7]
Let $A$ be an Artin algebra and $M, N \in \text{mod}(A)$. Then, the following statements hold.

1. $\psi(M) = \text{pd}(M)$ if $M$ has finite projective dimension.
2. $\psi(M) \leq \psi(M \oplus N)$.
3. $\psi(M^k) = \psi(M)$ for $k \in \mathbb{N}$.
4. If $N$ is direct summand of $\Omega^n(M)$ where $n \leq \phi(M)$ and $\text{dp}(N) < \infty$, then $\text{dp}(N) + n \leq \psi(M)$.
5. $\psi(M) \leq \psi(\Omega(M)) + 1$.

The following definitions were introduced in [7].

**Definition 2.10.** [7]
Let $A$ be an Artin algebra. We recall that the $\phi$-dimension and $\psi$-dimension of $A$ are defined as follows:

- $\phi\text{dim}(A) = \sup\{\phi(M) \mid M \in \text{mod}A\}$.
- $\psi\text{dim}(A) = \sup\{\psi(M) \mid M \in \text{mod}A\}$

The following results give properties of the Igusa-Todorov functions for an Artin algebra $A$ with $\text{id}(A) < \infty$. For the proof see [11].

**Proposition 2.11.** [11, Corollary 3.17]
Let $A$ be an Artin algebra such that $\text{id}(A) = n < \infty$, then

$$\text{findim}(A) \leq \phi\text{dim}(A) \leq \psi\text{dim}(A) \leq n.$$
Proposition 2.12. [11, Corollary 4.7]
If \( A \) is a Gorenstein algebra, then

1. \( \phi \dim(A) = \psi \dim(A) = \text{findim}(A) < \infty \).

2. Let \( m \) be the minimum natural number such that \( A \) is an \( m \)-Gorenstein algebra then:

\[
\phi \dim(A) = \psi \dim(A) = \text{findim}(A) = m.
\]

2.4. Path rings

A quiver \( Q \) consists of:

- The set \( Q_0 \) of vertices of \( Q \).
- The set \( Q_1 \) of arrows of \( Q \).
- Two functions: \( s : Q_1 \to Q_0 \) giving the start or source of the arrow, and \( t : Q_1 \to Q_0 \) giving the target of the arrow.

We say that a quiver \( Q \) is finite if \( Q_0 \) and \( Q_1 \) are finite. A path of length \( n \) in \( Q \) is a sequence of arrows \( \alpha_n \alpha_{n-1} \ldots \alpha_3 \alpha_2 \alpha_1 \) such that \( t(\alpha_{i+1}) = s(\alpha_i) \). We also agree to associate with each point \( a \in Q_0 \) a path of length 0, called the trivial path at \( a \), and denoted by \( e_a \). For the composition of paths, we use the convention of concatenating paths from right to left. We denote by \( P(v, w) \) the set of paths with start \( v \) and target \( w \).

By the previous facts, we observe that \( Q \) can be considered as a category.

Note 1. Given a finite dimensional \( k \)-algebra \( A \) and a finite acyclic quiver \( Q \), we denote by \( \text{Rep}_A(Q) \) the category of functors from \( Q \) to \( \text{mod} A \).

Note 2. We denote by \( AQ \) the path algebra with quiver \( Q \) and coefficients over \( A \), i.e. \( A \otimes_k kQ \).

Remark 2.13. The categories \( \text{Rep}_A(Q) \) and \( \text{Rep}_k(AQ) \) are equivalent for any finite dimensional \( k \)-algebra \( A \) and any finite acyclic quiver \( Q \).

Definition 2.14. Let \( A \) be a finite dimensional \( k \)-algebra and a finite acyclic quiver \( Q \). Given \( M \) an \( A \)-module and \( v \) a vertex of \( Q \), then we denote by \( M^v \) the \( AQ \)-module such that:

- \( M^v(v) = M \), \( M^v(w) = 0 \) if \( w \neq v \), and
- \( M^v(\alpha) = 0 \) for every arrow in \( Q_1 \).

