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# Between Strongly $\theta$-continuous and Weakly Continuous Functions

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ABSTRACT: In this paper, we investigate some properties of $\theta_g$-continuous and weakly $g$-continuous functions in a topological spaces. Moreover, the relationships with other related functions are investigated.

Key Words: $\theta$-continuous, strongly $\theta$-continuous, weakly $g$-continuous, $\theta_g$-continuous.

## Contents



## 1. Introduction

Let $(X, \tau)$ be a topological space with no separation axioms assumed. If $A \subseteq X$, $Cl(A)$ (or $cl(A)$) and $Int(A)$ (or $int(A)$) will denote the closure and interior of $A$ in $(X, \tau)$, respectively.

In 1968, Veličko [14] introduced the class of $\theta$-open sets. A set $A$ is said to be $\theta$-open [14] if every point of $A$ has an open neighborhood whose closure is contained in $A$. The $\theta$-interior [14] of $A$ in $X$ is the union of all $\theta$-open subsets of $A$ and is denoted by $Int_\theta(A)$. Naturally, the complement of a $\theta$-open set is said to be $\theta$-closed. Equivalently $Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \phi, U \in \tau \text{ and } x \in U\}$ and a set $A$ is $\theta$-closed if and only if $A = Cl_\theta(A)$. Note that all $\theta$-open sets form a topology on $X$, coarser than $\tau$ that is $\tau_\theta \subseteq \tau$ , denoted by $\tau_\theta$ and that a space $(X, \tau)$ is regular if and only if $\tau = \tau_\theta$. Note also that the $\theta$-closure of a given set need not be a $\theta$-closed set.

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\theta$-continuous [6] (resp. strongly $\theta$-continuous [12], weakly continuous [7]) if for each $x \in X$ and each open set $V$ in $Y$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(Cl(U)) \subseteq Cl(V)$ (resp. $f(Cl(U)) \subseteq V$, $f(U) \subseteq Cl(V)$). Some other locally closed set related continuity has been discussed in [1,2,3,5,9,10]. In the present paper, we investigate some properties of $\theta_g$-continuous and weakly $g$-continuous functions in a topological spaces. Moreover, the relationships with other related functions are investigated.

**Definition 1.1.** *[10] Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then*

1. *$A$ is generalized closed (briefly g-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.*
2. *$A$ is generalized open(briefly g-open) if $X \setminus A$ is g-closed.*
3. *$(X, \tau)$ is $T_{\frac{1}{2}}$-space if every g-closed set is closed.*

The intersection of all $g$-closed sets containing $A$ is called the $g$-closure of $A$ [4] and denoted by $Cl^*(A)$ or $cl^*(A)$, and the $g$-interior of $A$ is the union of all $g$-open sets contained in $A$ and is denoted by $Int^*(A)$ or $int^*(A)$. $A$ is said to be $\tau^*$-closed if $Cl^*(A) = A$. The complement of a $\tau^*$-closed set is called a $\tau^*$-open set.

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## 2. Preliminaries

Let us start by the following Lemma:

**Lemma 2.1.** *[4] Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then*

1. *$A \subseteq Cl^*(A) \subseteq Cl(A)$.*
2. *$Cl^*$ is a Kuratowski closure operator on $X$ and $\tau^* = \{A : Cl^*(X - A) = X - A\}$ is a topology on $X$ generated by $Cl^*$ in the usual manner.*
3. *$\tau \subseteq \tau^*$ with equality if and only if $(X, \tau)$ is $T_{\frac{1}{2}}$-space.*

Let $(X, \tau)$ be a topological space and $A \subseteq X$. A point $x$ of $X$ is called a $\theta_g$-cluster point of $A$ if $Cl^*(U) \cap A \neq \emptyset$ for every open set $U$ of $X$ containing $x$. The set of all $\theta_g$-cluster points of $A$ is called the $\theta_g$-closure of $A$ and is denoted by $Cl^*_\theta(A)$. $A$ is said to be $\theta_g$-closed if $Cl^*_\theta(A) = A$. The complement of a $\theta_g$-closed set is called a $\theta_g$-open set. The family of all $\theta_g$-open sets in $(X, \tau)$ will be denoted by $\tau^*_\theta$.

**Definition 2.2.** *Let $(X, \tau)$ be a topological space. A point $x$ of $X$ is called a $\theta_g$-interior point of $A$ if there exists an open set $U$ containing $x$ such that $Cl^*(U) \subseteq A$. The set of all $\theta_g$-interior points of $A$ is called the $\theta_g$-interior of $A$ and is denoted by $Int^*_\theta(A)$.*

**Remark 2.3.** *For a set $A$ of $X$, $Int^*_\theta(X - A) = X - Cl^*_\theta(A)$ so that $A$ is $\theta_g$-open if and only if $A = Int^*_\theta(A)$. In this respect, $Int^*_\theta \sim^X Cl^*_\theta$ [11].*

**Lemma 2.4.** *Let $(X, \tau)$ be a topological space and let $A \subseteq X$. Then*
*$Cl^*(A) \subseteq Cl(A) \subseteq Cl^*_\theta(A) \subseteq Cl_\theta(A)$ and $\tau_\theta \subseteq \tau^*_\theta \subseteq \tau \subseteq \tau^*$.*

**Lemma 2.5.** *[14] Let $(X, \tau)$ be a topological space. Then*

1. *for each $A \in \tau$, $Cl_\theta(A) = Cl(A)$.*
2. *$X$ is regular if and only if $\tau = \tau_\theta$.*

**Theorem 2.6.** *Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then $A \in \tau^*_\theta$ if and only if for each $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq Cl^*(U) \subseteq A$.*

*Proof.* Suppose that $A \in \tau^*_\theta$ and let $x \in A$. Then $X - A$ is $\theta_g$-closed and $x \notin X - A$. Thus, $x \notin Cl^*_\theta(X - A)$ and hence there is $U \in \tau$ such that $x \in U$ and $Cl^*(U) \cap (X - A) = \emptyset$. Therefore, we have $x \in U \subseteq Cl^*(U) \subseteq A$.

