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Impulsive Fractional Dynamic Equation with Non-local Initial Condition on Time Scales

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ABSTRACT: In this manuscript the existence and uniqueness of an impulsive fractional dynamic equation on time scales involving non-local initial condition with the help of Caputo nabla derivative has been investigated. The existency of the solution is based on the nonlinear alternatives Leray-Schauder's type fixed point theorem along with the Banach contraction theorem. The comparison of Caputo nabla derivative and Riemann-Liouville nabla derivative of fractional order is also discussed in the context of time scale.

Key Words: Caputo nabla derivative and Riemann-Liouville nabla derivative, nonlinear alternatives Leray-Schauder's type fixed point theorem, Arzela-Ascoli theorem, Banach contraction theorem.

Contents

1. Introduction

In our real world, some situation may arise which cannot be modeled in entirely continuous phenomena or in entirely discrete phenomena, in such situation we need a common domain which justify both the conditions. On the basis of unification of these conditions Stefan Hilger introduced a common domain called time scale $\mathbb{T}$ which unifies both continuous and discrete calculus [16,17]. Dynamic equations on time scale were introduced to solve this kind of model which is a combination of both the differential and difference equation. Many researchers worked on dynamic equation in linear and non linear form involving local initial and boundary conditions. Some authors have discussed the dynamic equation using the tool of fractional calculus due to the accuracy and advantage in the physical interpretation. For detailed study of fractional dynamic equations by Caputo, Rieamann-Liouville, Caputo-Hadamard and many others, readers can go through the manuscripts [4,6,8,9,12,29,24,26,30,32] and the references therein.

In real world situation, we have seen some equations where the systems are allowed to undergo some sudden perturbation, whose duration can be negligible in comparison with the duration of the process. In this case the solution of these equations may have jump discontinuities at time $\theta_1 < \theta_2 < \theta_3 < \cdots$, given in the form $p(\theta_k^+) - p(\theta_k^-) = \mathscr{I}_k(\theta_k, p(\theta_k^-))$. The dynamic equations having jump discontinuities for their solutions are called impulsive dynamic equations. The theory have interesting applications in the branch of mathematical modeling of different types of real world situation which require sudden changes at a particular time of their evolution, for example: natural disaster, particular diseases, etc. Work related to impulsive dynamic equations can be seen in the manuscripts [13,18,19,26] and the reference therein. In recent time, several researchers and authors have shown their attention in the topic of impulsive dynamic equations on time scales. However a few number of works have been seen in the impulsive dynamic equation by using the fractional calculus on time scales with non local initial condition.

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In the manuscript [26], the authors discussed the impulsive dynamic equation in terms of non local initial condition, whereas in [18,19], the authors investigated the fractional dynamic equation with instantaneous and non-instantaneous impulses with local initial condition by using the tools of delta (Hilger) derivative. A dynamic model with non local initial conditions are applicable in all branches of science and engineering, due to the advantage of using non local initial conditions such as, the measurement at more places which can be incorporated to get a better model. For detailed study of the advantages of non-local initial conditions, one can see [2] and the references therein.

So motivated by aforementioned work, it is worthwhile to study the impulsive fractional dynamic equation with non-local initial condition of the type;

$$\begin{cases} {}^{C}\mathcal{D}^{w}p(\theta) = \mathcal{L}\big(\theta, p(\theta), {}^{C}\mathcal{D}^{w}p(\theta)\big), & \theta \in \mathcal{I}_T, \theta \neq \theta_k \\ p(\theta_k^+) - p(\theta_k^-) = \mathscr{I}_k\big(\theta_k, p(\theta_k^-), & k = 1,2,3,\cdots,n \\ p(0) = \phi(p), \end{cases} \tag{1.1}$$

where $k \in \mathbb{N} \cup \{0\}$, and $\mathcal{I}_T = [0,T] \cap \mathbb{T}$, for $T \in \mathbb{T}$, denote the time scale interval. $\mathcal{L} : \mathcal{I}_T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a left dense(ld) continuous function, and ${}^{C}\mathscr{D}^{w}p(\theta)$ denote the Caputo nabla derivative of order $w \in (0,1)$ which is discussed later in the paper. We assume that $0 < \theta_0 < \theta_1 < \theta_2 < \theta_3 < \ldots < \theta_n < \theta_{n+1} = T$, which represents the impulse at a certain moment, and the term $p(\theta_k^+) = \lim_{d\to 0} p(\theta + d)$ and $p(\theta_k^-) = \lim_{d\to 0} p(\theta - d)$ represents the right and left limits of the function p at $\theta = \theta_k$ in the context of time scales. $\mathscr{I}_k$ is a continuous real valued function on $\mathbb{R}$ for each $k = 1,2,3,\cdots,m$ and $\mathscr{I}_k(\theta_k, p(\theta_k^-))$ are the impulses acted on the time scale interval $\mathcal{I}_T$ which will be specified later.
The manuscript is organized as follows. In Section 2, we have presented some auxiliary results related to fractional dynamic equation on time scale, which will be required to show our main findings. In Section 3, we compare the Riemann-Liouville and Caputo nabla derivative in the context of time scale. In Section 4, we have given the existence and uniqueness theorem of an impulsive fractional dynamic equation with non local initial condition. In Section 5, we provided an example, which makes the manuscript easier to understand. Finally, conclusion of the paper is presented in Section 6.