Definition 2.15. Let \( Q \) be a finite acyclic quiver. We denote by:

- \( \bar{P}^v \), if \( P \) is a projective \( A \)-module, the following \( AQ \) module:
Definition 2.16. Let $P$ be a finite dimensional $k$-algebra and $Q$ be a finite acyclic quiver. If $\iota: M \hookrightarrow P$ is a monomorphism of $A$-modules where $P$ is a projective module and $v$ a vertex in $Q$, then we denote by $MP^v$ the following $AQ$-module:

- $\tilde{I}^v$, if $I$ is an injective $A$-module, the following $AQ$ module:

\[
\begin{align*}
\tilde{I}^v(w) &= \begin{cases} 
I & \text{if } w = v; \\
\oplus_{\lambda \in P(v,w)} P_{\lambda} & \text{if } w \neq v \text{ and } P(v,w) \neq \emptyset; \\
0 & \text{otherwise},
\end{cases} \\
\tilde{I}^v(\alpha) &= \begin{cases} 
f_{\alpha} & \text{if } \alpha \in P(w_1,w_2) \text{ and } P(w_2,v) \neq \emptyset; \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]

where $f_{\alpha} = \sum_{\lambda \in P(v,w_1)} \tilde{I}^v(w_1) P_{\lambda} \overset{\lambda}{\hookrightarrow} P_{\alpha} I_{\lambda} \overset{1_{\lambda}}{\hookrightarrow} \tilde{I}^v(w_2)$.

\[\circ \quad MP^v(w) = \begin{cases} 
M & \text{if } w = v; \\
\oplus_{\lambda \in P(v,w)} P_{\lambda} & \text{if } w \neq v \text{ and } P(v,w) \neq \emptyset; \\
0 & \text{otherwise},
\end{cases} \]

where $P_{\lambda} = P$ for all $\lambda \in P(v,w)$.

\[\circ \quad MP^v(\alpha) = \begin{cases} 
t_{\alpha} & \text{if } \alpha \text{ starts in } v; \\
f_{\alpha} & \text{if } \alpha \in P(w_1,w_2) \text{ and } P(v,w_1) \neq \emptyset; \\
0 & \text{otherwise},
\end{cases} \]

where:

\[\circ \quad t_{\alpha} = M \overset{\iota_{\alpha}}{\hookrightarrow} P_{\alpha} \overset{\oplus_{\lambda \in P(v,w)} P_{\lambda}}{\hookrightarrow} M P^v(\alpha) \overset{1_{\lambda}}{\hookrightarrow} MP^v(w_2).\]

Remark 2.17. Given $M$, $N$, $P$ and $P'$ $A$-modules where $P$ and $P'$ are projectives and the morphisms $\iota: M \to P$ and $\iota': N \to P'$ are inclusions, then $MP^v \cong NP^v$ if and only if the following commutative diagram exists:

\[
\begin{array}{c}
M \\ \iota \downarrow \\
P \\
\end{array} \cong
\begin{array}{c}
N \\ \iota' \\
P' \\
\end{array}
\]
The following theorem can be found in [4] in a more general version:

**Theorem 2.18.** Let $Q$ be a finite quiver without oriented cycles and $A$ a finite dimensional $k$-algebra. A representation $P$ of $AQ$ is projective if and only if the following conditions are satisfied:

1. For every $v \in Q$, $P(v)$ is a projective $A$-module.
2. For every $v \in Q$, the morphism $\oplus_{t(\alpha)=v} P(s(\alpha)) \to P(v)$ (where $P(s(\alpha)) \to P(v)$ is $P(\alpha)$) is a split monomorphism.

The next theorem is the dual version of the previous one.

**Theorem 2.19.** Let $Q$ be a finite quiver without oriented cycles and $A$ be a finite dimensional $k$-algebra. A representation $I$ of $AQ$ is injective if and only if the following conditions are satisfied:

1. For every $v \in Q$, $I(v)$ is an injective $A$-module.
2. For every $v \in Q$, the morphism $I(v) \to \oplus_{s(\alpha)=v} I(t(\alpha))$ (where $I(v) \to I(t(\alpha))$ is $I(\alpha)$) is a split epimorphism.

