$n$-absorbing and Strongly $n$-absorbing Second Submodules

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ABSTRACT: In this paper, we introduce the concepts of $n$-absorbing and strongly $n$-absorbing second submodules as a dual notion of $n$-absorbing submodules of modules over a commutative ring and obtain some related results. In particular, we investigate some results concerning strongly 2-absorbing second submodules.

Key Words: Strongly $n$-absorbing second submodule, $n$-absorbing second submodule, Weakly strongly 2-absorbing second submodule.

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1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. Let $N$ be a submodule of an $R$-module $M$. For $r \in R$, $(N :_M r)$ will denote $(N :_M r) = \{m \in M : rm \in N\}$. Clearly, $(N :_M r)$ is a submodule of $M$ containing $N$.

Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be prime if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [17]. A non-zero submodule $S$ of $M$ is said to be second if for each $a \in R$, the homomorphism $S \rightarrow S$ is either surjective or zero [26]. In this case $\text{Ann}_R(S)$ is a prime ideal of $R$. A proper submodule $N$ of $M$ is said to be completely irreducible if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$ [19].

The concept of 2-absorbing ideals was introduced in [11] and then extended to $n$-absorbing ideals in [1]. A proper ideal $I$ of $R$ is a 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Let $I$ be a proper ideal of $R$ and $n$ a positive integer. $I$ is called an $n$-absorbing ideal of $R$ if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are $n$ of the $x_i$'s whose product is in $I$.

The authors in [15] and [24], extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule $N$ of $M$ is called a 2-absorbing submodule of $M$ if
whenever \( abm \in N \) for some \( a, b \in R \) and \( m \in M \), then \( am \in N \) or \( bm \in N \) or \( ab \in (N :_R M) \). A proper submodule \( N \) of \( M \) is said to be a weakly 2-absorbing submodule of \( M \) if whenever \( a, b \in R \) and \( m \in M \) with \( 0 \neq abm \in N \), then \( ab \in (N :_R M) \) or \( am \in N \) or \( bm \in N \) [15]. A proper submodule \( N \) of \( M \) is called \( n \)-absorbing submodule of \( M \) if whenever \( a_1, \ldots, a_n, m \in N \) for \( a_1, \ldots, a_n \in R \) and \( m \in M \), then either \( a_1 \cdots a_n \in (N :_R M) \) or there are \( n - 1 \) of \( a_i \)’s whose their product with \( m \) is in \( N \) [16]. Several authors investigated properties of 2-absorbing, and some generalization of 2-absorbing submodules, for example [15, 16, 22, 23, 24, 25].

In [2], the authors introduced the dual notion of 2-absorbing submodules (that is, 2-absorbing (resp. strongly 2-absorbing) second submodules) of \( M \) and investigated some properties of these classes of modules. A non-zero submodule \( N \) of \( M \) is said to be a 2-absorbing second submodule of \( M \) if whenever \( a, b \in R \), \( L \) is a completely irreducible submodule of \( M \), and \( abN \subseteq L \), then \( aN \subseteq L \) or \( bN \subseteq L \) or \( ab \in \text{Ann}_R(N) \). A non-zero submodule \( N \) of \( M \) is said to be a strongly 2-absorbing second submodule of \( M \) if whenever \( a, b \in R \), \( K \) is a submodule of \( M \), and \( abN \subseteq K \), then \( aN \subseteq K \) or \( bN \subseteq K \) or \( ab \in \text{Ann}_R(N) \). Also, in [3, 4], the authors introduced and investigated some generalization of 2-absorbing second and strongly 2-absorbing second submodules.

The purpose of this paper is to introduce the concepts of \( n \)-absorbing and strongly \( n \)-absorbing second submodules as dual notion of \( n \)-absorbing submodules of modules and provide some information concerning these new classes of modules. Furthermore, we study some properties of strongly 2-absorbing second submodules of an \( R \)-module \( M \). Also, we introduce the concept of weakly strongly 2-absorbing second submodules of \( M \) as a dual notion of weakly 2-absorbing submodules and obtain some related results.

2. \( n \)-absorbing and strongly \( n \)-absorbing second submodules

**Definition 2.1.** Let \( N \) be a non-zero submodule of an \( R \)-module \( M \) and \( n \) be a positive integer. We say that \( N \) is an \( n \)-absorbing second submodule of \( M \) if whenever \( a_1, \ldots, a_n, N \subseteq L \) for \( a_1, \ldots, a_n \in R \) and a completely irreducible submodule \( L \) of \( M \), then either \( a_1 \cdots a_n \in \text{Ann}_R(N) \) or there are \( n - 1 \) of \( a_i \)’s whose their product with \( N \) is a subset of \( L \).

**Remark 2.2.** Let \( N \) and \( K \) be two submodules of an \( R \)-module \( M \). To prove \( N \subseteq K \), it is enough to show that if \( L \) is a completely irreducible submodule of \( M \) such that \( K \subseteq L \), then \( N \subseteq L \) [9, 2.1].

We recall that an \( R \)-module \( M \) is said to be a cyclic module if \( \text{Soc}_R(M) \) is a large and simple submodule of \( M \) [27]. (Here \( \text{Soc}_R(M) \) denotes the sum of all minimal submodules of \( M \).) A submodule \( L \) of \( M \) is a completely irreducible submodule of \( M \) if and only if \( M/L \) is a cyclic \( R \)-module [19, 12.1.1].

**Proposition 2.3.** Let \( N \) be an \( n \)-absorbing second submodule of an \( R \)-module \( M \). Then we have the following.

(a) If \( L \) is a completely irreducible submodule of \( M \) such that \( N \not\subseteq L \), then \( (L :_R N) \) is an \( n \)-absorbing ideal of \( R \).
(b) If $M$ is a coyclic module, then $\text{Ann}_R(N)$ is an $n$-absorbing ideal of $R$.

(c) If $a \in R$, then $a^nN = a^{n+1}N$.