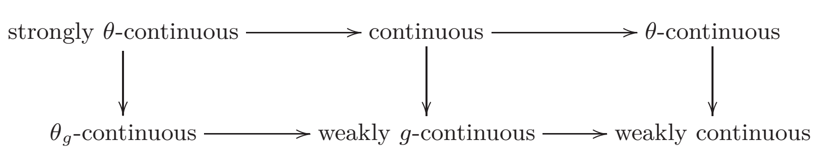
Conversely, suppose for each $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq Cl^*(U) \subseteq A$ and suppose on that contrary that $A \notin \tau^*_\theta$. Then $X - A$ is not $\theta_g$-closed and $Cl^*_\theta(X - A) \neq X - A$. Choose $x \in Cl^*_\theta(X - A) - (X - A)$. Since $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq Cl^*(U) \subseteq A$. Thus we have $x \in U \in \tau$ and hence $Cl^*(U) \cap (X - A) = \emptyset$. Therefore $x \notin Cl^*_\theta(X - A)$, a contradiction. □

**Corollary 2.7.** *In a topological space $(X, \tau)$ every open and $\tau^*$-closed set is $\theta_g$-open.*

*Proof.* Let $(X, \tau)$ be a topological space and let $A$ be open and $\tau^*$-closed set in $(X, \tau)$. Let $x \in A$. Since $A$ is $\tau^*$-closed, then $Cl^*(A) = A$. Take $U = A$. Then $U \in \tau$ and $x \in U = Cl^*(U) = A \subseteq A$. Thus by Theorem 2.6, it follows that $A$ is $\theta_g$-open. □

**Definition 2.8.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly $g$-continuous (resp. $\theta_g$-continuous ) if for each $x \in X$ and each open set $V$ in $Y$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq cl^*(V)$ (resp. $f(cl^*(U)) \subseteq cl^*(V)$).*

By the above definitions, we have the following diagram and none of these implications is reversible as shown by examples.



The following strict implications are well-known:

$$\begin{array}{ccccc} \text{strongly } \theta\text{-continuous} & \longrightarrow & \text{continuous} & \longrightarrow & \theta\text{-continuous} \\ & & & & \downarrow \\ & & & & \text{weakly continuous} \end{array}$$

**Example 2.9.** *Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{1\}, \{2, 3\}\}$ and $Y = \{a, b, c\}$, $\sigma = \{\emptyset, Y, \{a\}\}$. Then $\sigma^* = \{\emptyset, Y, \{a\}, \{b, c\}, \{a, c\}, \{a, b\}, \{b\}, \{c\}\}$ We define a function $f : (X, \tau) \to (Y, \sigma)$ as $f = \{(1, b), (2, a), (3, c)\}$. Then $f$ is weakly continuous function but not weakly $g$-continuous.*

1. *Let $1 \in X$ such that $f(1) = b \in V = Y$, then there exists an open set $U = \{1\} \in \tau$ containing 1 such that $f(U) = \{b\} \subseteq cl(V) = Y$.*
2. *Let $2 \in X$ such that $f(2) = a \in V = \{a\}$ or $V = Y$, then there exists an open set $U = \{2, 3\}$ containing 2 such that $f(U) = \{a, c\} \subseteq cl(V) = Y$.*
3. *Let $3 \in X$ such that $f(3) = c \in V = Y$, then there exists an open set $U = \{2, 3\}$ containing 3 such that $f(U) = \{a, c\} \subseteq cl(V) = Y$.*

*By (1), (2) and (3) $f$ is weakly continuous. On the other hand, let $2 \in X$ such that $f(2) = a \in V = \{a\}$. But, for every open set $U$ containing 2 where $U = \{2, 3\}$ or $U = X$. Then $\{a, c\} \subseteq f(U) \nsubseteq cl^*(V) = \{a\}$. Therefore, $f : (X, \tau) \to (Y, \sigma)$ is not weakly $g$-continuous.*

**Example 2.10.** *Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{2\}, \{3\}, \{2, 3\}\}$ and $Y = \{a, b, c\}$, $\sigma = \{\emptyset, Y, \{a\}\}$. Then $\tau^* = \{\emptyset, X, \{2\}, \{3\}, \{2, 3\}\}$ and $\sigma^* = \{\emptyset, Y, \{a\}, \{b, c\}, \{a, c\}, \{a, b\}, \{b\}, \{c\}\}$. We define a function $f : (X, \tau) \to (Y, \sigma)$ as $f = \{(1, b), (2, a), (3, c)\}$. Then $f$ is weakly $g$-continuous but not $\theta_g$-continuous. We show that $f$ is weakly $g$-continuous*

1. *$1 \in X$ and $f(1) = b \in V = Y$. If we take $U = X$, then $f(U) = Y \subseteq cl^*(V)$.*
2. *$2 \in X$ and $f(2) = a \in V = \{a\} \subseteq cl^*(V) = \{a\}$ or $V = Y \subseteq cl^*(V) = Y$. If we take $U = \{2\}$, then $f(U) = \{a\} \subseteq cl^*(V)$ for all $V$ containing $a$.*
3. *$3 \in X$ and $f(3) = c \in V = Y \subseteq cl^*(V) = Y$. If we take $U = \{3\}$, then $f(U) = \{c\} \subseteq cl^*(V)$ for all $V$ containing $c$.*

*Then for each $x \in X$ and each open set $V$ in $Y$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq cl^*(V)$.*