**Remark 2.20.** From Theorem 2.18 we can see that every indecomposable projective module is of the form $\overline{P}^v$ where $P$ is an indecomposable projective module and $v \in Q_0$. Dually from Theorem 2.19 one can see that every indecomposable injective module is of the form $\tilde{I}^v$ where $I$ is an indecomposable injective module and $v \in Q_0$.

### 3. Igusa-Todorov functions for path rings

The objective for this section is to relate the $\phi$-dimension and $\psi$-dimension of $A$ and $AQ$.

Let us begin, by computing the syzygies for a particular class of $AQ$-modules.

**Proposition 3.1.** Let $A$ be a $k$-algebra and $Q$ be a finite acyclic quiver. If $M$ is a finitely generated $A$-module and $v$ a vertex of $Q$, then following results holds:

1. If $v$ is a sink, then $\Omega^k_{AQ}(M^v) = (\Omega^k_A(M))^v$.
2. If $v$ is not a sink, then $\Omega^k_{AQ}(M^v) = \Omega^k_A(M)P_{k-1}^v$, where

   $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$

   is the minimal projective resolution of $M$.

**Proof:**

1. Is clear.
2. Consider the AQ-module \((\Omega_A M)P^v_0\). By Definition 2.16, it has the following shape:

\[
(\Omega_A M)P^v_0(w) = \begin{cases} 
\Omega_A M, & \text{if } w = v; \\
\oplus_{\lambda \in \mathcal{P}(v, w)} P_\lambda, & \text{if } w \neq v \text{ and } \mathcal{P}(v, w) \neq \emptyset; \\
0, & \text{otherwise.}
\end{cases}
\]

where \(P_\lambda = P_0\) for all \(\lambda \in \mathcal{P}(v, w)\).

Let \(\alpha\) be an arrow starting at \(v\). The following diagram shows locally the syzygy of \(M^v\) at the arrow \(\alpha\):

\begin{align*}
0 &\longrightarrow \Omega(M) \overset{i}{\longrightarrow} P_0 \underset{i}{\longrightarrow} M \overset{1}{\longrightarrow} 0 \\
0 &\longrightarrow P_0 \overset{1}{\longrightarrow} P_0 \overset{1}{\longrightarrow} 0 \overset{0}{\longrightarrow} 0
\end{align*}

If \(\alpha\) is an arrow that does not start at \(v\), but \(\alpha\) belongs to any path starting at \(v\), then there exists the following commutative diagram with exact rows:

\begin{align*}
0 &\longrightarrow P_0 \overset{1}{\longrightarrow} P_0 \overset{1}{\longrightarrow} 0 \overset{0}{\longrightarrow} 0 \\
0 &\longrightarrow P_0 \overset{1}{\longrightarrow} P_0 \overset{1}{\longrightarrow} 0 \overset{0}{\longrightarrow} 0
\end{align*}

and therefore \(\Omega_{AQ}(M^v) = (\Omega_A M)P^v_0\).

Suppose that \(\Omega^k(M^v) = (\Omega^k(M)P_{k-1})^v\).

If \(\alpha\) is an arrow that start at \(v\), we obtain the following commutative diagram with exact rows, such that the first one locally represents \(\Omega^k_{AQ}(M^v)\) at \(\alpha\):

\begin{align*}
0 &\longrightarrow \Omega^{k+1}(M) \overset{i_k}{\longrightarrow} P_k \overset{f_k}{\longrightarrow} \Omega^k M \overset{0}{\longrightarrow} 0 \\
0 &\longrightarrow P_k \overset{j_k}{\longrightarrow} P_k \oplus P_{k-1} \overset{0}{\longrightarrow} P_{k-1} \overset{0}{\longrightarrow} 0
\end{align*}

where \(j_k = (1_{P_k}, 0)\). We deduce that the following diagram commutes:

\[
\begin{array}{ccc}
\Omega^{k+1}(M) & \overset{i_k}{\longrightarrow} & \Omega^{k+1}(M) \\
\downarrow & & \downarrow \\
P_k & \overset{h_k}{\longrightarrow} & P_k
\end{array}
\]
where $h_k$ is the first coordinate of the map $j'_k$. The second coordinate of $(i'_{k-1} \circ f_k, 1_{P_{k-1}})$ is an isomorphism, then $h_k$ must be a monomorphism. Therefore the rows of the previous diagrams are isomorphisms. This implies that the left column can be changed by the right one in the diagram above.