2. Auxilary results

Definition 2.1. *[29] A function $\rho : \mathbb{T} \to \mathbb{R}$, defined by $\rho(\theta) = \{\zeta \in \mathbb{T} : \zeta < \theta\}$ is said to be backward jump operator. Any $\theta \in \mathbb{T}$ is said to be left dense if $\rho(\theta) = \theta$ and if $\rho(\theta) = \theta - 1$, then θ is said to be a left scattered point on $\mathbb{T}$. If $\mathbb{T}$ has a minimum right scattered point say y, then set $\mathbb{T}_{\mathcal{V}} = \mathbb{T} \setminus \{y\}$, else $\mathbb{T}_{\mathcal{V}} = \mathbb{T}$.*

Definition 2.2. *[26] A funtion $x : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be left dense continuous function, if $x\big(\cdot, u, v\big)$ is left dense continuous on $\mathbb{T}$ for each ordered pair $(\theta, \zeta) \in \mathbb{R} \times \mathbb{R}$. And $x(\theta, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}$ for fixed point $\theta \in \mathbb{T}$.*
***Remark**: The set of all left dense continuous function from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $\mathcal{C}\big(\mathbb{T}, \mathbb{R}\big)$.*

Definition 2.3. *[19] Consider a function $g : \mathbb{T} \to \mathbb{R}$. Let $\mathcal{G}$ be a function such that $\mathcal{G}_{\nabla}(\theta) = g(\theta)$, for each $\theta \in \mathbb{T}_{\mathcal{V}}$, then the nabla integral is presented by*

$$\int_a^\theta g(x)\nabla x = \mathcal{G}(x) - \mathcal{G}(a).$$

Proposition 2.4. *[6] Let g be an increasing continuous function on the time scale interval $[0,T] \cap \mathbb{T}$. If $\mathcal{G}$ is the extension of g in the real line interval $[0,T]$, $T \in \mathbb{R}$, one can get*

$$\mathcal{G}(\theta) = \begin{cases} g(\theta), & \text{if } \theta \in \mathbb{T}, \\ g(\zeta), & \text{if } \theta \in (\theta, \rho(\theta)) \notin \mathbb{T}, \end{cases}$$

then

$$\int_r^u g(\theta)\nabla\theta \leq \int_r^u \mathcal{G}(\theta)d\theta, \tag{2.1}$$

for $r, u \in [0,T] \cap \mathbb{T}$, such that $r < u$.

Definition 2.5. *[29, Higher order nabla derivative] Assume $\mathcal{H} : \mathbb{T}_{\mathcal{V}} \to \mathbb{R}$ is a ld continuous function on a time scale $\mathbb{T}$. The second order nabla derivative $\mathcal{H}_{\nabla\nabla} = \mathcal{H}_{\nabla}^{(2)}$ can be define, provided $\mathcal{H}_{\nabla}$ is differentiable on ${\mathbb{T}_{\mathcal{V}}}^{(2)} = \mathbb{T}_{\mathcal{V}\mathcal{V}}$ with derivative $\mathcal{H}_{\nabla}^{(2)} = (\mathcal{H}_{\nabla})_{\nabla} : {\mathbb{T}_{\mathcal{V}}}^{(2)} \to \mathbb{R}$. Similarly, proceeding upto n^{th} order we get $\mathcal{H}_{\nabla}^{(n)} : {\mathbb{T}_{\mathcal{V}}}^{n} \to \mathbb{R}$, it is attained by cut out n right scattered left end points from $\mathbb{T}$.*

Definition 2.6. *[29] Let $\mathcal{H} : \mathbb{T}_{\mathcal{V}^n} \to \mathbb{R}$ be a ld continuous function, such that $\mathcal{H}_{\nabla}^{(n)}(\theta)(n^{th}$ order of nabla derivative) is exists. Then the Caputo nabla derivative is defined as*

$$^{C}\mathcal{D}_{a}^{w}\mathcal{H}(\theta) = \frac{1}{\Gamma(n-w)} \int_{a}^{\theta} (\theta - \rho(\zeta))^{n-w-1} \mathcal{H}_{\nabla}^{(n)}(\zeta)\nabla\zeta,$$

for $n = [w] + 1$. If $w \in (0, 1)$, then

$$^{C}\mathcal{D}_{a}^{w}\mathcal{H}(\theta) = \frac{1}{\Gamma(1-w)} \int_{a}^{\theta} (\theta - \rho(\zeta))^{-w} \mathcal{H}_{\nabla}\nabla\zeta,$$

where [] is used to denote the greatest integer function.