*We show that $f : (X, \tau) \to (Y, \sigma)$ is not $\theta_g$-continuous. Let $2 \in X$ and $V = \{a\} \in \sigma$ such that $f(2) = a \in V \in \sigma$. But, for every open set $U \subseteq X$ such that $2 \in U$, where $U = \{2\}$ or $U = \{2, 3\}$ or $U = X$, $Cl^*(U) = \{1, 2\}$ or $Cl^*(U) = X$. Then, for all open set $U$ containing we have 2, $f(Cl^*(U) \nsubseteq cl^*(V) = \{a\}$. Therefore, $f : (X, \tau) \to (Y, \sigma)$ is not $\theta_g$-continuous.*

**Example 2.11.** *Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{2\}, \{3\}, \{2, 3\}\}$ and $Y = \{a, b, c\}$, $\sigma = \{\emptyset, Y, \{a, b\}\}$. Then $\tau^* = \{\emptyset, X, \{2\}, \{3\}, \{2, 3\}\}$ and $\sigma^* = \{\emptyset, Y, \{a, b\}, \{b\}, \{a\}\}$ We define a function $f : (X, \tau) \to (Y, \sigma)$ as $f = \{(1, a), (2, b), (3, c)\}$. Then $f$ is $\theta_g$-continuous but not continuous.*

1. *Let $1 \in X$ and $V = \{a, b\}$ or $V = Y$ such that $f(1) = a \in V$, then there exists an open set $U = X \in \tau$ containing 1 such that $f(cl^*(U)) \subseteq cl^*(V) = Y$.*
2. *Let $2 \in X$ and $V = \{a, b\}$ or $V = Y$ such that $f(2) = b \in V$, then there exists an open set $U = \{2\}$ containing 2 such that $f(cl^*(U)) \subseteq cl^*(V) = Y$.*
3. *Let $3 \in X$ and $V = Y$ such that $f(3) = c \in V$, then there exists an open set $U = \{2, 3\}$ containing 3 such that $f(cl^*(U)) \subseteq cl^*(V) = Y$.*

*By (1), (2) and (3) $f$ is $\theta_g$-continuous. On the other hand, let $1 \in X$ and $V = \{a, b\} \in \sigma$ such that $f(1) = a \in V$. But, for every open set $U \subseteq X$ such that $1 \in U = X$. Then $f(U) = Y \nsubseteq V = \{a, b\}$. Therefore, $f : (X, \tau) \to (Y, \sigma)$ is not continuous.*

## 3. Characterizations of $\theta_g$-continuous functions

In this section, we obtain several characterizations of $\theta_g$-continuous functions in topological spaces.

**Theorem 3.1.** *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. *$f$ is $\theta_g$-continuous;*

2. $Cl^*_\theta(f^{-1}(B)) \subseteq f^{-1}(cl^*_\theta(B))$ *for every subset* $B$ *of* $Y$;
3. $f(Cl^*_\theta(A)) \subseteq cl^*_\theta(f(A))$ *for every subset* $A$ *of* $X$.

*Proof.* (1) $\Rightarrow$ (2): Let $B$ be any subset of $Y$. Suppose that $x \notin f^{-1}(cl^*_\theta(B))$. Then $f(x) \notin cl^*_\theta(B)$ and there exists an open set $V$ containing $f(x)$ such that $cl^*(V) \cap B = \emptyset$. Since $f$ is $\theta_g$-continuous, there exists an open set $U$ containing $x$ such that $f(Cl^*(U)) \subseteq cl^*(V)$. Therefore, we have $f(Cl^*(U)) \cap B = \emptyset$ and $Cl^*(U) \cap f^{-1}(B) = \emptyset$. This shows that $x \notin Cl^*_\theta(f^{-1}(B))$. Thus, we obtain $Cl^*_\theta(f^{-1}(B)) \subseteq f^{-1}(cl^*_\theta(B))$.
(2) $\Rightarrow$ (1): Let $x \in X$ and $V$ be an open set of $Y$ containing $f(x)$. Then we have $cl^*(V) \cap (Y - cl^*(V)) = \emptyset$ and $f(x) \notin cl^*_\theta(Y - cl^*(V))$. Therefore, $x \notin f^{-1}(cl^*_\theta(Y - cl^*(V)))$ and by (2) we have $x \notin Cl^*_\theta(f^{-1}(Y - cl^*(V)))$. There exists an open set $U$ containing $x$ such that $Cl^*(U) \cap f^{-1}(Y - cl^*(V)) = \emptyset$ and hence $f(Cl^*(U)) \subseteq cl^*(V)$. Therefore, $f$ is $\theta_g$-continuous.
(2) $\Rightarrow$ (3): Let $A$ be any subset of $X$. Then we have $Cl^*_\theta(A) \subseteq Cl^*_\theta(f^{-1}(f(A))) \subseteq f^{-1}(cl^*_\theta(f(A)))$ and hence $f(Cl^*_\theta(A)) \subseteq cl^*_\theta(f(A))$.
(3) $\Rightarrow$ (2): Let $B$ be a subset of $Y$. We have $f(Cl^*_\theta(f^{-1}(B))) \subseteq cl^*_\theta(f(f^{-1}(B))) \subseteq cl^*_\theta(B)$ and hence $Cl^*_\theta(f^{-1}(B)) \subseteq f^{-1}(cl^*_\theta(B))$. □

**Theorem 3.2.** *For a function* $f : (X, \tau) \to (Y, \sigma)$, *the following implications: (1)* $\Leftrightarrow$ *(2)* $\Rightarrow$ *(3)* $\Leftrightarrow$ *(4) hold. Moreover, the implication (4)* $\Rightarrow$ *(1) holds if* $(Y, \sigma)$ *is a* $T_{\frac{1}{2}}$*-space.*

1. $f$ *is* $\theta_g$*-continuous;*
2. $f^{-1}(V) \subseteq Int^*_\theta(f^{-1}(cl^*(V)))$ *for every open set* $V$ *of* $Y$;
3. $Cl^*_\theta(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ *for every open set* $V$ *of* $Y$;
4. *For each* $x \in X$ *and each open set* $V$ *of* $Y$ *containing* $f(x)$, *there exists an open set* $U$ *of* $X$ *containing* $x$ *such that* $f(Cl^*(U)) \subseteq cl(V)$.