If $\alpha$ is an arrow that does not start at $v$, but it belongs to a path that starts at $v$, then the following diagram with exact rows commutes:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & P_k & \rightarrow & P_k \oplus P_{k-1} & \rightarrow & P_{k-1} & \rightarrow & 0 \\
0 & \rightarrow & P_k & \rightarrow & P_k \oplus P_{k-1} & \rightarrow & P_{k-1} & \rightarrow & 0 \\
\end{array}
\]

therefore the result follows.

\[
\mbox{\rule{1cm}{0.1mm}}
\]

Remark 3.2. Assume the hypothesis of Proposition 3.1. Let $M$ be a finitely generated $A$-module. By using the fact that $0P^v$ is a projective $AQ$-module, if $M$ is a finite dimensional $A$-module, it follows that:

1. If $v$ is a sink, then $\text{pd}(M^v) = \text{pd}(M)$.
2. If $v$ is not a sink, then $\text{pd}(M^v) = \text{pd}(M) + 1$ and $\text{pd}(MP^v) = \text{pd}(M) + 1$.

In the following proposition we compute the Igusa-Todorov function $\phi$ for $AQ$-modules of the form $M^v$.

Proposition 3.3. Let $A$ be a $k$-algebra and $Q$ a finite acyclic quiver. If $M$ is an $A$-module and $v$ a vertex of $Q$, then the following results hold:

1. If $v$ is a sink, then $\phi(M^v) = \phi(M)$.
2. If $v$ is not a sink, then $\phi(M^v) = \phi(M) + 1$.

Proof:

1. It is clear by the first part of Proposition 3.1.
2. Let $M$ be an $A$-module such that $\phi(M) = k$ and $M = \oplus_{i \in I} M_i^I$; the decomposition into indecomposables modules of $M$.

Consider the linear combination $\sum_{i \in I} \alpha_i \overline{\Omega}^k (\{M_i\}) = 0$ for $k \geq 1$, such that $\sum_{i \in I} \alpha_i \overline{\Omega}^{k-1} (\{M_i\}) \neq 0$. If we prove that the previous condition implies that
\[
\sum_{i \in I} \alpha_i \Omega^k([M^v_i]) = 0 \quad \text{and} \quad \sum_{i \in I} \alpha_i \Omega^k([M^v_i]) \neq 0,
\]
we obtain the inequality \( \phi(M^v) \geq \phi(M) + 1 \).

**Claim:**

If \( \sum_{i \in I} \alpha_i \Omega^k([M^v_i]) = 0 \) and \( \sum_{i \in I} \alpha_i \Omega^{k-1}([M^v_i]) \neq 0 \), we obtain the inequality \( \phi(M^v) \geq \phi(M) + 1 \).

Suppose that \( \sum_{i \in I} \alpha_i \Omega^k([M_i]) = 0 \). Then there exist projectives \( P_1 \) and \( P_2 \) such that \( \bigoplus_{i \in I_1} \Omega^k(M_i)^{\beta_i} \oplus P_1 \cong \bigoplus_{i \in I_2} \Omega^k(M_i)^{\beta_i} \oplus P_2 \), where \( I = I_1 \cup I_2 \) is a disjoint union and the exponents \( \beta_i = |\alpha_i| \) for every \( i \in I \). Then it follows the next commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & \bigoplus_{i \in I_1} \Omega^k(M_i)^{\beta_i} & \to & P & \to & \bigoplus_{i \in I_1} \Omega^k(M_i)^{\beta_i} & \oplus & P_1 & \to & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 & \to & \bigoplus_{i \in I_2} \Omega^k(M_i)^{\beta_i} & \to & P' & \to & \bigoplus_{i \in I_2} \Omega^k(M_i)^{\beta_i} & \oplus & P_2 & \to & 0
\end{array}
\]

Using Remark 2.17 in the left square of the previous diagram, we obtain the following:

\[
(\bigoplus_{i \in I_1} \Omega^k(M_i)^{\beta_i}) P^v \cong (\bigoplus_{i \in I_2} \Omega^k(M_i)^{\beta_i}) P'^v.
\]

Therefore by Proposition 3.1 \( \sum \alpha_i \Omega^{k+1}([M^v_i]) = 0 \).

