



Common Fixed Point and Common Coupled Fixed Point Theorems for Weakly Monotone Mappings

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ABSTRACT: In this paper, we have established two common fixed point theorems that are variants of the fixed point theorem of Boyd and Wong. By applying the newly established fixed point theorems, two common coupled fixed point theorems have been proved for a pair of weakly increasing mappings. Examples are given to substantiate our theorems.

Key Words: Coupled fixed point, coupled common fixed point, weakly increasing mappings, partially ordered set.

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1. Introduction

The most celebrated fixed point theorem in metric fixed point theory, known as the Banach contraction principle, has undergone several extensions and generalizations due to the simple nature of the statement and its wide range of applicability. Some of its significant generalizations are done by introducing classes of generalized contraction mappings due to Rakotch, Boyd and Wong, Browder, Geraghty, Caristi, and Jaggi [17,10,6,7,31,22]. In 2003, Ran and Reurings [32] started a new direction for the study of fixed points by establishing an analogous result of the Banach fixed point theorem in partially ordered metric spaces (a metric space with a partial order on it). Followed by this, Nieto and Lopez [27,28] established several fixed point theorems in partially ordered metric spaces using the continuity of function as well as the order completion property of the domain. Nowadays, a large number of research projects have been carried out in this direction [2,3,8,11,12].

In 1988, the concept of coupled fixed point was introduced by Guo and Lakshmikantham [19] as an extension of fixed point. They have established some coupled fixed point theorems for both continuous and discontinuous operators. The coupled fixed point theorems proposed by Gnana Bhaskar and Lakshmikantham [18] in 2006 gained more attention from researchers. In 2009, Lakshmikantham and Ćirić generalized the results in [18] by introducing the concepts of coupled coincidence and coupled common fixed points in [26]. Following these research works, several authors carried out studies on coupled fixed points, coupled coincidence points, and coupled common fixed points [4,5,13,14,23,24,21,34,20,1,9,30,29]. In this paper, we establish common fixed point theorems for a pair of weakly increasing mappings that are variants of Boyd and Wong’s fixed point theorem in partially ordered metric spaces. Using these results, we have proved common coupled fixed point theorems for a pair of weakly increasing mappings in partially ordered metric spaces.

Some useful definitions and results follow:

Definition 1.1. An element $x \in X$ is said to be a common fixed point of the mappings $f, g : X \rightarrow X$ if $f(x) = x = g(x)$.

Definition 1.2. An element $(x, y) \in X \times X$ is said to be a common coupled fixed point of the mappings $F, G : X \times X \rightarrow X$ if $F(x, y) = x = G(x, y)$ and $F(y, x) = y = G(y, x)$.

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Definition 1.3. [15,16] Let (X, \leq) be a partially ordered set. Let $f, g : X \rightarrow X$ and $F, G : X \times X \rightarrow X$ be mappings on X .

1. f, g are said to be weakly increasing if $f(x) \leq g(f(x))$ and $g(x) \leq f(g(x))$ for all $x \in X$.
2. f, g are said to be weakly decreasing if $f(x) \geq g(f(x))$ and $g(x) \geq f(g(x))$ for all $x \in X$.
3. F, G are said to be weakly increasing if

$$F(x, y) \leq G(F(x, y), F(y, x))$$

and

$$G(x, y) \leq F(G(x, y), G(y, x))$$

for all $x, y \in X$.

4. F, G are said to be weakly decreasing if

$$F(x, y) \geq G(F(x, y), F(y, x))$$

and

$$G(x, y) \geq F(G(x, y), G(y, x))$$

for all $x, y \in X$.

In the above definitions 1 and 2, if $f = g$, then we say that f is weakly increasing and weakly decreasing respectively. Similarly in definitions 3 and 4 if $F = G$ then we say that F is weakly increasing and weakly decreasing respectively.

Definition 1.4. [25] The function $\phi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function, if the following properties hold:

1. ϕ is continuous and non decreasing
2. $\phi(t) = 0$ if and only if $t = 0$.