*Proof.* (1) $\Rightarrow$ (2): Let $V$ be any open set in $Y$ and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists an open set $U$ containing $x$ such that $f(Cl^*(U)) \subseteq cl^*(V)$. Therefore, $x \in U \subseteq Cl^*(U) \subseteq f^{-1}(cl^*(V))$. This shows that $x \in Int^*_\theta(f^{-1}(cl^*(V)))$. Therefore, we obtain $f^{-1}(V) \subseteq Int^*_\theta(f^{-1}(cl^*(V)))$.
(2) $\Rightarrow$ (1): Let $x \in X$ and $V \in \sigma$ containing $f(x)$. Then, by (2) $f^{-1}(V) \subseteq Int^*_\theta(f^{-1}(cl^*(V)))$. Since $x \in f^{-1}(V)$, there exists an open set $U$ containing $x$ such that $Cl^*(U) \subseteq f^{-1}(cl^*(V))$. Therefore, $f(Cl^*(U)) \subseteq cl^*(V)$ and hence $f$ is $\theta_g$-continuous.
(2) $\Rightarrow$ (3): Let $V$ be any open set in $Y$ and $x \notin f^{-1}(cl(V))$. Then $f(x) \notin cl(V)$ and there exists an open set $W$ containing $f(x)$ such that $W \cap V = \emptyset$; hence $cl^*(W) \cap V \subseteq cl(W) \cap V = \emptyset$. Therefore, we have $f^{-1}(cl^*(W)) \cap f^{-1}(V) = \emptyset$. Since $x \in f^{-1}(W)$, by (2) $x \in Int^*_\theta(f^{-1}(cl^*(W)))$. There exists an open set $U$ containing $x$ such that $Cl^*(U) \subseteq f^{-1}(cl^*(W))$. Thus we have $Cl^*(U) \cap f^{-1}(V) = \emptyset$ and hence $x \notin Cl^*_\theta(f^{-1}(V))$. This shows that $Cl^*_\theta(f^{-1}(V)) \subseteq f^{-1}(cl(V))$.
(3) $\Rightarrow$ (4): Let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. Then $V \cap (Y - cl(V)) = \emptyset$ and $f(x) \notin cl(Y - cl(V))$. Therefore $x \notin f^{-1}(cl(Y - cl(V)))$ and by (3) $x \notin Cl^*_\theta(f^{-1}(Y - cl(V)))$. There exists an open set $U$ containing $x$ such that $Cl^*(U) \cap f^{-1}(Y - cl(V)) = \emptyset$. Therefore, we obtain $f(Cl^*(U)) \subseteq cl(V)$.
(4) $\Rightarrow$ (3): Let $V$ be any open set of $Y$. Suppose that $x \notin f^{-1}(cl(V))$. Then $f(x) \notin cl(V)$ and there exists an open set $W$ containing $f(x)$ such that $W \cap V = \emptyset$. By (4), there exists an open set $U$ containing $x$ such that $f(Cl^*(U)) \subseteq cl(W)$. Since $V \in \sigma$, $cl(W) \cap V = \emptyset$ and $f(Cl^*(U)) \cap V \subseteq cl(W) \cap V = \emptyset$. Therefore, $Cl^*(U) \cap f^{-1}(V) = \emptyset$ and hence $x \notin Cl^*_\theta(f^{-1}(V))$. This shows that $Cl^*_\theta(f^{-1}(V)) \subseteq f^{-1}(cl(V))$.
(4) $\Rightarrow$ (1): Since $(Y, \sigma)$ is a $T_{\frac{1}{2}}$-space, $cl(V) = cl^*(V)$ for every open set $V$ of $Y$ and hence $f$ is $\theta_g$-continuous. □

**Proposition 3.3.** *A function* $f : (X, \tau) \to (Y, \sigma)$ *from a* $T_{\frac{1}{2}}$*-space to a* $T_{\frac{1}{2}}$*-space is* $\theta_g$*-continuous if and only if it is* $\theta$*-continuous.*

*Proof.* This follows from the Lemma 2.1 (3). □

## 4. Some properties of $\theta_g$-continuous functions

**Definition 4.1.** *A topological space* $(X, \tau)$ *is said to be* $\theta_g$*-*$T_2$ *(resp. g-Urysohn) if for each distinct points* $x, y \in X$, *there exist two* $\theta_g$*-open (resp. open) sets* $U, V \in X$ *containing* $x$ *and* $y$, *respectively, such that* $U \cap V = \emptyset$ *(resp.* $Cl^*(U) \cap Cl^*(V) = \emptyset$*).*

**Theorem 4.2.** *If* $f, g : (X, \tau) \to (Y, \sigma)$ *are* $\theta_g$*-continuous functions and* $(Y, \sigma)$ *is g-Urysohn , then* $A = \{x \in X : f(x) = g(x)\}$ *is a* $\theta_g$*-closed set of* $(X, \tau)$.

*Proof.* We prove that $X - A$ is a $\theta_g$-open set. Let $x \in X - A$. Then $f(x) \neq g(x)$. Since $Y$ is $g$-Urysohn, there exist open sets $V_1$ and $V_2$ containing $f(x)$ and $g(x)$, respectively, such that $cl^*(V_1) \cap cl^*(V_2) = \emptyset$. Since $f$ and $g$ are $\theta_g$-continuous, there exists an open set $U_1$ containing $x$ such that $f(Cl^*(U_1)) \subseteq cl^*(V_1))$ and there exists an open set $U_2$ containing $x$ such that $g(Cl^*(U)) \subseteq cl^*(V_2))$. Let $U = U_1 \cap U_2$ which is an open set $U$ containing $x$ such that $f(Cl^*(U)) \subseteq cl^*(V_1))$ and $g(Cl^*(U)) \subseteq cl^*(V_2))$. Hence we obtain that $Cl^*(U) \subseteq f^{-1}(cl^*(V_1))$ and $Cl^*(U) \subseteq g^{-1}(cl^*(V_2))$. From here we have $Cl^*(U) \subseteq f^{-1}(cl^*(V_1)) \cap g^{-1}(cl^*(V_2))$. Moreover $f^{-1}(cl^*(V_1)) \cap g^{-1}(cl^*(V_2)) \subseteq X - A$. This shows that $X - A$ is $\theta_g$-open.