Definition 2.7. *[29] Let $\mathcal{H}$ be any ld continuous function define on the set $\mathbb{T}_{\mathcal{V}}$, then the Riemann-Liouville nabla derivative of order $w \in (0, 1)$ is defined as*

$$\mathcal{D}_{\theta_0}^{w} x(t) = \frac{1}{\Gamma(1-w)} \Big(\int_{\theta_0}^{\theta} \big(\theta - \rho(\zeta)\big)^{-w} x(\zeta)\nabla\zeta \Big)^{\nabla}.$$

Definition 2.8. *[6, Definition 13] Let $\mathcal{H} : \mathcal{I}_{\mathcal{T}} \to \mathbb{R}$ be an integrable function. Then the Riemann-Liouville nabla fractional integral of $\mathcal{H}$ is given by*

$$\mathcal{D}_{\theta_0}^{-w}\mathcal{H}(\theta) = \mathcal{J}_{\theta_0}^{w}\mathcal{H}(\theta) = \frac{1}{\Gamma(w)} \int_{\theta_0}^{\theta} (\theta - \rho(\zeta))^{w-1} \mathcal{H}(\zeta)\nabla\zeta.$$

The Rieamm-Liouville nabla integral always satisfies the condition

$$\mathcal{J}_{\theta_0}^{w}\mathcal{J}_{\theta_0}^{u}\mathcal{H}(\theta) = \mathcal{J}_{\theta_0}^{w+u}\mathcal{H}(\theta).$$

Lemma 2.9. *[6, Definition 13] If $p(\theta)$ be a ld continuous function, then*

$$\begin{cases} \mathcal{D}^{u}\mathcal{J}^{w}p(\theta) = p(\theta) \\ \mathcal{D}^{u}\mathcal{J}^{w}p(\theta) = \mathcal{J}^{w-u}p(\theta). \end{cases}$$

Definition 2.10. *[19] Let $\mathcal{D} \subset \mathcal{C}(\mathbb{T}, \mathbb{R})$ be a set. Then $\mathcal{D}$ is a relatively compact, if it is bounded and equicontinuous simultaneously.*

Definition 2.11. *[26] A mapping $\mathcal{H} : A \to B$ is completely continuous, if for a bounded subset $\mathcal{B} \subseteq A$, $\mathcal{H}(\mathcal{B})$ is relatively compact in A.*

Definition 2.12. *[18, Nonlinear alternatives Leray-Schauder's type fixed point theorem] Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $G : U \to C$ is a compact map. Then either,*
(i) G has a fixed point in U; or
(ii) there is a point $u \in \delta U$ and $\lambda \in (0, 1)$ with $u = \lambda G(u)$.

Definition 2.13. *[26] If $\mathcal{L} : \mathcal{I}_{\mathcal{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a ld continuous function, then for $w \in (0, 1)$, a function p is a solution of*

$$^{C}\mathcal{D}^{w}p(\theta) = \mathcal{L}\big(\theta, p(\theta), ^{C}\mathcal{D}^{w}p(\theta)\big), p(\theta)|_{\theta=0} = \phi(p)$$

if and only if p is the solution of the integral equation

$$p(\theta) = \phi(p) + \frac{1}{\Gamma(w)} \int_{0}^{\theta} \big(\theta - \rho(x)\big)^{w-1} \mathcal{L}\big(x, p(x), ^{C}\mathcal{D}^{w}p(x)\big)\nabla x. \tag{2.2}$$

3. Comparison of Riemann-Liouville and Caputo nabla derivative

Proposition 3.1. *For any $w \in \mathbb{R}$, let $m-1 < w < m, m \in \mathbb{N}$ such that ${}^{C}\mathscr{D}^{w}_{\theta_0}\mathcal{G}(\theta)$ is exist in the time scale $\mathbb{T}$, then*

$$
{}^{C}\mathscr{D}^{w}_{\theta_0}\mathcal{G}(\theta) = \mathcal{J}^{m-w}_{\theta_0}\mathcal{G}^{(m)}_{\nabla}(\theta).
$$

Proof. The proof is obvious from the definition 2.12 and 2.13. □

Theorem 3.2. *For any $\theta \in \mathbb{T}_{\mathcal{V}^n}$, the Caputo nabla derivative and Riemann-Liouville nabla derivative of the order w, where $m = [w] + 1$ satisfies the following relation:*

$$
{}^{C}\mathcal{D}^{w}_{\alpha}\mathcal{G}(\theta) = \mathcal{D}^{w}_{\alpha}\big(\mathcal{G}(\theta) - \sum_{v=0}^{m-1}\frac{(\theta-\alpha)^v}{\Gamma(v+1)}\mathcal{G}^{(v)}_{\nabla}(\alpha)\big),
$$

for a fixed point $\alpha \in \mathbb{T}$.
The proof of this theorem is based on the Taylor's theorem defined in [30, Theorem 10].

Proof. Let us consider a ld continuous function $\mathcal{G}$ which is n times nabla differentiable, then for any fixed $\alpha \in \mathbb{T}$ and $m \in \mathbb{N}\cup\{0\}, m < n$ one can get

$$
\begin{aligned}
\mathcal{G}(\theta) &= \sum_{v=0}^{m-1}\frac{(\theta-\alpha)^v}{\Gamma(v+1)}\mathcal{G}^{(v)}_{\nabla}(\alpha) + \frac{1}{\Gamma(m)}\int_{\alpha}^{\theta}(\theta-\rho(\zeta)\mathcal{G}^{(m)}_{\nabla}(\zeta)\nabla\zeta \\
&= \sum_{v=0}^{m-1}\frac{(\theta-\alpha)^v}{\Gamma(v+1)}\mathcal{G}^{(v)}_{\nabla}(\alpha) + \mathcal{J}^{m}_{\alpha}\mathcal{G}^{(m)}_{\nabla}(\theta). \qquad (3.1)
\end{aligned}
$$