We use the following notations:

$\Phi = \{\phi : [0, \infty) \rightarrow [0, \infty) \mid \phi \text{ is an altering distance function}\}$,

and $\Psi = \{\psi : [0, \infty) \rightarrow [0, \infty) \mid \psi \text{ is right upper semi-continuous with the condition } \psi(0) = 0 \text{ and } \psi(t) < \phi(t) \text{ for all } t > 0 \text{ where } \phi \in \Phi\}$.

Definition 1.5. [33] An ordered metric space (X, \preceq, d) is said to have the sequential monotone property if it verifies:

- (i) If $\{x_m\}$ is a non decreasing sequence and $\{x_m\} \xrightarrow{d} x$, then $x_m \preceq x$ for all m .
- (ii) If $\{y_m\}$ is a non increasing sequence and $\{y_m\} \xrightarrow{d} y$, then $y \preceq y_m$ for all m .

Definition 1.6. A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is said to be right upper semi continuous, if $r_n \downarrow r \geq 0$ then $\limsup_{n \rightarrow \infty} \psi(r_n) \leq \psi(r)$.

The following is a generalization of Banach fixed point theorem by Boyd and Wong.

Theorem 1.7. [6] Let (X, d) be a complete metric space and suppose $f : X \rightarrow X$ satisfies

$$d(f(x), f(y)) \leq \psi(d(x, y)), \text{ for each } x, y \in X,$$

where $\psi : \bar{P} \rightarrow [0, \infty)$ is upper semi continuous from the right on \bar{P} and satisfies $\psi(t) < t$ for all $t \in \bar{P} \setminus \{0\}$. Then f has a unique fixed point x_0 and $f^n(x) \rightarrow x_0$ for each $x \in X$.

In the above theorem P denote the range of the metric d .

Note: In this paper we use $\mathcal{F}(f, g)$, $\mathcal{CF}(f, g)$ to denote the set of all fixed points and set of coupled fixed points of the mappings f and g respectively.

2. Main Results

In this section four theorems are established in which two are common fixed point theorems and two are common coupled fixed point theorems for weakly monotone mappings.

Theorem 2.1. *Let (X, \preceq, d) be a partially ordered complete metric space, f and g be self maps on X and the pair (f, g) be weakly increasing with respect to \preceq such that*

$$\phi(d(fx, gy)) \leq \psi(d(x, y)), \quad (2.1)$$

for all comparable $x, g(y) \in X$ where $\phi \in \Phi$ and $\psi \in \Psi$. Suppose either

(a) f (or g) is continuous or

(b) (X, \preceq, d) satisfies the property that if $\{x_n\}$ is an increasing sequence in X and $x_n \rightarrow x$ as $n \rightarrow \infty$ then $x_n \preceq x, \forall n$,

then f and g have a common fixed point. Moreover, if x^* and y^* are comparable whenever $x^*, y^* \in \mathcal{F}(f, g)$, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary element.

Define $x_1 = f(x_0)$, $x_2 = g(x_1)$.

Continuing like this we get a sequence $\{x_n\}$ in X such that $\forall n \in \mathbb{N} \cup \{0\}$

$$x_{2n+1} = f(x_{2n}) \text{ and } x_{2n+2} = g(x_{2n+1}).$$

Since f and g are weakly increasing, we have

$$x_{2n+1} = f(x_{2n}) \preceq g(f(x_{2n})) = g(x_{2n+1}) = x_{2n+2}$$

and

$$x_{2n+2} = g(x_{2n+1}) \preceq f(g(x_{2n+1})) = f(x_{2n+2}) = x_{2n+3}.$$

Therefore sequence $\{x_n\}$ is monotone increasing in X .

Case 1: $x_m = x_{m+1}$ for some $m \in \mathbb{N} \cup \{0\}$.

Then either $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N} \cup \{0\}$ or $x_{2n} = x_{2n-1}$ for some $n \in \mathbb{N}$.

Case 1(i): Suppose $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N} \cup \{0\}$.

That is $x_{2n} = f(x_{2n})$.

Since $\psi(0) = 0$, we have

$$\begin{aligned} \phi[d(x_{2n+1}, x_{2n+2})] &= \phi[d(f(x_{2n}), g(x_{2n+1}))] \\ &\leq \psi(d(x_{2n}, x_{2n+1})) \\ &= 0 \end{aligned}$$

Therefore by the property of ϕ we get $x_{2n+1} = x_{2n+2}$.