Proof. (a) Since $N \not\subseteq L_1$, we have $(L : R N) \neq R$. Let $a_1, a_2, \ldots, a_n, a_{n+1} \in R$ and $a_1a_2\ldots a_{n+1} \in (L : R N)$. Then $a_1a_2\ldots a_nN \subseteq (L : M a_{n+1})$. Thus there are $n - 1$ of $a_i$'s whose their product with $N$ is a subset of $(L : M a_{n+1})$, where $1 \leq i \leq n$ or $a_1a_2\ldots a_nN = 0$ because by [10, 2.1], $(L : M a_{n+1})$ is a completely irreducible submodule of $M$. Therefore, there are $n$ of $a_i$'s whose their product lies in $(L : R N)$ for some $1 \leq i \leq n + 1$ or $a_1\ldots a_n \in (L : R N)$ as needed.

(b) Since $M$ is coyclic, the zero submodule of $M$ is a completely irreducible submodule of $M$. Thus the result follows from part (a).

(c) It is clear that $a^{n+1}N \subseteq a^nN$. Let $L$ be a completely irreducible submodule of $M$ such that $a^{n+1}N \subseteq L$. Then $a^nN \subseteq (L : M a)$. Since $N$ is $n$-absorbing second submodule and $(L : M a)$ is a completely irreducible submodule of $M$ by [10, 2.1], $a^{n-1}N \subseteq (L : M a)$ or $a^nN = 0$. Therefore, $a^nN \subseteq L$. This implies that $a^nN \subseteq a^{n+1}N$ by Remark 2.2.

\begin{definition}
Let $N$ be a non-zero submodule of an $R$-module $M$ and $n$ be a positive integer. We say that $N$ is a strongly $n$-absorbing second submodule of $M$ if whenever $a_1\ldots a_nN \subseteq K$ for $a_1, \ldots, a_n \in R$ and a submodule $K$ of $M$, then either $a_1\ldots a_n \in \text{Ann}_R(N)$ or there are $n - 1$ of $a_i$’s whose their product with $N$ is a subset of $K$.
\end{definition}

Clearly every strongly $n$-absorbing second submodule is an $n$-absorbing second submodule. It is natural to ask the following question:

\begin{question}
Let $M$ be an $R$-module. Is every $n$-absorbing second submodule of $M$ a strongly $n$-absorbing second submodule of $M$?
\end{question}

\begin{note}
Let $a_1, a_2, \ldots, a_n \in R$. We denote by $\hat{a}_i$ the element $a_1\ldots a_{i-1}a_{i+1}\ldots a_n$. In this case, the definition of an $n$-absorbing (resp. a strongly $n$-absorbing) second submodule can be reformulated as: a non-zero submodule $N$ of an $R$-module $M$ is called $n$-absorbing (resp. strongly $n$-absorbing) second if whenever $a_1, \ldots, a_n \in R$ and $L$ is a completely irreducible submodule (resp. $K$ is a submodule) of $M$ with $a_1\ldots a_nN \subseteq L$ (resp. $a_1\ldots a_nN \subseteq K$), then either $a_1\ldots a_n \in \text{Ann}_R(N)$ or $\hat{a}_iN \subseteq L$ (resp. $\hat{a}_iN \subseteq K$) for some $1 \leq i \leq n$.
\end{note}

\begin{proposition}
Let $M$ be an $R$-module and let \{\(K_\lambda\)\}_{\lambda \in \Lambda}$ be a chain of $n$-absorbing second submodules of $M$. Then $\cup_{\lambda \in \Lambda}K_\lambda$ is an $n$-absorbing second submodule of $M$.
\end{proposition}

Proof. Let $a_1, \ldots, a_n \in R$, $L$ be a completely irreducible submodule of $M$, and $a_1\ldots a_n(\cup_{\lambda \in \Lambda}K_\lambda) \subseteq L$. Assume that $\hat{a}_i(\cup_{\lambda \in \Lambda}K_\lambda) \not\subseteq L$. Then for each $1 \leq i \leq n$ there is $\beta_i \in \Lambda$, where $\hat{a}_iK_{\beta_i} \not\subseteq L$. Hence, for every $K_{\beta_i} \subseteq K_{\alpha_i}$, we have $\hat{a}_iK_{\alpha_i} \not\subseteq L$. Therefore, for each submodule $K_\alpha$ such that $K_{\beta_i} \subseteq K_\alpha$ (for each $1 \leq i \leq n$), we have $\hat{a}_iK_\alpha \not\subseteq L$ for each $1 \leq i \leq n$. Thus $a_1\ldots a_nK_\alpha = 0$ as $K_\alpha$ is an $n$-absorbing
second submodules of $M$. Let $K_n$ be a submodule of $M$ such that $K_{\beta_i} \subseteq K_n$ for each $1 \leq i \leq n$. As $\{K_\lambda\}_{\lambda \in \Lambda}$ is a chain, we have

$$\bigcup_{\lambda \in \Lambda} K_\lambda = (\bigcup_{\lambda \in \Lambda} K_{\lambda}) \cup (\bigcup_{\lambda \in \Lambda} K_{\lambda}) = K_n \cup \bigcup_{\lambda \in \Lambda} K_\lambda.$$

Therefore $a_1...a_n(\bigcup_{\lambda \in \Lambda} K_\lambda) = 0$, as needed. \qed

**Definition 2.7.** We say that an $n$-absorbing second submodule $N$ of an $R$-module $M$ is a maximal $n$-absorbing second submodule of a submodule $K$ of $M$, if $N \subseteq K$ and there does not exist an $n$-absorbing second submodule $H$ of $M$ such that $N \subset H \subset K$.

**Lemma 2.8.** Let $M$ be an $R$-module. Then every $n$-absorbing second submodule of $M$ is contained in a maximal $n$-absorbing second submodule of $M$.