Now taking the Riemann-Liouville derivative $\mathcal{D}^{w}_{\alpha}$ of order w in both side of Equation (3.1) and using Lemma 2.9 and Proposition 3.1,

$$
\begin{aligned}
\mathcal{D}^{w}_{\alpha}\mathcal{G}(\theta) =& \mathcal{D}^{w}_{\alpha}\sum_{v=0}^{m-1}\frac{(\theta-\alpha)^v}{\Gamma(v+1)}\mathcal{G}^{(v)}_{\nabla}(\alpha) + \mathcal{D}^{w}_{\alpha}\mathcal{J}^{(m)}_{\alpha}\mathcal{G}^{(m)}_{\nabla}(\theta) \\
=& \mathcal{D}^{w}_{\alpha}\sum_{v=0}^{n-1}\frac{(\theta-\alpha)^v}{\Gamma(v+1)}\mathcal{G}^{(v)}_{\nabla}(\alpha) + \mathcal{J}^{m-w}_{\alpha}\mathcal{G}^{(m)}_{\nabla}(\theta) \\
=& \mathcal{D}^{w}_{\alpha}\sum_{v=0}^{m-1}\frac{(\theta-\alpha)^v}{\Gamma(v+1)}\mathcal{G}^{(v)}_{\nabla}(\alpha) + {}^{C}\mathcal{D}^{w}_{\alpha}\mathcal{G}(\theta). \qquad (3.2)
\end{aligned}
$$

From the above we obtain

$$
\begin{aligned}
{}^{C}\mathcal{D}^{w}_{\alpha}\mathcal{G}(\theta) =& \mathcal{D}^{w}_{\alpha}\mathcal{G}(\theta) - \mathcal{D}^{w}_{\alpha}\sum_{v=0}^{m-1}\frac{(\theta-\alpha)^v}{\Gamma(v+1)}\mathcal{G}^{(v)}_{\nabla}(\alpha) \\
=& \mathcal{D}^{w}_{\alpha}\big(\mathcal{G}(\theta) - \sum_{v=0}^{m-1}\frac{(\theta-\alpha)^v}{\Gamma(v+1)}\mathcal{G}^{(v)}_{\nabla}(\alpha)\big). \qquad (3.3)
\end{aligned}
$$

□

Proposition 3.3. *If $w \in (0,1)$, then $m = 1$. Hence, from the equation (3.3)*

$$
{}^{C}\mathcal{D}^{w}_{\alpha}(\theta) = \mathcal{D}^{w}_{\alpha}\big(\mathcal{G}(\theta) - \mathcal{G}(\alpha)\big).
$$

Case 1: If the initial condition $\mathcal{G}(\alpha) \to 0$, as $\alpha \to 0$, then

$$
{}^{C}\mathcal{D}^{w}_{\alpha}(\theta) = \mathcal{D}^{w}_{\alpha}\mathcal{G}(\theta). \qquad (3.4)
$$

Thus the Caputo nbala derivative of order $w \in (0,1)$ coincide with the Riemann-Liouville nabla derivative. Case 2: If $w \in \mathbb{N}$, and applying the equation (3.1) in equation (3.2), then using the Lemma 2.9, we obtain

$$\begin{aligned} {}^{C}\mathcal{D}_{\alpha}^{m}\mathcal{G}(\theta) =& \mathcal{D}_{\alpha}^{m}\big(\mathcal{G}(\theta) - \sum_{v=0}^{m-1}\frac{(\theta-\alpha)^{v}}{\Gamma(v+1)}\mathcal{G}(\alpha)\big) \\ =& \mathcal{D}_{\alpha}^{m}\mathcal{J}_{\alpha}^{m}\mathcal{G}_{\nabla}^{(m)}(\theta) \\ =& \mathcal{G}_{\nabla}^{(m)}(\theta). \end{aligned}$$

Thus the Caputo nabla derivative is coincide with the nabla derivative.

Remark 3.4. *When the initial condition is given in terms of real order in any types of dynamic equation involving impulses, the application of Caputo nabla derivative is mostly preferable over the Riemann-Liouville derivative due to its physical interpretation see [6,20]. However in terms of integral order the accuracy of Caputo nabla derivative and Riemann-Liouville nabla derivative are almost same.*

4. Existence and uniqueness of impulsive fractional dynamic equation

The dynamic equation (1.1) can be compared with a model of population dynamics with a stop start phenomena, where $p(\theta)$ is a population of a particular species of insect at a time θ. If we include a toxic effect on that particular species, and we noticed the change of population which is presented by the Caputo derivative operator ${}^{C}\mathscr{D}^{w}p(\theta)$(at the initial stage of time) of order w, with respect to time θ on the interval $\mathcal{J}_{\mathcal{T}} = [0,T]\cap\mathbb{T}$. Now we consider the case where at certain moments $\theta_1, \theta_2, \theta_3, \cdots$ such that $0 < \theta_1 < \theta_2 <, \cdots, \theta_m < \theta_{m+1} = T, \lim_{k\to\infty}\theta_k = \infty$, impulse effect act on the population "momentarily", so that the population $p(\theta)$ varies by jump. And $p(\theta_k^{+})$ and $p(\theta_k^{-})$ present the population of the species before and after the impulsive effect at the time θ_k.