□

**Definition 4.3.** *A topological space $(X, \tau)$ is said to be $g$-regular if for each closed set $F$ and each point $x \notin F$, there exist an open set $V$ and an $\tau^*$-open set $U \in \tau^*$ such that $x \in V$ , $F \subseteq U$ and $U \cap V = \emptyset$.*

**Example 4.4.** *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ with $\tau^* = \{\emptyset, X, \{a\}, \{b, c\}, \{a, c\}, \{a, b\}, \{b\}, \{c\}\}$, then $(X, \tau)$ is a $g$-regular space which is not regular.*

**Lemma 4.5.** *A topological space $(X, \tau)$ is $g$-regular if and only if for each open set $U$ containing $x$ there exists an open set $V$ such that $x \in V \subseteq Cl^*(V) \subseteq U$.*

**Lemma 4.6.** *A function $f : (X, \tau) \to (Y, \sigma)$ is weakly $g$-continuous if and only if for each open set $V$, $f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$.*

*Proof. Necessity.* Let $V$ be any open set of $Y$ and $x \in f^{-1}(V)$. Since $f$ is weakly $g$-continuous, there exists an open set $U$ such that $x \in U$ and $f(U) \subseteq cl^*(V)$. Hence $x \in U \subseteq f^{-1}(cl^*(V))$ and $x \in Int(f^{-1}(cl^*(V)))$. Therefore, we obtain $f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$.

*Sufficiency.* Let $x \in X$ and $V$ be an open set of $Y$ containing $f(x)$. Then $x \in f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$. Let $U = Int(f^{-1}(cl^*(V)))$. Then $f(U) \subseteq f(Int(f^{-1}(cl^*(V)))) \subseteq f(f^{-1}(cl^*(V))) \subseteq cl^*(V)$. Hence $f$ is weakly $g$-continuous.

□

**Lemma 4.7.** *If a space $(Y, \sigma)$ is a $T_{\frac{1}{2}}$-space and a function $f : (X, \tau) \to (Y, \sigma)$ is weakly $g$-continuous, then $Cl^*(f^{-1}(G)) \subseteq f^{-1}(cl^*(G))$ for every open set $G$ in $Y$.*

*Proof.* Suppose there exists a point $x \in Cl^*(f^{-1}(G)) - f^{-1}(cl^*(G))$. Then $f(x) \notin cl^*(G)$, we have $f(x) \notin G$. Since $Y$ is a $T_{\frac{1}{2}}$-space $f(x) \notin cl(G)$. Hence there exists an open set $W$ containing $f(x)$ such that $W \cap G = \emptyset$. Since $G$ is open, $G \cap cl(W) = \emptyset$ and hence we have $G \cap cl^*(W) = \emptyset$. Since $f$ is weakly $g$-continuous, there exists an open set $U \subseteq X$ containing $x$ such that $f(U) \subseteq cl^*(W)$. Thus we obtain $f(U) \cap G = \emptyset$. On the other hand, since $x \in Cl^*(f^{-1}(G))$, we have $x \in Cl(f^{-1}(G))$ and hence $U \cap f^{-1}(G) \neq \emptyset$. Thus $f(U) \cap G \neq \emptyset$, a contradiction. Hence $Cl^*(f^{-1}(G)) \subseteq f^{-1}(cl^*(G))$ for every open set $G$ in $Y$. □

**Theorem 4.8.** *Let $(Y, \sigma)$ be a $T_{\frac{1}{2}}$-space. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. *$f$ is weakly $g$-continuous;*
2. *$Cl(f^{-1}(V)) \subseteq f^{-1}(cl^*(V))$ for every open set $V$ of $Y$;*
3. *$f$ is weakly continuous.*

*Proof.* (1) ⇒ (2): Let $V$ be any open set of $Y$. Suppose that $x \notin f^{-1}(cl^*(V))$. Then $f(x) \notin cl^*(V)$. Since $(Y, \sigma)$ is a $T_{\frac{1}{2}}$-space, $f(x) \notin cl(V)$ and there exists $W \in \sigma$ containing $f(x)$ such that $W \cap V = \emptyset$, hence $cl^*(W) \cap V = cl(W) \cap V = \emptyset$. Since $f$ is weakly $g$-continuous, there exists $U \in \tau$ containing $x$ such that $f(U) \subseteq cl^*(W)$. Therefore, we have $f(U) \cap V = \emptyset$ and $U \cap f^{-1}(V) = \emptyset$. Since $U \in \tau$, $U \cap Cl(f^{-1}(V)) = \emptyset$ and hence $x \notin Cl(f^{-1}(V))$. Therefore, we obtain $Cl(f^{-1}(V)) \subseteq f^{-1}(cl^*(V))$.
(2) ⇒ (3): Let $V$ be any open set of $Y$. Since $(Y, \sigma)$ is a $T_{\frac{1}{2}}$-space, by (2) we have $Cl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$. It follows from Theorem 7 of [13] that $f$ is weakly continuous.
(3) ⇒ (1): Let $f$ be weakly continuous. By Theorem 1 of [7] $f^{-1}(V) \subseteq Int(f^{-1}(cl(V)))$ for every open set $V$ of $Y$. Since $(Y, \sigma)$ is a $T_{\frac{1}{2}}$-space, $cl(V) = cl^*(V)$ and we have $f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$. Therefore, by Lemma 4.6 $f$ is weakly $g$-continuous. □