**Proof.** This is proved easily by using Zorn’s Lemma and Proposition 2.6. \qed

**Theorem 2.9.** Every Artinian $R$-module $M$ has only a finite number of maximal $n$-absorbing second submodules.

**Proof.** Suppose that the result is false. Let $\Sigma$ denote the collection of non-zero submodules $N$ of $M$ such that $N$ has an infinite number of maximal $n$-absorbing second submodules. The collection $\Sigma$ is non-empty because $M \in \Sigma$ and hence has a minimal member, $S$ say. Then $S$ is not $n$-absorbing second submodule. Thus there exist $a_1, a_2, ..., a_n \in R$, and a completely irreducible submodule $L$ of $M$ such that $a_1...a_n S \subseteq L$ but $\hat{a}_i S \not\subseteq L$ (for each $1 \leq i \leq n$) and $a_1...a_n S \neq 0$. Let $V$ be a maximal $n$-absorbing second submodule of $M$ contained in $S$. Then $\hat{a}_i V \subseteq L$ for some $1 \leq i \leq n$ or $a_1...a_n V = 0$. Thus $V \subseteq (L : M \hat{a}_i)$ or $V \subseteq (0 : M a_1...a_i)$. Therefore, $V \subseteq (L : S \hat{a}_i)$ or $V \subseteq (0 : S a_1...a_n)$. Hence every maximal $n$-absorbing second submodule of $S$ is a maximal $n$-absorbing second submodule of $(L : S \hat{a}_i)$ or $(0 : S a_1...a_n)$. By the choice of $S$, the modules $(L : S \hat{a}_i)$ and $(0 : S a_1...a_n)$ have only finitely many maximal $n$-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module $S$ which is a contradiction. \qed

**Definition 2.10.** We say that a strongly $n$-absorbing second submodule $N$ of an $R$-module $M$ is a maximal strongly $n$-absorbing second submodule of a submodule $K$ of $M$, if $N \subseteq K$ and there does not exist a strongly $n$-absorbing second submodule $H$ of $M$ such that $N \subset H \subset K$.

**Remark 2.11.** One can check that, by using the same technique, that the results in Proposition 2.6, Lemma 2.8, and Theorem 2.9 about $n$-absorbing second submodules is also true for strongly $n$-absorbing second submodules.

An $R$-module $M$ is said to be a *comultiplication module* if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 : M I)$, equivalently, for each submodule $N$ of $M$, we have $N = (0 : M \Ann_R(N))$ [5].
A proper ideal $I$ is a strongly $n$-absorbing ideal of $R$ if whenever $I_1,...,I_{n+1} \subseteq I$ for ideals $I_1,...,I_{n+1}$ of $R$ then there are $n$ of the $I_i$'s whose their product is in $I$ [1]. Clearly a strongly $n$-absorbing ideal of $R$ is also an $n$-absorbing ideal of $R$. Anderson and Badawi conjectured that these two concepts are equivalent, e.g., they proved that an ideal $I$ of a Prüfer domain $R$ is strongly $n$-absorbing if and only if $I$ is an $n$-absorbing ideal of $R$ [1, Corollary 6.9].

**Theorem 2.12.** Let $N$ be a submodule of an $R$-module $M$. Then we have the following.

(a) If $N$ is a strongly $n$-absorbing second submodule of $M$, then $Ann_R(N)$ is an $n$-absorbing ideal of $R$.

(b) If $M$ is a comultiplication $R$-module and $Ann_R(N)$ is a strongly $n$-absorbing ideal of $R$, then $N$ is a strongly $n$-absorbing second submodule of $M$.

*Proof.* (a) Let $N$ be a strongly $n$-absorbing second submodule of $M$. Assume that $a_1,...,a_{n+1} \in R$, with $a_1...a_{n+1} \in Ann_R(N)$. For each $1 \leq i \leq n$, let $\tilde{a}_i$ be the element of $R$ which is obtained by eliminating $a_i$ from $a_1...a_n$. Then $a_1...a_{n}N \subseteq a_1...a_{n-1}N$ implies that $\tilde{a}_iN \subseteq a_1...a_{n-1}N$ for some $1 \leq i \leq n$ because $N$ is strongly $n$-absorbing second. Thus $\tilde{a}_ia_{n+1}N = 0$ that is, $Ann_R(N)$ is $n$-absorbing.

(b) Assume that $Ann_R(N)$ is a strongly $n$-absorbing ideal of $R$. Let $a_1,...,a_n \in R$ and $K$ be a submodule of $M$ such that $a_1...a_nN \subseteq K$ and $a_1...a_nN \neq 0$. Then $a_1...a_nAnn_R(K)N = 0$. Now as $Ann_R(N)$ is a strongly $n$-absorbing ideal of $R$, $\tilde{a}_iAnn_R(K) \subseteq Ann_R(N)$ since $a_1...a_n \notin Ann_R(N)$. Thus $Ann_R(K) \subseteq Ann_R(\tilde{a}_iN)$. It follows that $\tilde{a}_iN \subseteq K$ since $M$ is a comultiplication $R$-module that is, $N$ is strongly $n$-absorbing second submodule of $M$.

**Theorem 2.13.** Let $N$ be a strongly $n$-absorbing second submodule of an $R$-module $M$. Then $rN$ is a strongly $n$-absorbing second submodule of $M$ for all $r \in R \setminus Ann_R(N)$.

*Proof.* Let $a_1...a_n rN \subseteq K$ for some $a_1,...,a_n \in R$ and a submodule $K$ of $M$. Then $a_1a_2...a_nN \subseteq (K :_M r)$. So either $a_1...a_n \in Ann_R(N)$ or there are $n - 1$ of $\tilde{a}_i$’s whose their product with $N$ is a subset of $(K :_M r)$. If $a_1...a_n \in Ann_R(N)$, since $Ann_R(N) \subseteq Ann_R(rN)$ we are done. In other case, there are $n - 1$ of $\tilde{a}_i$’s whose their product with $N$ is a subset of $(K :_M r)$ implies that there is a product of $n - 1$ of the $\tilde{a}_i$’s with $rN$ is a subset of $K$. Thus $rN$ is a strongly $n$-absorbing second submodule of $M$.