Consider a set of all ld continuous function $\mathcal{C}(\mathcal{J}_T,\mathbb{R})$ from $\mathcal{J}_T$ to $\mathbb{R}$. Set $\mathcal{J}_0 = [0,\theta_1]$, and $\mathcal{J}_k = [\theta_k,\theta_{k+1}]$. for each $k = 1,2,3,\cdots,m$.
Consider

$$\mathcal{PC}(\mathcal{J}_T,\mathbb{R}) = \{p : \mathcal{J}_k \to \mathbb{R}, p \in \mathcal{C}(\mathcal{J}_T,\mathcal{R}), \text{ and } p(\theta_k^{+}) \text{ and } p(\theta_k^{-}) \text{ exist with } p(\theta_k^{-}) = p(\theta_k), k = 1,2,3,\cdots,m\},$$

and

$$\mathcal{PC}^1(\mathcal{J}_T,\mathbb{R}) = \{p : \mathcal{J}_k \to \mathbb{R}, p \in \mathcal{C}^1(\mathcal{J}_T,\mathbb{R}), k = 1,2,3,\cdots,m\},$$

where $\mathcal{PC}^1(\mathcal{J}_T,\mathbb{R})$ is the set of all function from $\mathcal{J}_k$ to $\mathbb{R}$. which is ld continuously nabla differentiable function.

The set $\mathcal{PC}(\mathcal{J}_T,\mathbb{R})$ is a Banach space coupled with the norm $||p||_{\mathcal{PC}} = \sup_{\theta\in\mathcal{J}_T}|p(\theta)|$.

Definition 4.1. *A function $p \in \mathcal{PC}^1(\mathcal{J}_T,\mathbb{R})$ is called a solution of the equation (1.1), if p satisfies the equation (1.1) on $\mathcal{J}_T$ involving the condition $p(\theta_k^{+}) - p(\theta_k^{-}) = \mathscr{I}_k(\theta_k, p(\theta_k^{-}))$ and $p(0) = \phi(T)$.*

Lemma 4.2. *Consider a ld continuous function $\mathscr{H} : \mathcal{J}_T \to \mathbb{R}$. Then the solution of the problem is*

$$\begin{cases} {}^{C}\mathscr{D}^{w}p(\theta) = \mathscr{H}(\theta), & \theta \in \mathcal{J}_T, \theta \neq \theta_k \\ p(\theta_k^{+}) - p(\theta_k^{-}) = \mathscr{I}_k\big(\theta_k, p(\theta_k^{-})\big), & k = 1,2,3,\cdots,m \\ p(0) = \phi(p), \end{cases} \tag{4.1}$$

specified by the integral equation

$$p(\theta)\begin{cases} \phi(p) + \frac{1}{\Gamma(w)}\int_0^{\theta}\big(\theta-\rho(\zeta)\big)^{w-1}\mathscr{H}(\zeta)\nabla\zeta, & \theta \in \mathcal{J}_0 \\ \phi(p) + \frac{1}{\Gamma(w)}\sum_{i=1}^{k}\int_{\theta_{i-1}}^{\theta_i}\big(\theta_i-\rho(\zeta)\big)^{w-1}\mathscr{H}(\zeta)\nabla\zeta + \\ \frac{1}{\Gamma(w)}\int_{\theta_k}^{\theta}\big(\theta-\rho(\zeta)\big)^{w-1}\mathscr{H}(\zeta)\nabla\zeta + \sum_{i=1}^{k}\mathscr{I}_i\big(\theta_i, p(\theta_i^{-})\big), & \theta \in \mathcal{J}_k. \end{cases} \tag{4.2}$$

Proof. If $\theta \in \mathfrak{J}_0$, then the solution of the equation (4.1) is given by

$$p(\theta) = \phi(p) + \frac{1}{\Gamma(w)} \int_0^\theta \big(\theta - \rho(\zeta)\big)^{w-1} \mathscr{H}(\zeta)\nabla\zeta. \tag{4.3}$$

For $\theta \in \mathfrak{J}_1$, the problem

$$\begin{cases} {}^C\mathscr{D}^w p(\theta) = \mathscr{H}(\theta), \\ p(\theta_1^+) - p(\theta_1^-) = \mathscr{I}_1\big(\theta_1, p(\theta_1^-)\big), \end{cases}$$

hold the solution

$$p(\theta) = p(\theta_1^+) + \frac{1}{\Gamma(w)} \int_{\theta_1}^\theta \big(\theta - \rho(\zeta)\big)^{w-1} \mathscr{H}(\zeta)\nabla\zeta. \tag{4.4}$$

Again,

$$p(\theta_1^+) - p(\theta_1^-) = \mathscr{I}_1\big(\theta_1, p(\theta_1)\big), \tag{4.5}$$

Applying equation (4.5) in equation (4.4) then

$$p(\theta) = p(\theta_1^-) + \mathscr{I}_1\big(\theta_1, p(\theta_1^-) + \frac{1}{\Gamma(w)} \int_{\theta_1}^\theta \big(\theta - \rho(\zeta)\big)^{w-1} \mathscr{H}(\zeta)\nabla\zeta,$$

which follows that

$$\begin{aligned} p(\theta) =& \phi(p) + \mathscr{I}_1\big(\theta_1, p(\theta_1^-)\big) + \frac{1}{\Gamma(w)} \int_{\theta_1}^\theta \big(\theta - \rho(\zeta)\big)^{w-1} \mathscr{H}(\zeta)\nabla\zeta \\ &+ \frac{1}{\Gamma(w)} \int_0^\theta \big(\theta - \rho(\zeta)\big)^{w-1} \mathscr{H}(\zeta)\nabla\zeta, \ \theta \in \mathfrak{J}_1. \end{aligned}$$