**Definition 4.9.** *An topological space $(X, \tau)$ is said to be $g$-extremally disconnected if the $g$-closure of every open subset of $X$ is open.*

**Proposition 4.10.** *Let $(X, \tau)$ be an $g$-regular space. Then $f : (X, \tau) \to (Y, \sigma)$ is $\theta_g$-continuous if and only if it is weakly $g$-continuous.*

*Proof.* Every $\theta_g$-continuous function is weakly $g$-continuous. Suppose that $f$ is weakly $g$-continuous. Let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. Then, there exists an open set $U$ containing $x$ such that $f(U) \subseteq cl^*(V)$. Since $X$ is $g$-regular, by Lemma 4.5 there exists an open set $W$ such that $x \in W \subseteq Cl^*(W) \subseteq U$. Therefore, we obtain $f(Cl^*(W)) \subseteq cl^*(V)$. This shows that $f$ is $\theta_g$-continuous. □

**Theorem 4.11.** *Let a topological space $(Y, \sigma)$ be a $T_{\frac{1}{2}}$-space and $g$-extremally disconnected. Then $f : (X, \tau) \to (Y, \sigma)$ is $\theta_g$-continuous if and only if it is weakly $g$-continuous.*

*Proof.* It is clear that every $\theta_g$-continuous function is weakly $g$-continuous. Conversely, suppose that $f$ is weakly $g$-continuous. Let $x \in X$ and $V$ be an open set of $Y$ containing $f(x)$. Then by Lemma 4.6, $x \in f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$. Let $U = Int(f^{-1}(cl^*(V)))$. Since $(Y, \sigma)$ is an $T_{\frac{1}{2}}$-space and $g$-extremally disconnected, by using Lemma 4.7,

$$\begin{aligned} f(Cl^*(U)) =& f(Cl^*(Int(f^{-1}(cl^*(V)))) \\ \subseteq& f(Cl^*(f^{-1}(cl^*(V))) \\ \subseteq& f(f^{-1}(cl^*(cl^*(V))) \subseteq cl^*(V). \end{aligned}$$

Hence $f$ is $\theta_g$-continuous. □

**Corollary 4.12.** *Let a topological space $(Y, \sigma)$ be a $T_{\frac{1}{2}}$-space and $g$-extremally disconnected. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. *$f$ is $\theta_g$-continuous;*
2. *$f$ is weakly $g$-continuous;*
3. *$f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V))$ for every open set $V$ in $Y$;*
4. *$f^{-1}(V) \subseteq Int(f^{-1}(cl(V)))$ for every open set $V$ of $Y$;*
5. *$f$ is weakly continuous.*

*Proof.* By Theorem 4.11, we have the equivalence of (1) and (2). The equivalences of (2), (3) and (4) follow from Lemma 4.6 and Lemma 2.1 (1). The equivalence of (4) and (5) is shown in Theorem 1 of [7]. □

A subset $A$ of a topological space $(X, \tau)$ is said to be pre-$g$-open if $A \subseteq Int(Cl^*(A))$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be pre-$g$-continuous if the inverse image of every open set of $Y$ is pre-$g$-open in $X$.

**Theorem 4.13.** *If $f : (X, \tau) \to (Y, \sigma)$ is a pre-$g$-continuous function and $Cl^*(f^{-1}(U)) \subseteq f^{-1}(cl^*(U))$ for every open set $U$ in $Y$, then $f$ is $\theta_g$-continuous.*

*Proof.* Let $x \in X$ and $U$ be an open set in $Y$ containing $f(x)$. By hypothesis, $Cl^*(f^{-1}(U)) \subseteq f^{-1}(cl^*(U))$. Since $f$ is pre-$g$-continuous, $f^{-1}(U)$ is pre-$g$-open in $X$ and so $f^{-1}(U) \subseteq Int(Cl^*(f^{-1}(U)))$. Since $x \in f^{-1}(U) \subseteq Int(Cl^*(f^{-1}(U)))$, there exists an open set $V$ containing $x$ such that $x \in V \subseteq Cl^*(V) \subseteq Cl^*(f^{-1}(U)) \subseteq f^{-1}(cl^*(U))$ and so $f(Cl^*(V)) \subseteq cl^*(U)$ which implies that $f$ is $\theta_g$-continuous. □

The following corollary follows from Lemmas 4.6 and 4.7 also Theorems 4.8 and 4.13.

**Corollary 4.14.** *Let $f : (X, \tau) \to (Y, \sigma)$ be pre-$g$-continuous and $(Y, \sigma)$ is a $T_{\frac{1}{2}}$-space. The following properties are equivalent:*

1. *$f$ is $\theta_g$-continuous;*
2. *$Cl^*(f^{-1}(V)) \subseteq f^{-1}(cl^*(V))$ for every open set $V$ in $Y$;*
3. *$Cl(f^{-1}(V)) \subseteq f^{-1}(cl^*(V))$ for every open set $V$ in $Y$;*
4. *$f$ is weakly $g$-continuous.*

## 5. Preservation theorems

A subset $A$ of a space $X$ is said to be quasi $H^*$-closed relative to $X$ if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of $A$ by open sets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $A \subseteq \cup\{Cl^*(V_\alpha) : \alpha \in \Lambda_0\}$. A space $X$ is said to be quasi $H^*$-closed if $X$ is quasi $H^*$-closed relative to $X$

**Theorem 5.1.** *If $f : (X, \tau) \to (Y, \sigma)$ is $\theta_g$-continuous and $K$ is quasi $H^*$-closed relative to $X$, then $f(K)$ is quasi $H^*$-closed relative to $Y$.*