An $R$-module $M$ is said to be a multiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$ [12].

**Corollary 2.14.** Let $R$ be a principal ideal domain and $M$ be a multiplication strongly $n$-absorbing second $R$-module. Then every submodule of $M$ is a strongly $n$-absorbing second submodule of $M$. 

Proof. This follows from Theorem 2.13.

Proposition 2.15. Let \( f : M \to \hat{M} \) be a monomorphism of \( R \)-modules. Then we have the following.

(a) If \( N \) is a strongly \( n \)-absorbing second submodule of \( M \), then \( f(N) \) is a strongly \( n \)-absorbing second submodule of \( \hat{M} \).

(b) If \( \hat{N} \) is a strongly \( n \)-absorbing second submodule of \( f(M) \), then \( f^{-1}(\hat{N}) \) is a strongly \( n \)-absorbing second submodule of \( M \).

Proof. (a) Since \( N \neq 0 \) and \( f \) is a monomorphism, we have \( f(N) \neq 0 \). Let \( a_1, a_2, \ldots, a_n \in R \), \( K \) be a submodule of \( M \), and \( a_1a_2\ldots a_nf(N) \subseteq \hat{K} \). Then \( a_1a_2\ldots a_iN \subseteq f^{-1}(\hat{K}) \). As \( N \) is strongly \( n \)-absorbing second submodule, \( \hat{a}_iN \subseteq f^{-1}(\hat{K}) \) for some \( 1 \leq i \leq n \) or \( a_1a_2\ldots a_nN = 0 \). Therefore,

\[ \hat{a}_i f(N) \subseteq f(f^{-1}(\hat{K})) = f(M) \cap \hat{K} \subseteq \hat{K} \]

or \( a_1a_2\ldots a_nf(N) = 0 \), as needed.

(b) If \( f^{-1}(\hat{N}) = 0 \), then \( f(M) \cap \hat{N} = f(f^{-1}(\hat{N})) = f(0) = 0 \). By assumption, \( \hat{N} \subseteq f(M) \). Therefore \( \hat{N} = 0 \), a contradiction. Therefore, \( f^{-1}(\hat{N}) \neq 0 \). Now let \( a_1, a_2, \ldots, a_n \in R \), \( K \) be a submodule of \( M \), and \( a_1a_2\ldots a_nf^{-1}(\hat{N}) \subseteq K \). Then

\[ a_1a_2\ldots a_n \hat{N} = a_1a_2\ldots a_n(f(M) \cap \hat{N}) = a_1a_2\ldots a_n f(f^{-1}(\hat{N})) \subseteq f(K). \]

Thus as \( \hat{N} \) is strongly \( n \)-absorbing second submodule, \( \hat{a}_i \hat{N} \subseteq f(K) \) for some \( 1 \leq i \leq n \) or \( a_1a_2\ldots a_n \hat{N} = 0 \). Therefore, \( \hat{a}_i f^{-1}(\hat{N}) \subseteq f^{-1}(f(K)) = K \) or \( a_1a_2\ldots a_nf^{-1}(\hat{N}) = 0 \), as desired.

Theorem 2.16. Let \( M \) be an \( R \)-module. If \( N_1 \) is a strongly \( n_1 \)-absorbing second submodule of \( M \) for each \( 1 \leq i \leq k \), then \( N_1 + \ldots + N_k \) is a strongly \( n \)-absorbing second submodule of \( M \) for \( n = n_1 + \ldots + n_k \). In particular, if \( N_1, \ldots, N_n \) are second submodules of \( M \), then \( N_1 + \ldots + N_n \) is a strongly \( n \)-absorbing second submodule of \( M \).

Proof. Let \( a_1, \ldots, a_n \in R \) and \( K \) be a submodule of \( M \) with \( a_1\ldots a_n(N_1 + \ldots + N_k) \subseteq K \) such that \( \hat{a}_i(N_1 + \ldots + N_k) \not\subseteq K \) for each \( 1 \leq i \leq n \). As \( a_1\ldots a_n(N_1 + \ldots + N_k) \subseteq K \), we have \( a_1\ldots a_nN_j \subseteq K \) for every \( 1 \leq j \leq k \). Therefore, \( a_1\ldots a_n \in \text{Ann}_{R}(N_j) \) for every \( 1 \leq j \leq k \) since \( N_j \) is a strongly \( n_j \)-absorbing second submodule of \( M \) and \( n_j \leq n \). Therefore \( a_1\ldots a_n \in \text{Ann}_{R}(N_1) \cap \ldots \cap \text{Ann}_{R}(N_k) = \text{Ann}_{R}(N_1 + \ldots + N_k) \), that is, \( N_1 + \ldots + N_k \) is strongly \( n \)-absorbing second. The “in particular” statement follows from the fact that every second submodule is a strongly \( n \)-absorbing second submodule.

Let \( N \) be a non-zero submodule of an \( R \)-module \( M \). It is clear that if \( N \) is an \( n \)-absorbing (resp. a strongly \( n \)-absorbing) second submodule, then it is an \( m \)-absorbing (resp. a strongly \( m \)-absorbing) second submodule of \( M \) for every integer
Corollary 2.17. Let $M$ be an $R$-module. Then we have the following.

(a) If $N_1, ..., N_k$ are strongly $n$-absorbing second submodules of $M$, then $\omega_M(N_1 + \ldots + N_k) \leq \omega_M(N_1) + \ldots + \omega_M(N_k)$.

(b) If $N_1, ..., N_n$ are second submodules of $M$, then $\omega_M(N_1 + \ldots + N_n) \leq n$.

Proof. This follows from Theorem 2.16.