Generalizing in this way, by using the principle of mathematical induction, for $\theta \in \mathfrak{J}_k, \ k = 1, 2, 3, \cdots, m$ one can conclude that

$$\begin{aligned} p(\theta) =& \phi(p) + \frac{1}{\Gamma(w)} \int_{\theta_k}^\theta \big(\theta - \rho(\zeta)\big)^{w-1} \mathscr{H}(\zeta)\nabla\zeta + \sum_{i=1}^k \frac{1}{\Gamma(w)} \int_{\theta_{i-1}}^{\theta_i} \big(\theta - \rho(\zeta)\big)^{w-1} \mathscr{H}(\zeta)\nabla\zeta \\ &+ \sum_{i=1}^k \mathscr{I}_i(\theta_i, p(\theta_i), \ k = 1, 2, 3, \cdots, m. \end{aligned}$$

To establish the existence and uniqueness solution of the equation (1.1), we need to assume the following conditions:

$(\mathcal{A}_1)$ The mapping $\mathfrak{L} : \mathfrak{J}_T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a ld continuous and there must have constants $\mathscr{K} > 0$ and $0 < \mathscr{G} < 1$ which satisfies

$$|\mathfrak{L}(\theta, \zeta_1, \zeta_2) - \mathfrak{L}(\theta, \eta_1, \eta_2)| \leq \mathscr{K}|\zeta_1 - \eta_1| + \mathscr{G}|\zeta_2 - \eta_2|, \ \forall \theta \in \mathfrak{J},$$

$\zeta_i, \eta_i \in \mathbb{R}$ for i = 1, 2.

$(\mathcal{A}_2)$ There exist constants $\mathscr{A} > 0, \mathscr{F} > 0$ and $0 < \mathscr{E} < 1$, such that

$$|\mathfrak{L}(\theta, \zeta, \eta)| \leq \mathscr{A} + \mathscr{F}|\zeta| + \mathscr{E}|\eta|, \ \forall \zeta, \eta \in \mathbb{R}.$$

$(\mathcal{A}_3)$ The function $\mathscr{I}_k(\theta, p)$ is continuous for all $k = 1, 2, 3, \cdots, m$ and satisfies the following:

(I) There exist a positive constant $\mathscr{M}_k$ for $k = 1, 2, 3, \cdots, m$ such that

$$|\mathscr{I}_k(\theta, p)| \leq \mathscr{M}_k, \forall \theta \in \mathfrak{J}_k, p \in \mathbb{R}.$$

(II) There exists a positive constants $\mathscr{L}_k$, for $k = 1, 2, 3, \cdots, m$ such that

$$|\mathscr{I}_k(\theta, p) - \mathscr{I}_k(\theta, h)| \leq \mathscr{L}_k|p - h|, \ \forall \theta \in \mathfrak{J}_k, \ p, h \in \mathbb{R}.$$

$(\mathcal{A}_4)$ There must have a non negative increasing function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|\phi(\theta)| \leq \mu(|\theta|) \text{ for every } \theta \in \mathfrak{J}_T,$$

and a positive constant $\mathscr{H}$ such that

$$|\phi(\theta) - \phi(\zeta)| \leq \mathscr{H}|\theta - \zeta| \text{ for } \theta, \zeta \in \mathfrak{J}_T.$$

The next theorem is based on the Banach contraction theorem [24]. □

Theorem 4.3. *If all the conditions* $(\mathcal{A}_1)$ - $(\mathcal{A}_4)$ *and*

$$\sum_{i=1}^m \mathscr{L}_i + \mathscr{H} + \frac{\mathscr{K}T^w(m+1)}{(1-\mathscr{G})\Gamma(w+1)} < 1 \text{ are hold,}$$

then equation (1.1) must contain a solution on the interval $\mathfrak{J}_T$.

Proof. Let ${}^C\mathcal{D}^w p(\theta) = h(\theta)$. Consider a set $\Pi \subseteq \mathcal{PC}(\mathfrak{J}_k, \mathbb{R})$, such that

$$\Pi = \{p \in \mathcal{PC}^1(\mathfrak{J}_k, \mathbb{R}) : ||p||_{\mathcal{PC}} \leq \sigma\},$$

and an operator $\mathcal{X} : \Pi \to \Pi$ such that

$$(\mathcal{X}p)(\theta) = \phi(p) + \frac{1}{\Gamma(w)} \int_0^\theta \big(\theta - \rho(\zeta)\big)^{w-1} \mathcal{L}(\theta, p(\theta), {}^C\mathscr{D}^w p(\theta)) \nabla\zeta,$$