*Proof.* Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a $\theta_g$-continuous function and $K$ is quasi $H^*$-closed relative to $X$. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $f(K)$ by open sets of $Y$. For each point $x \in K$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since $f$ is $\theta_g$-continuous, there exists an open set $U_x$ containing $x$ such that $f(Cl^*(U_x)) \subseteq cl^*(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of $K$ by open sets of $X$ and hence there exists a finite subset $K_*$ of $K$ such that $K \subseteq \cup_{x\in K_*} Cl^*(U_x)$. Therefore, we obtain $f(K) \subseteq \cup_{x\in K_*} cl^*(V_{\alpha(x)})$. This shows that $f(K)$ is quasi $H^*$-closed relative to $Y$. □

**Definition 5.2.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\theta_g$-irresolute if for every $\theta_g$-open set $U$ in $Y$, $f^{-1}(U)$ is $\theta_g$-open in $X$.*

**Theorem 5.3.** *Every $\theta_g$-continuous function $f : (X, \tau) \to (Y, \sigma)$ is $\theta_g$-irresolute.*

*Proof.* Let $f$ be a $\theta_g$-continuous function and $U$ be a $\theta_g$-open set in $Y$. Let $x \in f^{-1}(U)$. Then, $f(x) \in U$. Since $U$ is $\theta_g$-open, there exists an open set $V$ in $Y$ such that $f(x) \in V \subseteq cl^*(V) \subseteq U$. By $\theta_g$-continuity of $f$, there exists an open set $W$ in $X$ containing $x$ such that $f(Cl^*(W)) \subseteq cl^*(V) \subseteq U$. Thus $x \in W \subseteq Cl^*(W) \subseteq f^{-1}(U)$. Hence $f^{-1}(U)$ is $\theta_g$-open and hence $f$ is $\theta_g$-irresolute. □

**Definition 5.4.** *(1) A topological space $(X, \tau)$ is said to be $\theta_g$-compact if every cover of $X$ by $\theta_g$-open sets admits a finite subcover.*
*(2) A subset $A$ of a topological space $(X, \tau)$ is said to be $\theta_g$-compact relative to $X$ if every cover of $A$ by $\theta_g$-open sets of $X$ admits a finite subcover.*

**Proposition 5.5.** *In a topological space $(X, \tau)$ every quasi $H^*$-closed set is $\theta_g$-compact.*

*Proof.* More generally, we show that if $A$ is quasi $H^*$-closed relative to a space $X$, then $A$ is $\theta_g$-compact relative to $X$. Let $A \subseteq \cup\{V_\alpha : \alpha \in \Lambda\}$, where each $V_\alpha$ is $\theta_g$-open, and $A$ be quasi $H^*$-closed relative to $X$, then for each $x \in A$ there exists an $\alpha(x) \in \Lambda$ with $x \in V_{\alpha(x)}$. Then there exists an open set $U_{\alpha(x)}$ with $x \in U_{\alpha(x)}$ such that $Cl^*(U_{\alpha(x)}) \subseteq V_{\alpha(x)}$. Since $\{U_{\alpha(x)} : x \in A\}$ is a cover of $A$ by open set in $X$, then there is a finite subset $\{x_1, x_2, ..., x_n\} \subseteq A$ such that $A \subseteq \cup\{Cl^*(U_{\alpha(x_i)}) : i = 1, 2, ..., n\} \subseteq \cup\{V_{\alpha(x_i)} : i = 1, 2, ..., n\}$. Hence $A$ is $\theta_g$-compact relative to $X$. □

**Theorem 5.6.** *If $f : (X, \tau) \to (Y, \sigma)$ is a $\theta_g$-irresolute surjection and $(X, \tau)$ is $\theta_g$-compact, then $Y$ is $\theta_g$-compact.*

*Proof.* Let $\mathcal{V}$ be a $\theta_g$-open covering of $Y$. Then, since $f$ is $\theta_g$-irresolute, the collection $\mathcal{U} = \{f^{-1}(U) : U \in \mathcal{V}\}$ is a $\theta_g$-open covering of $X$. Since $X$ is $\theta_g$-compact, there exists a finite subcollection $\{f^{-1}(U_i) : i = 1, ..., n\}$ of $\mathcal{U}$ which covers $X$. Now since $f$ is onto, $\{U_i : i = 1, ..., n\}$ is a finite subcollection of $\mathcal{V}$ which covers $Y$. Hence $Y$ is a $\theta_g$-compact space. □

**Corollary 5.7.** *The $\theta_g$-continuous surjective image of a $\theta_g$-compact space is $\theta_g$-compact.*

**Definition 5.8.** *A topological space $(X, \tau)$ is said to be $g$-Lindelöf if for every open cover $\{U_\alpha : \alpha \in \Lambda\}$ of $X$ there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Lambda$ such that $X = \cup_{n\in\mathbb{N}} Cl^*(U_{\alpha_n})$.*

**Theorem 5.9.** *Let $f : (X, \tau) \to (Y, \sigma)$ be a $\theta_g$-continuous (resp. weakly $g$-continuous) surjection. If $X$ is $g$-Lindelöf (resp. Lindelöf), then $Y$ is $g$-Lindelöf.*

*Proof.* Suppose that $f$ is $\theta_g$-continuous and $X$ is $g$-Lindelöf. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $Y$. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since $f$ is $\theta_g$-continuous, there exists an open set $U_{\alpha(x)}$ of $X$ containing $x$ such that $f(Cl^*(U_{\alpha(x)})) \subseteq cl^*(V_{\alpha(x)})$. Now $\{U_{\alpha(x)} : x \in X\}$ is an open cover of the $g$-Lindelöf space $X$. So there exists a countable subset $\{U_{\alpha(x_n)} : n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}}(Cl^*(U_{\alpha(x_n)}))$. Thus $Y = f(\cup_{n\in\mathbb{N}}(Cl^*(U_{\alpha(x_n)}))) \subseteq \cup_{n\in\mathbb{N}} f(Cl^*(U_{\alpha(x_n)})) \subseteq \cup_{n\in\mathbb{N}} cl^*(V_{\alpha(x_n)})$. This shows that $Y$ is $g$-Lindelöf. In case $X$ is Lindelöf the proof is similar. □

A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $\theta_g$-closed if for each $\theta_g$-closed set $F$ in $X$, $f(F)$ is $\theta_g$-closed in $Y$. The following characterization of $\theta_g$-closed functions will be used in the sequel.