That is, $x_{2n} = x_{2n+1} = x_{2n+2} = g(x_{2n+1})$, which gives that $x_{2n} = g(x_{2n})$.

Therefore x_{2n} is a common fixed point of f and g .

Case 1(ii): Suppose $x_{2n} = x_{2n-1}$ for some $n \in \mathbb{N}$.

In similar steps as in Case 1(i) we get x_{2n-1} is a common fixed point of f and g .

Case 2: $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Consider,

$$\begin{aligned} \phi[d(x_{2n+1}, x_{2n+2})] &= \phi[d(f(x_{2n}), g(x_{2n+1}))] \\ &\leq \psi(d(x_{2n}, x_{2n+1})) \\ &< \phi[d(x_{2n}, x_{2n+1})] \end{aligned} \quad (2.2)$$

also,

$$\begin{aligned}\phi[d(x_{2n+3}, x_{2n+2})] &= \phi[d(f(x_{2n+2}), g(x_{2n+1}))] \\ &\leq \psi(d(x_{2n+2}, x_{2n+1})) \\ &< \phi[d(x_{2n+2}, x_{2n+1})].\end{aligned}$$

Since ϕ is monotone increasing we have

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}) \text{ and } d(x_{2n+3}, x_{2n+2}) < d(x_{2n+2}, x_{2n+1}).$$

That is, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of non negative reals, so there exist $s \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = s.$$

Suppose $s > 0$.

By (2.2) we have

$$\phi[d(x_{2n+1}, x_{2n+2})] \leq \psi(d(x_{2n}, x_{2n+1})).$$

By taking upper limit on both sides we get

$$\phi[s] \leq \psi(s),$$

a contradiction. Therefore $s = 0$.

Hence, to prove that $\{x_n\}$ is a Cauchy sequence in X , it is enough to prove that $\{x_{2n}\}$ is a Cauchy sequence in X .

On the contrary assume that there exist $\epsilon > 0$ and two sub sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n(k) > m(k) \geq k$ such that

$$d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon, \quad \forall k \in \mathbb{N} \tag{2.3}$$

and by choosing $n(k)$ to be the smallest number exceeding $m(k)$ for which (2.3) holds we get,

$$d(x_{2m(k)}, x_{2n(k)-2}) < \epsilon. \tag{2.4}$$

By (2.3) and (2.4) we get

$$\begin{aligned}\epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\ &< \epsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}).\end{aligned}$$

By taking limit as $k \rightarrow \infty$ we get

$$\lim_{k \rightarrow \infty} d(x_{2m(k)}, x_{2n(k)}) = \epsilon \tag{2.5}$$

$$\text{and } \lim_{k \rightarrow \infty} d(x_{2m(k)}, x_{2n(k)-2}) = \epsilon. \tag{2.6}$$

Next consider

$$\begin{aligned}d(x_{2m(k)}, x_{2n(k)}) &\leq d(x_{2m(k)}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2n(k)}).\end{aligned}$$

By taking limit as $k \rightarrow \infty$ and by (2.5) we get

$$\lim_{k \rightarrow \infty} d(x_{2m(k)+1}, x_{2n(k)}) = \epsilon. \tag{2.7}$$

Now consider

$$\begin{aligned} d(x_{2m(k)+1}, x_{2n(k)}) &\leq d(x_{2m(k)+1}, x_{2m(k)+2}) + d(x_{2m(k)+2}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}) \\ &\leq d(x_{2m(k)+1}, x_{2m(k)+2}) + d(x_{2m(k)+2}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2n(k)}) \\ &\quad + d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}). \end{aligned}$$

By taking limit as $k \rightarrow \infty$ and by (2.7) we get

$$\lim_{k \rightarrow \infty} d(x_{2m(k)+2}, x_{2n(k)+1}) = \epsilon. \quad (2.8)$$

Next consider

$$\begin{aligned} \phi[d(x_{2n(k)+1}, x_{2m(k)+2})] &= \phi[d(f(x_{2n(k)}), g(x_{2m(k)+1}))] \\ &\leq \psi(d(x_{2n(k)}, x_{2m(k)+1})) \end{aligned}$$

By (2.7) and (2.8) we have

$$\phi[\epsilon] < \psi(\epsilon),$$

a contradiction.