Suppose that \( \sum \alpha_i \Omega^k([M^v_i]) = 0 \). This implies, by Remark 2.17, that the following diagram is commutative with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & \bigoplus_{i \in I_1} \Omega^k(M_i)^{\beta_i} & \to & P & \to & \bigoplus_{i \in I_1} \Omega^{k-1}(M_i)^{\beta_i} & \oplus & P'_1 & \to & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 & \to & \bigoplus_{i \in I_2} \Omega^k(M_i)^{\beta_i} & \to & P' & \to & \bigoplus_{i \in I_2} \Omega^{k-1}(M_i)^{\beta_i} & \oplus & P'_2 & \to & 0
\end{array}
\]

and we obtain the relation \( \sum \alpha_i \Omega^{k-1}([M_i]) = 0 \), which is a contradiction.

\[\square\]

As a consequence of the previous results, we obtain the following theorem that is trivial in the case \( Q \) has one vertex:
**Theorem 3.4.** Let $A$ be a finite dimensional $k$-algebra. If $Q$ is a finite acyclic quiver with at least two vertices, then the inequality below holds:

$$\phi \dim AQ \geq \phi \dim A + 1.$$ 

**Proof:**

It follows from Proposition 3.3. 

**4. Special path rings**

In this section we study the $\phi$-dimension for algebras $AQ$ when $A$ is a Gorenstein algebra. Using these computations, we show an example of an algebra of $\Omega^\infty$-infinite representation type with finite $\phi$-dimension.

The following theorem can be found in [2].

**Theorem 4.1.** [2, Proposition 2.2] If $A$ and $B$ are finite dimensional algebras over a field $k$, then

$$\max(\text{id}_A A, \text{id}_B B) \leq \text{id}_{A \otimes B}(A \otimes_k B) \leq \text{id}_A A + \text{id}_B B.$$ 

The last theorem shows that tensor products of Gorenstein algebras are Gorenstein algebras. Now, by the above theorem and Theorem 3.4 we obtain the next result:

**Theorem 4.2.** Let $A$ be a finite dimensional $k$-algebra and $Q$ be a finite acyclic quiver with at least two vertices. Then $A$ is $n$-Gorenstein if and only if $AQ$ is $(n+1)$-Gorenstein.

**Proof:**

Let $n$ be the minimum non-negative integer such that $A$ is an $n$-Gorenstein algebra. By Corollary 2.12 and Theorem 3.4 we obtain $\phi \dim AQ \geq \phi \dim A + 1 = n + 1$. Again by Corollary 2.12 we have that $\text{id}_{AQ} AQ = \phi \dim AQ \geq n + 1$. On the other hand, the inequality $\text{id}_{AQ} AQ \leq \text{id}_A A + 1 < \infty$ follows from Theorem 4.1. This proves that $\text{id}_{AQ} AQ = n + 1$, thus $AQ$ is $(n + 1)$-Gorenstein with minimum $n$.

Now suppose that $AQ$ is an $(n + 1)$-Gorenstein algebra. Let $I$ be an indecomposable injective $A$-module with $\text{pd}(\tilde{I}) = v \leq n + 1$ for $v$ a source (therefore it is not a sink of $Q$). This implies that $\text{pd} I = k - 1$, by Remark 3.2. The proof is analogous if we consider the injective dimension of an indecomposable projective $A$-module. Thus $A$ is a $t$-Gorenstein algebra with $t \leq n$. 

As a consequence of the previous theorem we obtain another proof for the following result of [12].