**Theorem 5.10.** *A surjective function $f:(X,\tau)\to(Y,\sigma)$ is $\theta_g$-closed if and only if for each set $B\subseteq Y$ and for each $\theta_g$-open set $U$ containing $f^{-1}(B)$, there exists a $\theta_g$-open set $V$ containing $B$ such that $f^{-1}(V)\subseteq U$.*

*Proof. Necessity.* Suppose that $f$ is $\theta_g$-closed. Since $U$ is $\theta_g$-open in $X$, $X-U$ is $\theta_g$-closed and so $f(X-U)$ is $\theta_g$-closed in $Y$. Now, $V=Y-f(X-U)$ is $\theta_g$-open, $B\subseteq V$ and $f^{-1}(V)=f^{-1}(Y-f(X-U))=X-f^{-1}(f(X-U))\subseteq X-(X-U)=U$.

*Sufficiency.* Let $A$ be a $\theta_g$-closed set in $X$. To prove that $f(A)$ is $\theta_g$-closed, we shall show that $Y-f(A)$ is $\theta_g$-open. Let $y\in Y-f(A)$. Then $f^{-1}(y)\cap f^{-1}(f(A))=\emptyset$ and so $f^{-1}(y)\subseteq X-f^{-1}(f(A))\subseteq X-A$. By hypothesis there exists a $\theta_g$-open set $V$ containing $y$ such that $f^{-1}(V)\subseteq X-A$. So $A\subseteq X-f^{-1}(V)$ and hence $f(A)\subseteq f(X-f^{-1}(V))=Y-V$. Thus $V\subseteq Y-f(A)$ and so the set $Y-f(A)$ being the union of $\theta_g$-open sets is $\theta_g$-open. □

**Theorem 5.11.** *Let $f:(X,\tau)\to(Y,\sigma)$ be a $\theta_g$-closed surjection such that for each $y\in Y$, $f^{-1}(y)$ is $\theta_g$-compact relative to $X$. If $Y$ is $\theta_g$-compact, then $X$ is $\theta_g$-compact.*

*Proof.* Let $\mathcal{U}=\{U_\alpha:\alpha\in\Lambda\}$ be a $\theta_g$-open covering of $X$. Since for each $y\in Y$, $f^{-1}(y)$ is $\theta_g$-compact relative to $X$, we can choose a finite subset $\Lambda_y$ of $\Lambda$ such that $\{U_\beta:\beta\in\Lambda_y\}$ is a covering of $f^{-1}(y)$. Now, by Theorem 5.10, there exists a $\theta_g$-open set $V_y$ containing $y$ such that $f^{-1}(V_y)\subseteq\cup\{U_\beta:\beta\in\Lambda_y\}$. The collection $\mathcal{V}=\{V_y:y\in Y\}$ is a $\theta_g$-open covering of $Y$. In view of $\theta_g$-compactness of $Y$ there exists a finite subcollection $\{V_{y_1},...,V_{y_n}\}$ of $\mathcal{V}$ which covers $Y$. Then the finite subcollection $\{U_\beta:\beta\in\Lambda_{y_i}, i=1,...,n\}$ of $\mathcal{U}$ covers $X$. Hence $X$ is a $\theta_g$-compact space. □

**Theorem 5.12.** *Let $f:(X,\tau)\to(Y,\sigma)$ be a function, $D$ be a dense subset in the topological space $(Y,\sigma^*)$ and $f(X)\subseteq D$. Then the following properties are equivalent:*

1. *$f:(X,\tau)\to(Y,\sigma)$ is $\theta_g$-continuous;*
2. *$f:(X,\tau)\to(D,\sigma_D)$ is $\theta_g$-continuous.*

*Proof.* (1) $\Rightarrow$ (2): Let $x\in X$ and $W$ be any open set of $D$ containing $f(x)$, that is $f(x)\in W\in\sigma_D$ where $\sigma_D=\{U\cap D\}$ and $U\in\sigma$. Then exists $V\in\sigma$ such that $W=D\cap V$. Since $f:(X,\tau)\to(Y,\sigma)$ is $\theta_g$-continuous and $f(x)\in V\in\sigma$, there exists $U\in\tau$ such that $x\in U$ and $f(Cl^*(U))\subseteq cl^*(V)$. If $D$ is a dense subset in the topological space $(Y,\sigma^*)$, then $D$ is a dense subset in the topological space $(Y,\sigma)$ since $cl^*(D)\subseteq cl(D)$. Since $\sigma\subseteq\sigma^*$, $V\in\sigma^*$. So, $cl^*(D\cap V)=cl^*(V)$ since $D$ is dense. Thus $f(Cl^*(U))\subseteq cl^*(V)\cap f(X)\subseteq cl^*(D\cap V)\cap D\subseteq cl^*(V)\cap D$. Since $W=D\cap V$, $cl^*_D(W)=cl^*(V)\cap D$, $f(Cl^*(U))\subseteq cl^*_D(W)$. Hence we obtain that $f:(X,\tau)\to(D,\sigma_D)$ is $\theta_g$-continuous.
(2) $\Rightarrow$ (1): Let $x\in X$ and $V$ be any open set $Y$ containing $f(x)$. Since $f(x)\in D\cap V$ and $D\cap V\in\sigma_D$, by (2) there exists $U\in\tau$ containing $x$ such that $f(Cl^*(U))\subseteq cl^*_D(D\cap V)=cl^*(D\cap V)\cap D\subseteq cl^*(V)$. This shows that $f$ is $\theta_g$-continuous. □

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