Thus $\{x_{2n}\}$ is a Cauchy sequence in X , hence $\{x_n\}$ is a Cauchy sequence in X .

Since X is a complete metric space, there exist some $x \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = x.$$

(a) Suppose that f is continuous.

We have $x_{2n+1} = f(x_{2n})$.

By taking limit as $n \rightarrow \infty$, we get $x = f(x)$

Since f and g are weakly increasing we get x and $g(x)$ are comparable.

Now consider

$$\begin{aligned} \phi[d(x, g(x))] &= \phi[d(f(x), g(x))] \\ &\leq \psi(d(x, x)) \\ &= 0. \end{aligned}$$

By the property of ϕ we get $x = g(x)$.

Thus x is a common fixed point of f and g .

Similarly, when g is continuous we get x is a common fixed point of f and g .

(b) Suppose X satisfies the property that if $\{x_n\}$ is an increasing sequence in X and $x_n \rightarrow x$ as $n \rightarrow \infty$ then $x_n \preceq x, \forall n$.

By the property of X , we have $x_n \preceq x$ for all n , which gives that $x_{2n+2} \preceq x$ for all n .

Now consider

$$\begin{aligned} \phi[d(f(x), x_{2n+2})] &= \phi[d(f(x), g(x_{2n+1}))] \\ &\leq \psi(d(x, x_{2n+1})). \end{aligned}$$

By taking the upper limit as $n \rightarrow \infty$ and by the property of ϕ and ψ we get

$$\phi[d(f(x), x)] = 0,$$

which gives that $x = f(x)$.

Since f and g are weakly increasing we get x and $g(x)$ are comparable.

Now consider

$$\begin{aligned} \phi[d(x, g(x))] &= \phi[d(f(x), g(x))] \\ &\leq \psi(d(x, x)) \\ &= 0. \end{aligned}$$

Thus $x = g(x)$.

Therefore x is a common fixed point of f and g .

Suppose $x, y \in X$ are two different common fixed points of f and g .

Assume that x and y are comparable.

Now consider

$$\begin{aligned}\phi[d(x, y)] &= \phi[d(f(x), g(y))] \\ &\leq \psi(d(x, y)) \\ &< \phi[d(x, y)],\end{aligned}$$

a contradiction.

Therefore $x = y$.

Thus there exist a unique common fixed point of f and g .

Hence the proof. □

We illustrate the theorem with the following example.

Example 2.2. Let $X = [0, 1]$ with usual metric.

Define $f : X \rightarrow X$ and $g : X \rightarrow X$ as

$$f(x) = \frac{x}{2} \text{ and } g(x) = \begin{cases} \frac{x}{2}, & \text{if } x \neq 1 \\ \frac{1}{4}, & \text{if } x = 1. \end{cases}$$

Define a partial order \preceq on X as

$$\preceq := \left\{ (x, x), \left(\frac{x}{2}, \frac{x}{2^n} \right) \mid x \in X, n \in \mathbb{N} \right\}.$$

The pair of mappings (f, g) is weakly increasing and satisfy the contraction type condition (2.1) for

$$\phi(t) = t, \quad \psi(t) = \frac{t}{2}, \quad \forall t.$$

Here 0 is a common fixed point of f and g .

Theorem 2.3. Let (X, \preceq, d) be a partially ordered complete metric space, f and g be self maps on X and the pair (f, g) be weakly decreasing with respect to \preceq such that

$$\phi(d(fx, gy)) \leq \psi(d(x, y)),$$

for all comparable $x, g(y) \in X$ where $\phi \in \Phi$ and $\psi \in \Psi$. Suppose either

(a) f (or g) is continuous or

(b) (X, \preceq, d) satisfies the property that if $\{x_n\}$ is a decreasing sequence in X and $x_n \rightarrow x$ as $n \rightarrow \infty$ then $x \preceq x_n, \forall n$,

then f and g have a common fixed point. Moreover, if x^* and y^* are comparable whenever $x^*, y^* \in \mathcal{F}(f, g)$, then f and g have a unique common fixed point.