**Corollary 4.3.** [12, Lemma 4.1] The algebra $T_n(A)$ with $n \geq 2$ is $(m + 1)$-Gorenstein if and only if $A$ is $m$-Gorenstein, where:
\[ T_n(A) = \begin{pmatrix} A & 0 & \cdots & 0 & 0 \\ A & A & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & \cdots & A & 0 \\ A & A & \cdots & A & A \end{pmatrix} \]

**Proof:**

It is easy to see that \( T_n(A) \cong AQ \) where \( Q \) is an \( A_n \) Dynkin type quiver with its arrows in the same orientation. Then by Theorem 4.2 we obtain the thesis. \( \square \)

**Definition 4.4.** Given an Artin algebra \( A \), we define \( K_n \) as

\[ K_n = \{ [M] : \text{there exist } N \in \text{mod} A \text{ such that } \Omega^n(N) = M \} \]

**Definition 4.5.** Let \( A \) be an Artin algebra. The class \( K_n \) is of finite type if there exists an \( A \)-module \( N \) such that \( M \in \text{add} N \) for all \([M] \in K_n \). In this case \( A \) is called of \( \Omega^n \)-finite representation type. If \( A \) is not of \( \Omega^n \) finite representation type for any \( n \in \mathbb{N} \) we say that \( A \) is \( \Omega^\infty \)-infinite representation type.

In [11] was proved that if \( A \) is of \( \Omega^n \)-finite representation type for some \( n \) then \( \phi \dim(A) \) and \( \psi \dim(A) \) are both finite. A natural question to ask is which is the behaviour of these dimensions in the case that \( A \) is of \( \Omega^\infty \)-infinite representation type. we show in Example 4.1, that they can be finite.

**Proposition 4.6.** [11, Proposition 4.2]

Given \( G \) a Gorenstein projective module, \([G] \in K_i \) for every \( i \in \mathbb{N} \).

Therefore if \( A \) has infinite indecomposable Gorenstein projective modules up to isomorphism \( (A \text{ is CM-infinite}), A \) is of \( \Omega^\infty \)-infinite representation type.

**Proposition 4.7.** Let \( A \) be a Gorenstein algebra and \( Q \) be a finite acyclic quiver with at least two vertices. If \( M \) is a Gorenstein indecomposable \( A \)-module and \( v \) a sink of \( Q \), then \( M^v \) is a Gorenstein indecomposable \( AQ \)-module.

**Proof:**

Suppose that \( A \) is an \( n \)-Gorenstein algebra. By Theorem 4.2 \( AQ \) is an \((n + 1)\)-Gorenstein algebra. Since \( M \) is Gorenstein, there exists \( N \) a \( A \)-module such that \( \Omega_A^{n+1}(N) = M \). Using Proposition 3.1 it follows that \( M^v = \Omega_A^{n+1}(N^v) \), and by Proposition 2.4 the result is obtained. \( \square \)

The following example exhibits a family of algebras \( \{A_n\}_{n \in \mathbb{N}} \) such that \( \phi \dim(A_n) = n \) and every \( A_n \) is \( \Omega^\infty \)-infinite representation type.

**Example 4.1.** Let \( A \) be the radical square zero algebra with associated quiver \( Q \) as follows:

\[
\begin{array}{c}
1 \rightarrow 2
\end{array}
\]
It is clear that $A$ is a selfinjective $k$-algebra.

Now consider $A' = A^op \otimes A$. It is not difficult to see that $A'$ is a selfinjective algebra of infinite representation type. Consider $\Gamma$ a finite acyclic quiver, with a sink $v$. $A_k$ denotes the following algebras:

$$A_k = \begin{cases} A^e & \text{if } k = 0 \\ A_k = A_{k-1}\Gamma & \text{if } k \geq 1 \end{cases}$$

Then $A_k$ is a $k$-Gorenstein algebra, but is not $(k-1)$-Gorenstein for $k \geq 1$, in particular $\dot{\phi}\dim(A_k) = \psi\dim(A_k) = k$ and all of them are of $\Omega^\infty$-infinite representation type because $\mathfrak{g}P_A$ is of infinite representation type by Proposition 4.7.

References


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