Theorem 2.18. Let $M$ be an $R$-module and $N$ be a $P$-secondary submodule of $M$ such that $P^n \subseteq \text{Ann}_R(N)$. Then $N$ is a strongly $n$-absorbing second submodule of $M$. Moreover, $\omega_M(N) \leq n$. In particular, if $(0 :_M P^n)$ is a $P$-secondary submodule of $M$, then $(0 :_M P^n)$ is a strongly $n$-absorbing second submodule of $M$. Moreover, $\omega_M((0 :_M P^n)) \leq n$.

Proof. Assume that $a_1, ..., a_n \in R$ and $K$ be a submodule of $M$ with $a_1 \ldots a_n N \subseteq K$ such that $\hat{a}_i N \nsubseteq K$ for each $1 \leq i \leq n$. For every $1 \leq i \leq n$, as $\hat{a}_i a_i N \subseteq K$ with $\hat{a}_i N \nsubseteq K$ and $N$ is a $P$-secondary submodule of $M$, we have $a_i \in P$. Consequently, $a_1 \ldots a_n \in P^n \subseteq \text{Ann}_R(N)$, that is, $N$ is a strongly $n$-absorbing second submodule of $M$. The "In particular" statement follows from the fact that $P^n \subseteq \text{Ann}_R((0 :_M P^n))$.

Theorem 2.19. Let $R$ be a Noetherian ring and let $M$ be a finitely cogenerated $R$-module. Then every non-zero proper submodule of $M$ is a strongly $n$-absorbing second submodule of $M$ for some positive integer $n$.

Proof. Let $N$ be a $P$-secondary submodule of $M$. So $\text{Ann}_R(N)$ is a $P$-primary ideal of $R$. Since $R$ is a Noetherian ring, there exists a positive integer $m$ for which $P^m \subseteq \text{Ann}_R(N)$. Thus $N$ is a strongly $m$-absorbing second submodule of $M$ by Theorem 2.18. Now assume that $K$ is a non-zero submodule of $M$. Since $M$ is an Artinian $R$-module, $K$ has a secondary representation by [20, 6.11]. Let $K = N_1 + \ldots + N_k$ be a secondary representation of $K$, where each $N_i$ is a $P_i$-secondary submodule of $M$ for any $1 \leq i \leq n$. By the first part, each $N_i$ is an $m_i$-absorbing second submodule of $M$ for some positive integer $m_i$. Now $K$ is a strongly $n$-absorbing second submodule in which $n = m_1 + \ldots + m_k$ by Theorem 2.16. Therefore the result follows.
Theorem 2.20. Let $N$ be a strongly $n$-absorbing second submodule of an $R$-module $M$ with $n \geq 2$ and $\text{Ann}_R(N) \subseteq \sqrt{\text{Ann}_R(N)}$. Suppose that $r \in \sqrt{\text{Ann}_R(N)} \setminus \text{Ann}_R(N)$ and let $t \geq 2$ be the least positive integer such that $r^t \in \text{Ann}_R(N)$. Then $r^{t-1}N$ is a strongly $(n-t+1)$-absorbing second submodule of $M$.

Proof. Choose $2 \leq t \leq n$. Then $n-t+1 \geq 1$. Let $a_1, \ldots, a_{n-t+1} \in R$ and a submodule $K$ of $M$. Since $r^{t-1}a_1 \ldots a_{n-t+1}N \subseteq K$ and $N$ is a strongly $n$-absorbing second submodule of $M$, therefore either $r^{t-1}a_iN \subseteq K$ or $r^{t-1}a_1 \ldots a_{n-t+1}N \subseteq K$ or $a_1 \ldots a_{n-t+1} \in \text{Ann}_R(r^{n-1}N)$. If $r^{t-1}a_iN \subseteq K$ or $a_1 \ldots a_{n-t+1} \in \text{Ann}_R(r^{n-1}N)$, then we are done. Hence assume that $r^{t-1}a_iN \not\subseteq K$ and $a_1, a_2, a_{n-t+1} \not\in \text{Ann}_R(r^{n-1}N)$. Since $N$ is a strongly $n$-absorbing second submodule of $M$, therefore $r^{t-2}a_1 \ldots a_{n-t+1}N \subseteq K$. Now $r^t \in \text{Ann}_R(N)$ and $r^{t-1}a_1 \ldots a_{n-t+1}N \subseteq K$ imply $r^{t-2}a_1 \ldots a_{n-t}(a_{n-t+1} + r)N \subseteq K$. Again, since $N$ is a strongly $n$-absorbing second and $r^{t-1}a_iN \not\subseteq K$ for any $1 \leq i \leq (n-t)$ and $r^{t-2}a_1 \ldots a_{n-t}(a_{n-t+1} + r)N \not\subseteq \text{Ann}_R(N)$ (as $r^t \in \text{Ann}_R(N)$), we must have $r^{t-2}a_1 \ldots a_{n-t}(a_{n-t+1} + r)N = r^{t-2}a_1 \ldots a_{n-t+1}N + r^{t-1}a_1 \ldots a_{n-t}N \subseteq K$. As $r^{t-2}a_1 \ldots a_{n-t+1}N \subseteq K$, we have $r^{t-1}a_1 \ldots a_{n-t}N \subseteq K$, a contradiction, since we assumed that the product of $r^{t-1}$ with any $n-t$ of the $a_i$’s with $N$ is not a subset of $K$. Thus $r^{t-1}a_iN \subseteq K$ or $a_1 \ldots a_{n-t+1} \in \text{Ann}_R(r^{t-1}N)$, and hence $r^{t-1}N$ is a strongly $(n-t+1)$-absorbing second submodule of $M$. 

Remark 2.21. One can see, by using the same technique, that the results in Theorems 2.16, 2.13, and Corollary 2.14 about strongly $n$-absorbing second submodules in this section is also true for $n$-absorbing second submodules.