Proof. The proof is similar to that of Theorem 2.1. Here since the pair of mappings f, g are weakly decreasing we obtain a decreasing sequence $\{x_n\}$ instead of an increasing sequence. □

By taking X to be a totally ordered set, $f = g$ and $\phi \in \Phi$ the identity function on $[0, \infty)$, in Theorem 2.1 and Theorem 2.3 we get two new fixed point theorems. The following corollary is proposed by combining the two theorems.

Corollary 2.4. *Let (X, \preceq, d) be a totally ordered complete metric space and suppose $f : X \rightarrow X$ satisfies*

$$d(f(x), f(y)) \leq \psi(d(x, y)), \text{ for each } x, y \in X,$$

where $\psi : [0, \infty) \rightarrow [0, \infty)$ is upper semi continuous from the right and satisfies $\psi(t) < t$ for all $t > 0$. Then f has a unique fixed point provided either $f(x) \preceq f(f(x))$ or $f(x) \succeq f(f(x))$ for all $x \in X$.

Remark 2.5. *The above Corollary is the fixed point theorem of Boyd and Wong [Theorem 1.7] for weakly increasing (decreasing) mappings.*

Now we define a partial order and a metric on the product space by using the partial order and metric on the underlying space as follows:

Let (X, \preceq, d) be a partially ordered metric space.

Define a partial order \leq and a metric D on the set $X \times X$ as follows:

For all $(x, y), (u, v) \in X \times X$,

$$(x, y) \leq (u, v) \Leftrightarrow x \preceq u \text{ and } y \preceq v,$$

and

$$D((x, y), (u, v)) = d(x, u) + d(y, v)$$

.

Note 2.6. *It can be easily shown that*

1. (X, d) is complete if and only if $(X \times X, D)$ is complete.
2. (X, \preceq, d) has sequential monotone property if and only if $(X \times X, \leq, D)$ has sequential monotone property.
3. $F, G : X \times X \rightarrow X$ are weakly increasing (decreasing) with respect to the partial order \preceq if and only if the mappings $T_F, T_G : X \times X \rightarrow X \times X$ defined by $T_F(x, y) = (F(x, y), F(y, x))$ and $T_G(x, y) = (G(x, y), G(y, x))$ are weakly increasing (decreasing) with respect to the partial order \leq .
4. The mappings F and G are continuous if and only if the mappings T_F and T_G are continuous.
5. The mappings F and G have common coupled fixed point if and only if the mappings T_F and T_G have common fixed points.

Theorem 2.7. *Let (X, \preceq, d) be a partially ordered complete metric space, $F, G : X \times X \rightarrow X$ be the given mappings and the pair (F, G) be weakly increasing with two variables with respect to \preceq such that*

$$\phi(d(F(x, y), G(u, v)) + d(F(y, x), G(v, u))) \leq \psi(d(x, u) + d(y, v)), \quad (2.9)$$

for all comparable $(x, y), (G(u, v), G(v, u)) \in X \times X$ where $\phi \in \Phi$ and $\psi \in \Psi$. If the following conditions hold:

- (a) F (or G) is continuous or
- (b) (X, \preceq, d) satisfies the property that if $\{x_n\}$ is an increasing sequence in X and $x_n \rightarrow x$ as $n \rightarrow \infty$ then $x_n \preceq x, \forall n$,

then F and G have a common coupled fixed point. Moreover, if (x_1, y_1) and (x_2, y_2) are comparable whenever $(x_1, y_1), (x_2, y_2) \in \mathcal{CF}(F, G)$, then F and G have a unique common coupled fixed point.

Proof. By the hypotheses and using the properties in Note 2.6 we see that $(X \times X, \leq, D)$ is a partially ordered complete metric space and the mappings $T_F, T_G : X \times X \rightarrow X \times X$ defined by $T_F(x, y) = (F(x, y), F(y, x))$ and $T_G(x, y) = (G(x, y), G(y, x))$ are weakly increasing with respect to the partial order \leq .

By the definition of metric D , partial order \leq and the functions T_F, T_G on $X \times X$ we can deduce contractive condition (2.9) as:

$$\phi[D(T_F(x, y), T_G(u, v))] \leq \psi(D((x, y), (u, v))),$$

for all comparable $(x, y), T_G(u, v) \in X \times X$ where $\phi \in \Phi$ and $\psi \in \Psi$.