for $\theta \in \mathfrak{J}_0$. And

$$\begin{aligned}(\mathcal{X}p)(\theta) =&\phi(p) + \frac{1}{\Gamma(w)} \sum_{i=1}^k \int_{\theta_{i-1}}^{\theta_i} \big(\theta - \rho(\zeta)\big)^{w-1} \mathcal{L}(\theta, p(\theta), h(\theta)) \nabla\zeta + \sum_{i=1}^k \mathscr{I}_i(\theta_i, p(\theta_i^-))\\ &+ \frac{1}{\Gamma(w)} \int_{\theta_k}^\theta \big(\theta - \rho(\zeta)\big)^{w-1} \mathcal{L}(\theta, p(\theta), {}^C\mathscr{D}^w p(\theta)) \nabla\zeta,\end{aligned}$$

for $\theta \in \mathfrak{J}_k,\ k = 1, 2, 3, ..., m$.
Case 1: When $\theta \in \mathfrak{J}_k$, then for any $p \in \Pi$, we get

$$\begin{aligned}|(\mathcal{X}p)(\theta)| =&|\phi(p)| + \Big|\frac{1}{\Gamma(w)} \sum_{i=1}^k \int_{\theta_{i-1}}^{\theta_i} \big(\theta - \rho(\zeta)\big)^{w-1} h(\zeta) \nabla\zeta\Big|\\ &+ \Big|\sum_{i=1}^k \mathscr{I}_k(\theta_i, p(\theta_i^-))\Big| + \Big|\frac{1}{\Gamma(w)} \int_{\theta_k}^\theta \big(\theta - \rho(\zeta)\big)^{w-1} h(\zeta) \nabla\zeta\Big|,\end{aligned}$$

where $h \in \Pi,\ \theta \in \mathfrak{J}_T$, then from the Equation (1.1) we get $h = \mathcal{L}(\theta, p, h)$. Hence

$$\begin{aligned}|h| =&|\mathcal{L}(\theta, p, h)|\\ &\leq \mathscr{A} + \mathscr{F}|p(\theta)| + \mathscr{E}|h(\theta)|\\ &\leq \frac{\mathscr{A} + \mathscr{F}\sigma}{1 - \mathscr{E}}.\end{aligned} \tag{4.6}$$

Again taking the norm of $\mathcal{PC}(\mathfrak{J}_T, \mathbb{R})$, in (4.6) then

$$||h||_{\mathcal{PC}} \leq \frac{\alpha + \mathscr{F}\sigma}{1 - \mathscr{E}}, \text{ where } ||\mathscr{A}||_{\mathcal{PC}} = \alpha.$$

Using the proposition 2.4, along with the condition of Case 1, we get

$$\begin{aligned}||\mathcal{X}p||_{\mathcal{PC}} =& \sup_{\theta \in \mathfrak{J}_T} |\mathcal{X}p(\theta)|\\ &\leq \mu|p| + \sum_{i=1}^m \mathscr{M}_i + \frac{\mathscr{A} + \mathscr{F}|p|}{(1 - \mathscr{E})\Gamma(w)} \Big[\sum_{i=1}^m \int_{\theta_{i-1}}^{\theta_i} (\theta - \zeta)^{w-1} d\zeta + \int_{\theta_k}^\theta (\theta - \zeta)^{w-1} d\zeta\Big]\\ &\leq \mu\sigma + \sum_{i=1}^m \mathscr{M}_i + \frac{T^w(\alpha + \mathscr{F}\sigma)(m+1)}{\Gamma(w+1)(1 - \mathscr{E})}\\ &\leq \sigma,\end{aligned} \tag{4.7}$$

where

$$\sigma = \frac{\sum_{i=1}^{m} \mathscr{M}_i + \frac{(m+1)T^w\alpha}{\Gamma(w+1)(1-\mathscr{E})}}{1-\mu+\frac{(m+1)T^w\mathscr{F}}{\Gamma(w+1)(1-\mathscr{E})}}.$$

Case 2: If $\theta \in \mathcal{J}_0$, by the similar way one can obtain

$$\begin{aligned} ||\mathcal{X}p||_{\mathcal{PC}} &\leq \mu\sigma + \frac{T^w\big(\alpha + \mathscr{F}\sigma\big)}{(1-\mathscr{E})\Gamma(w+1)} \\ &\leq \sigma. \end{aligned} \tag{4.8}$$

Thus from (4.8), $||\mathcal{X}p||_{\mathcal{PC}} \leq \sigma$. Hence $\mathcal{X}(\Pi)$ is bounded. Again for $p, q \in \Pi$

$$\begin{aligned} &||\mathcal{X}p - \mathcal{X}q||_{\mathcal{PC}} \\ &= \sup_{\theta\in\mathcal{J}_k} |(\mathcal{X}p)(\theta) - (\mathcal{X}q)(\theta)| \\ &\leq \sum_{i=1}^{k} \big|\mathscr{I}_i(\theta_i, p(\theta_i^-)) - \mathscr{I}_i(\theta_i, q(\theta_i^-))\big| + \frac{1}{\Gamma(w)}\Big|\int_{\theta_k}^{\theta} \big(\theta-\rho(\zeta)\big)^{w-1}(h(\zeta)-g(\zeta))\nabla\zeta\Big| \\ &+ \frac{1}{\Gamma(w)}\Big|\sum_{i=1}^{k}\int_{\theta_{i-1}}^{\theta_i} \big(\theta-\rho(\zeta)\big)^{w-1}(h(\zeta)-g(\zeta))\nabla\zeta\Big| + |\phi(p)-\phi(q)|, \end{aligned} \tag{4.9}$$

where $g \in \Pi$, then $g(\theta) = \mathcal{L}(\theta, q(\theta), g(\theta))$, and for $\theta \in \mathcal{J}_T$, one can get

$$\begin{aligned} |h(\theta) - g(\theta)| =& \big|\mathcal{L}(\theta, p(\theta), h(\theta)) - \mathcal{L}(\theta, q(\theta), g(\theta))\big| \\ \leq& \mathscr{K}|p(\theta) - q(\theta)| + \mathscr{G}|h(\theta) - g(\theta)| \\ \leq& \frac{\mathscr{K}|p(\theta) - q(\theta)|}{1-\mathscr{G}}. \end{aligned} \tag{4.10}$$