3. Strongly and weakly strongly 2-absorbing second submodules

Recall that an $R$-module $M$ is said to be sum-irreducible precisely when it is nonzero and cannot be expressed as the sum of two proper submodules of itself [13, Definition and Exercise 7.2.8].

Theorem 3.1. Let $N$ be a strongly 2-absorbing second submodule of an $R$-module $M$. Then $aN = a^2N$ for all $a \in R \setminus \sqrt{\text{Ann}_R(N)}$. The converse holds, if $N$ is a sum-irreducible submodule of $M$.

Proof. Let $a \in R \setminus \sqrt{\text{Ann}_R(N)}$. Then $a^2 \in R \setminus \text{Ann}_R(N)$. Thus $aN = a^2N$ by [2, 3.3]. Conversely, let $N$ be a sum-irreducible submodule of $M$ and $abN \subseteq K$ for some $a, b \in R$ and a submodule $K$ of $M$. Assume that, $ab \in R \setminus \sqrt{\text{Ann}_R(N)}$. We show that $aN \subseteq K$ or $bN \subseteq K$. As $ab \in R \setminus \sqrt{\text{Ann}_R(N)}$, we have $a, b \in R \setminus \sqrt{\text{Ann}_R(N)}$. Thus $aN = a^2N$ by assumption. Let $x \in N$. Then $ax \in aN = a^2N$. Hence $ax = a^2y$ for some $y \in N$. This implies that $x - ay \in (0 : N a) \subseteq (K : N a)$. Thus $x = x - ay + ay \in (K : N a) + (K : N b)$. Therefore, $N \subseteq (K : N a) + (K : N b)$. Clearly, $(K : N a) + (K : N b) \subseteq N$. Thus as $N$ is sum-irreducible, $(K : N a) = N$ or $(K : N b) = N$ as needed. 

Proof. (a) Clearly, \( (0 :_M \text{Ann}_R(N)^2) \subseteq (0 :_M \text{Ann}_R(N)^3) \). As \( (0 :_M \text{Ann}_R(N)^3) \) is a strongly 2-absorbing second submodule of \( M \) and \( \text{Ann}_R(N)^2(0 :_M \text{Ann}_R(N)^3) \subseteq (0 :_M \text{Ann}_R(N)) \), we have \( \text{Ann}_R(N)(0 :_M \text{Ann}_R(N)^3) \subseteq (0 :_M \text{Ann}_R(N)) \) or \( \text{Ann}_R(N)^2(0 :_M \text{Ann}_R(N)^3) = 0 \). So in any case, \( \text{Ann}_R(N)^2(0 :_M \text{Ann}_R(N)^3) = 0 \). This implies that \( (0 :_M \text{Ann}_R(N)^3) \subseteq (0 :_M \text{Ann}_R(N)^2) \).

(b) As \( K \not\subseteq N \), we have \( (K + N)/N \not= 0 \). Let \( ab(K + N)/N \subseteq H/N \) for some \( a, b \in R \) and a submodule \( H/N \) of \( M/N \). Then \( ab(K + N) + N \subseteq H \). This implies that \( abK \subseteq H \). Now as \( K \) is a strongly 2-absorbing second submodule of \( M \), we have either \( aK \subseteq H \) or \( bK \subseteq H \) or \( abK = 0 \). Therefore, either \( a(K + N)/N \subseteq H/N \) or \( b(K + N)/N \subseteq H/N \) or \( ab((K + N)/N) = 0 \) as needed. To see the second assertion, let \( a \in \sqrt{(N :_R K + N)} \setminus (N :_R K) \). Then \( a^t K \subseteq N \) for some positive integer \( t \). Now as \( K \) is a strongly 2-absorbing second submodule of \( M \) and \( a \not\in (N :_R K) \), we have \( a \in \sqrt{\text{Ann}_R(K)} \). Hence \( \sqrt{(N :_R K + N)} \setminus (N :_R K) \subseteq \sqrt{\text{Ann}_R(K)} \setminus (N :_R K) \). The reverse inclusion is clear. \( \square \)

For a submodule \( N \) of an \( R \)-module \( M \) the second radical (or second socle) of \( N \) is defined as the sum of all second submodules of \( M \) contained in \( N \) and it is denoted by \( \text{sec}(N) \) (or \( \text{soc}(N) \)). In case \( N \) does not contain any second submodule, the second radical of \( N \) is defined to be \( (0) \) (see [14], [8]).

Theorem 3.2. Let \( N \) be a strongly 2-absorbing second submodule of an \( R \)-module \( M \). Then we have the following.

(a) \( \sqrt{\text{Ann}_R(N)^2} \subseteq \text{Ann}_R(N) \).

(b) If \( M \) is a finitely generated comultiplication \( R \)-module, then \( N \subseteq (0 :_M \text{Ann}_R^2(\text{sec}(N))) \).

(c) If \( \sqrt{\text{Ann}_R(N)} \not= \text{Ann}_R(N) \), then for each \( a \in \sqrt{\text{Ann}_R(N)} \setminus \text{Ann}_R(N) \), \( aN \) is a second \( R \)-module with \( \sqrt{\text{Ann}_R(N)} \subseteq \text{Ann}_R(aN) \). Furthermore, we have \( \{\text{Ann}_R(aN)\}_{a \in \sqrt{\text{Ann}_R(N)} \setminus \text{Ann}_R(N)} \) is a chain of prime ideals of \( R \).

Proof. (a) By [2, 3.5], \( \text{Ann}_R(N) \) is a 2-absorbing ideal of \( R \). Thus the result follows from [11, 2.4].

(b) By [7, 2.12], \( \text{Ann}_R(\text{sec}(N)) = \sqrt{\text{Ann}_R(N)} \). Thus \( \text{Ann}_R(\text{sec}(N))^2 \subseteq \text{Ann}_R(N) \), by part (a). Hence \( N \subseteq (0 :_M \text{Ann}_R^2(\text{sec}(N))) \).