Again since F (or G) is continuous, T_F (or T_G) is continuous and since (X, \preceq, d) satisfies the property that if $\{x_n\}$ is an increasing sequence in X and $x_n \rightarrow x$ as $n \rightarrow \infty$ then $x_n \preceq x, \forall n$, $(X \times X, \leq, D)$ also satisfies the same property.

Now apply Theorem 2.1 for the space $(X \times X, \leq, D)$, and for the mappings T_F and T_G so that we get a unique common fixed point of the mappings T_F and T_G .

Again by the Note 2.6, we get a unique common coupled fixed point of F and G . \square

We illustrate the new theorem with the following example.

Example 2.8. Let $X = [0, 1]$ with usual metric.

Define $F : X \times X \rightarrow X$ and $G : X \times X \rightarrow X$ as

$$F(x, y) = \frac{x+y}{2} \text{ and } G(x, y) = \begin{cases} \frac{x+y}{2}, & \text{if } (x, y) \neq (1, 1), \\ 0, & \text{if } (x, y) = (1, 1). \end{cases}$$

Define a partial order \preceq on X as

$$\preceq : = \{(x, x) \mid x \in X\}.$$

Corresponding partial order \sqsubseteq on $X \times X$ is:

$$\sqsubseteq : = \{((x, y), (x, y)) \mid x, y \in X\}.$$

The pair of mappings (F, G) is weakly increasing and satisfy the contraction type condition (2.9) for all $\psi \in \Psi$ and $\phi \in \Phi$

Here $\{(x, x) \mid x \in X \setminus \{1\}\}$ is the set of all common coupled fixed point of F and G .

Theorem 2.9. Let (X, \preceq, d) be a partially ordered complete metric space, $F, G : X \times X \rightarrow X$ be given mappings and the pair (F, G) be weakly decreasing with two variables with respect to \preceq such that

$$\phi(d(F(x, y), G(u, v)) + d(F(y, x), G(v, u))) \leq \psi(d(x, u) + d(y, v)),$$

for all comparable $(x, y), (G(u, v), G(v, u)) \in X \times X$ where $\phi \in \Phi$ and $\psi \in \Psi$. If the following conditions hold:

(a) F (or G) is continuous or

(b) (X, \preceq, d) satisfies the property that if $\{x_n\}$ is a decreasing sequence in X and $x_n \rightarrow x$ as $n \rightarrow \infty$ then $x \preceq x_n, \forall n$,

then F and G have a common coupled fixed point. Moreover, if (x_1, y_1) and (x_2, y_2) are comparable whenever $(x_1, y_1), (x_2, y_2) \in \mathcal{CF}(F, G)$, then F and G have a unique common coupled fixed point.

Proof: Using Theorem 2.3 and continuing as in Theorem 2.7 we get the result.