(c) Let \( a \in \sqrt{\text{Ann}_R(N)} \setminus \text{Ann}_R(N) \). Then \( aN \not= 0 \) and there exists a positive integer \( t \) such that \( a^t N = 0 \) but \( a^{t-1} N \not= 0 \). Now let \( b \in R \) such that \( abN \not= 0 \). We show that \( abN = aN \). As \( N \) is a strongly 2-absorbing second submodule of \( M \), \( abN \subseteq abN \) implies that \( aN \subseteq abN \) or \( bN \subseteq abN \). If \( aN \subseteq abN \), then we are done. If \( bN \subseteq abN \), then \( a^{t-1} bN \subseteq a^t bN = 0 \). By [2, 3.5], \( \text{Ann}_R(N) \) is a 2-absorbing ideal of \( R \). Hence \( a^{t-2} bN = 0 \). Continuing in this way we obtain, \( abN = 0 \) which is a contradiction.
By part (a), \( a\sqrt{\text{Ann}_R(N)} \subseteq \sqrt{\text{Ann}_R(N^2)} \subseteq \text{Ann}_R(N) \). Thus \( \sqrt{\text{Ann}_R(N)} \subseteq (\text{Ann}_R(N) : R a) = \text{Ann}_R(aN) \).

As \( \text{Ann}_R(N) \) is a 2-absorbing ideal of \( R \), \( \{\text{Ann}_R(aN)\}_{a \in \sqrt{\text{Ann}_R(N) \setminus \text{Ann}_R(N)}} \) is a chain of prime ideals of \( R \) by [11, 2.5], which completes the proof. \( \square \)

**Proposition 3.3.** Let \( N \) be a \( P \)-secondary submodule of an \( R \)-module \( M \). Then \( N \) is a strongly 2-absorbing second submodule of \( M \) if and only if \( P^2 \subseteq \text{Ann}_R(N) \).

**Proof.** This follows from Theorem 3.2 (a) and Theorem 2.18. \( \square \)

**Definition 3.4.** Let \( N \) be a non-zero submodule of an \( R \)-module \( M \). We say that \( N \) is a weakly strongly 2-absorbing second submodule of \( M \) if whenever \( a, b \in R \), \( K \) is a submodule of \( M \), \( abM \not\subseteq K \), and \( abN \subseteq K \), then \( aN \subseteq K \) or \( bN \subseteq K \) or \( ab \in \text{Ann}_R(N) \).

**Example 3.5.** Let \( M \) be an \( R \)-module. Clearly every strongly 2-absorbing second submodule of \( M \) is a weakly strongly 2-absorbing second submodule of \( M \). Also, evidently \( M \) is a weakly strongly 2-absorbing second submodule of itself. In particular, \( M = \mathbb{Z}_6 \oplus \mathbb{Z}_{10} \) is not strongly 2-absorbing second \( \mathbb{Z} \)-module but \( M \) is a weakly strongly 2-absorbing second \( \mathbb{Z} \)-submodule of \( M \).

**Theorem 3.6.** Let \( N \) be a weakly strongly 2-absorbing second submodule of an \( R \)-module \( M \) which is not a strongly 2-absorbing second submodule. Then \( \text{Ann}_R^2(N) \subseteq (N : R M) \).

**Proof.** Assume on the contrary that \( \text{Ann}_R^2(N) \not\subseteq (N : R M) \). We show that \( N \) is a strongly 2-absorbing second submodule of \( M \). Let \( a, b \in R \) and \( K \) be a submodule of \( M \) such that \( abN \subseteq K \). If \( abM \not\subseteq K \), then we are done because \( N \) is a weakly strongly 2-absorbing second submodule of \( M \). Thus suppose that \( abM \subseteq K \). If \( abM \not\subseteq N \), then \( abM \not\subseteq N \cap K \). Hence \( abN \subseteq N \cap K \) implies that \( aN \subseteq N \cap K \subseteq K \) or \( bN \subseteq N \cap K \subseteq K \) or \( abN = 0 \) as needed. So let \( abM \subseteq N \). If \( a\text{Ann}_R(N)M \not\subseteq K \), then \( a(b + \text{Ann}_R(N))M \not\subseteq K \). Thus \( a(b + \text{Ann}_R(N))N \subseteq K \) implies that \( aN \subseteq K \) or \( bN = (b + \text{Ann}_R(N))N \subseteq K \) or \( abN = a(b + \text{Ann}_R(N))N = 0 \), as required. So let \( a\text{Ann}_R(N)M \subseteq K \). Similarly, we can assume that \( b\text{Ann}_R(N)M \subseteq K \). Since \( \text{Ann}_R(N)^2 \not\subseteq (N : R M) \), there exist \( a_1, b_1 \in \text{Ann}_R(N) \) such that \( a_1b_1M \not\subseteq N \). Thus there exists a completely irreducible submodule \( L \) of \( M \) such that \( N \subseteq L \) and \( a_1b_1M \not\subseteq L \) by Remark 2.2. If \( ab_1M \not\subseteq L \), then \( a(b + b_1)M \not\subseteq L \cap K \). Thus \( a(b + b_1)N \subseteq L \cap K \) implies that \( aN \subseteq L \cap K \subseteq K \) or \( bN = (b + b_1)N \subseteq L \cap K \subseteq K \) or \( abN = a(b + b_1)N = 0 \) as needed. So let \( ab_1M \subseteq L \). Similarly, we can assume that \( a_1bM \subseteq L \). Therefore, \( (a + a_1)(b + b_1)M \not\subseteq L \cap K \). Hence, \( (a + a_1)(b + b_1)N \subseteq L \cap K \) implies that \( aN = (a + a_1)N \subseteq K \) or \( bN = (b + b_1)N \subseteq K \) or \( abN = (a + a_1)(b + b_1)N = 0 \), as desired. \( \square \)

Let \( M \) be an \( R \)-module. A submodule \( N \) of \( M \) is said to be **idempotent** (resp. **coidealmpotent**) if \( N = (N : R M)^2 M \) (resp. \( N = (0 : R \text{Ann}_R(N)^2) \)). Also, \( M \) is said to be **fully idempotent** (resp. **fully coidealmpotent**) if every submodule of \( M \) is idempotent (resp. coidealmpotent) [6].
Corollary 3.7. Let $M$ be a faithful $R$-module. Then we have the following.