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References

1. Afshari, Hojjat, Sabileh Kalantari, and Erdal Karapinar. "Solution of fractional differential equations via coupled fixed point." *Electron. J. Differ. Equ* 286.1 (2015): 2015.
2. Amini, A.H., Emami, H., *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, *Nonlinear Anal.* 72(5), (2010) 2238 – 2242.
3. Muhammad Arshad, Erdal Karapinar, Jamshaid Ahmad, *Some unique fixed point theorems for rational contractions in partially ordered metric spaces*, *Journal of Inequalities and Applications* 2013, 2013:248.
4. Berinde, V, *Coupled fixed point theorems for ϕ - contractive mixed monotone mappings in partially ordered metric spaces*, *Nonlinear Anal.* 75, 3218 - 3228 (2011)
5. Berinde, V, *Coupled coincidence point theorems for mixed monotone nonlinear operators*, *Comput. Math. Appl.* 64, 1770 - 1777 (2012)
6. D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, *Proc. Amer. Math. Soc.* 20(1969), 458-464.
7. F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, *Nederl. Akad. Wetensch. SeL A 71=Indag. Math.* 30(1968), 27-35.
8. I. Cabrera, J. Harjani, K. Sadarangani, *A fixed point theorem for contractions of rational type in partially ordered metric spaces*, *Ann Univ Ferrara* (2013) 59: 251 - 258.
9. Erhan, İ, et al. "Remarks on 'Coupled coincidence point results for a generalized compatible pair with applications'." *Fixed Point Theory Appl.* 2014,(2014)." (2014).
10. James Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, *Transactions of the American mathematical society*, vol. 215 (1976).
11. Sumit Chandok, Jong Kyu Kim, *Fixed point theorem in ordered metric spaces for generalized contraction mappings satisfying rational type expressions*, *Nonlinear Functional Analysis and Applications* 17 (2012) 3: 301 - 306.
12. Sumit Chandok, Erdal Karapinar, *Common Fixed Point of Generalized Rational Type Contraction Mappings in Partially Ordered Metric Spaces*, *Thai Journal of Mathematics* 11 (2013) 2 : 251 - 260.
13. Choudhury, BS, Kundu, A, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, *Nonlinear Anal.* 73, 2524-2531 (2010)
14. Choudhury, BS, Metiya, N, Kundu, A, *Coupled coincidence point theorems in ordered metric spaces*, *Ann. Univ. Ferrara* 57, 1-16 (2011)
15. B. C. Dhage, *Condensing mappings and applications to existence theorem for common solution of differential equations*, *Bull. Korean Math. Soc.* 36 (1999), No. 3, pp. 565 - 578.
16. Hui-Sheng Ding, Lu Li, Wei Long, *Coupled common fixed point theorems for weakly increasing mappings with two variables*, *J. Computational Analysis and Applications*, Vol. 15, No.8, 1381-1390, 2013
17. Michael A. Geraghty, *On contractive mappings*, *Proceedings of the American mathematical society*, vol. 40 (1973) 2.
18. T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and application*, *Nonlinear Analysis* 65 (2006) 1379 - 1393.
19. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
20. Gülyaz, Selma, Erdal Karapinar, and Ilker Savas Yüce. "A coupled coincidence point theorem in partially ordered metric spaces with an implicit relation." *Fixed Point Theory and Applications* 2013.1 (2013): 1-11.
21. Gülyaz, Selma, and Erdal KARAPINAR. "A coupled fixed point result in partially ordered partial metric spaces through implicit function." *Hacettepe Journal of Mathematics and Statistics* 42.4 (2013): 347-357.
22. D. S. Jaggi, *Some unique fixed point theorems*, *Indian J. Pure Appl. Math.* 8 (1977), 223 - 230.
23. Karapinar, Erdal. "Couple fixed point theorems for nonlinear contractions in cone metric spaces." *Computers & Mathematics with Applications* 59.12 (2010): 3656-3668.
24. Karapinar, Erdal, and Ravi P. Agarwal. "A note on 'Coupled fixed point theorems for $\alpha - \psi$ -contractive-type mappings in partially ordered metric spaces'." *Fixed Point Theory and Applications* 2013.1 (2013): 1-16.
25. Khan, MS, Swaleh, M, Sessa, S, *Fixed point theorems by altering distances between the points*, *Bull. Aust. Math. Soc.* 30(1), (1984) 1-9
26. V. Lakshmikantham, Ljubomir Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, *Nonlinear Analysis* 70 (2009) 4341 - 4349
27. J.J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, *Order* 22 (2005) 223 - 239.
28. J.J. Nieto, R.R. López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, *Acta Mathematica Sinica, English Series* 23 (12) (2007) 2205 - 2212.

29. Prajisha, E., and P. Shaini. "Coupled fixed point theorems for mappings satisfying Geraghty type contractive conditions." *Italian Journal of Pure and Applied Mathematics*: N. 46-2021 (827-835)
30. Prajisha, E., and P. Shaini. "Coupled fixed point theorems for mappings satisfying rational type conditions in partially ordered metric spaces." *Asian-European Journal of Mathematics* (2022): 2250194.
31. E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. 13(1962), 459-465.
32. A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proceedings of The American Mathematical Society 132 (2003) 1435 - 1443.
33. A. Roldan, J. Martinez-moreno, C. Roldan and E. Karapinar, *Some remarks on multidimensional fixed point theorems*, Fixed Point Theory, 15(2014), No. 2, 545 - 558
34. Samet, Bessem, et al. "Discussion on some coupled fixed point theorems." *Fixed Point Theory and Applications* 2013.1 (2013): 1-12.

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