(a) If $M$ is a fully coidempotent $R$-module and $N$ is a proper submodule of $M$, then $N$ is a weakly strongly 2-absorbing second submodule of $M$ if and only if $N$ is a strongly 2-absorbing second submodule.

(b) If $M$ is a fully idempotent $R$-module and $N$ is a non-zero submodule of $M$, then $N$ is a weakly 2-absorbing submodule if and only if $N$ is a 2-absorbing submodule.

Proof. (a) The sufficiency is clear. Conversely, assume on the contrary that $N \not\subseteq M$ is a weakly strongly 2-absorbing second submodule of $M$ which is not a strongly 2-absorbing second submodule. Then by Theorem 3.6, $\Ann^3_R(N) \subseteq \Ann_R(M)$. Hence as $M$ is faithful, $\Ann^3_R(N) = 0$. Since $N$ is a coidempotent submodule of $M$, this implies that $N = (0 :_M \Ann_R(N)^2) = (0 :_M \Ann_R(N)^3) = M$, a contradiction.

(b) The proof is similar to the part (a) by using [15, 2.5].

Theorem 3.8. Let $t \in R$ and $M$ be an $R$-module. Then we have the following.

(a) If $(0 :_M t) \subseteq tM$, then $(0 :_M t)$ is a strongly 2-absorbing second submodule if and only if it is a weakly strongly 2-absorbing second submodule.

(b) If $(tM :_R M) \subseteq \Ann_R(tM)$, then the submodule $tM$ is strongly 2-absorbing second if and only if it is weakly strongly 2-absorbing second.

Proof. (a) Suppose that $(0 :_M t)$ is a weakly strongly 2-absorbing second submodule of $M$, $a, b \in R$, and $K$ is a submodule of $M$ such that $ab(0 :_M t) \subseteq K$. If $abM \subseteq K$, then since $(0 :_M t)$ is weakly strongly 2-absorbing second, we have $a(0 :_M t) \subseteq K$ or $b(0 :_M t) \subseteq K$ or $ba \in \Ann_R((0 :_M t))$ which implies $(0 :_M t)$ is strongly 2-absorbing second. Therefore we may assume that $abM \subseteq K$. Clearly, $a(b + t)(0 :_M t) \subseteq K$. If $a(b + t)M \subseteq K$, then we have $(b + t)(0 :_M t) \subseteq K$ or $a(0 :_M t) \subseteq K$ or $ab \in \Ann_R((0 :_M t))$. Since $at \in \Ann_R((0 :_M t)$ therefore $b(0 :_M t) \subseteq K$ or $a(0 :_M t) \subseteq K$ or $ab \in \Ann_R((0 :_M t))$. Now suppose that $a(b + t)M \subseteq K$. Then since $abM \subseteq K$, we have $taM \subseteq K$ and so $tM \subseteq (K :_M a)$. Now $(0 :_M t) \subseteq tM$ implies that $(0 :_M t) \subseteq (K :_M a)$. Thus $a(0 :_M t) \subseteq K$ as needed. The converse is clear.

(b) Let $tM$ be a weakly strongly 2-absorbing second submodule of $M$ and assume that $a, b \in R$ and $K$ be a submodule of $M$ with $abtM \subseteq K$. Since $tM$ is a weakly strongly 2-absorbing second submodule, we can suppose that $abM \subseteq K$, otherwise $tM$ is strongly 2-absorbing. Now $abtM \subseteq tM \cap K$. If $abM \not\subseteq tM \cap K$, then as $tM$ is a weakly strongly 2-absorbing second submodule, we are done. Now let $abM \subseteq tM \cap K$. Then $abM \subseteq tM$. Thus $(tM :_R M) \subseteq \Ann_R(tM)$ implies that $ab \in \Ann_R(tM)$ as requested. The converse is clear.

Theorem 3.9. Consider the following statements for an $R$-module $M$. 
(a) Every non-zero submodule of $M$ is a weakly strongly 2-absorbing second submodule of $M$.

(b) Every proper submodule of $M$ is a weakly 2-absorbing submodule of $M$.

Then (a) $\Rightarrow$ (b). Moreover, (b) $\Rightarrow$ (a) if $M$ is faithful.

Proof. (a) $\Rightarrow$ (b). Let $N$ be a proper submodule of $M$, $a, b \in R$, and $m \in M$ with $0 \neq abm \in N$. If $abM \subseteq N$, then we are done. So suppose that $abM \not\subseteq N$. Since $0 \neq abm \in Rm$, we have $Rm \neq 0$. By assumption, $Rm$ is weakly strongly 2-absorbing second. Thus $aRm \subseteq N$ or $bRm \subseteq N$ or $abRm = 0$. Since, $abm \neq 0$, $am \in N$ or $bm \in N$ as desired.

(b) $\Rightarrow$ (a). Let $0 \neq N$ be a submodule of $M$, $a, b \in R$, and $K$ be a submodule of $M$ with $abN \subseteq K$, where $abM \not\subseteq K$. If $abN = 0$, then we are done. So suppose that $abN \neq 0$. Clearly, $K$ is a proper submodule of $M$. By assumption, $K$ is weakly 2-absorbing. Thus by [18, 3.4], $aN \subseteq K$ or $bN \subseteq K$ as needed.

Corollary 3.10. Let $M$ be a non-zero $R$-module such that every non-zero submodule of $M$ is weakly strongly 2-absorbing second. Then $R$ has at most three maximal ideals containing Ann($M$).

Proof. This follows from [21, 6.1] and Theorem 3.9 (a) $\Rightarrow$ (b).

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