Taking the norm of $\mathcal{PC}(\mathcal{J}_T, \mathbb{R})$, then (4.10) become

$$||h - g||_{\mathcal{PC}} \leq \frac{\mathscr{K}||p-q||_{\mathcal{PC}}}{1-\mathscr{G}}. \tag{4.11}$$

Using (4.11) in (4.9), and applying the Proposition 2.4 then

$$\begin{aligned} ||\mathcal{X}p - \mathcal{X}q||_{\mathcal{PC}} \leq& \sum_{i=1}^{m} \mathscr{L}_i|p(\theta_i^-) - q(\theta_i^-)| + \frac{\mathscr{K}|p(\zeta)-q(\zeta)|}{(1-\mathscr{G})\Gamma(w)}\int_{\theta_k}^{\theta}(\theta-\zeta)^{w-1}d\zeta \\ &+ \frac{\mathscr{K}|p(\zeta)-q(\zeta)|}{(1-\mathscr{G})\Gamma(w)}\sum_{i=1}^{m}\int_{\theta_{i-1}}^{\theta_i}(\theta-\zeta)^{w-1}d\zeta + \mathscr{H}|p-q| \\ &\leq ||p-q||_{\mathcal{PC}}\sum_{i=1}^{m}\mathscr{L}_i + \frac{\mathscr{K}T^w||p-q||_{\mathcal{PC}}}{(1-\mathscr{G})\Gamma(w+1)} + \frac{m\mathscr{K}T^w||p-q||_{\mathcal{PC}}}{(1-\mathscr{G})\Gamma(w+1)} \\ &+ \mathscr{H}||p-q||_{\mathcal{PC}} \\ &\leq \Big(\sum_{i=1}^{m}\mathscr{L}_i + \frac{\mathscr{K}T^w(m+1)}{(1-\mathscr{G})\Gamma(w+1)} + \mathscr{H}\Big)||p-q|_{\mathcal{PC}}. \end{aligned} \tag{4.12}$$

Similarly for $\theta \in \mathcal{J}_0$

$$||\mathcal{X}p - \mathcal{X}q||_{\mathcal{PC}} \leq \Big(\mathscr{H} + \frac{\mathscr{K}T^w}{(1-\mathscr{G})\Gamma(w+1)}\Big)||p-q||_{\mathcal{PC}}. \tag{4.13}$$

Thus from (4.12) and (4.13), we obtain

$$||\mathfrak{X}p - \mathfrak{X}q||_{\mathcal{PC}} \leq \mathscr{U}||p-q||_{\mathcal{PC}},$$

where $\mathscr{U} = \sum_{i=1}^{m}\mathscr{L}_i + \frac{\mathscr{K}T^w(m+1)}{(1-\mathscr{G})\Gamma(w+1)} + \mathscr{H}$. Since $\mathscr{U} < 1$, so the operator $\mathfrak{X}:\Pi \to \Pi$ is a contraction operator hence it has a fixed point by Banach contraction theorem, which is the solution of the equation (1.1).

The sufficient condition of the existence of solution of the equation (1.1) is based on the nonlinear alternatives Leray-Schauder's type fixed point theorem (Deinition 2.12). □

Theorem 4.4. *If the assumptions $(\mathcal{A}_1)-(\mathcal{A}_4)$ are satisfied and there exists a positive constant β such that*

$$\mu\beta + \sum_{i=1}^{m}\mathscr{M}_i + \frac{(m+1)T^w(\mathscr{A}+\mathscr{F}\beta)}{\Gamma(w+1)(1-\mathscr{E})} < \beta, \tag{4.14}$$

then Equation (1.1) has at least one solution in $\mathfrak{I}_T$.

Proof. The proof of the theorem is presented in the following steps:
Step 1: The operator $\mathfrak{X}:\Pi \to \Pi$ is continuous
Let $\{p_n\}$ be a sequence of Π such that $p_n \to p$, then for each $\theta \in \mathfrak{I}_k$, $k = 1,2,3,\cdots,m$

$$\begin{aligned}
&||\mathfrak{X}p_n - \mathfrak{X}q||_{\mathcal{PC}}\\
&= \sup_{\theta\in\mathfrak{I}_k}|(\mathfrak{X}p_n)(\theta) - (\mathfrak{X}q)(\theta)|\\
&\leq \sum_{i=1}^{m}\Big|\mathscr{I}_i(\theta_i, p_n(\theta_i^-)) - \mathscr{I}_i(\theta_i, p(\theta_i^-))\Big| + \Big|\frac{1}{\Gamma(w)}\int_{\theta_k}^{\theta}(\theta-\zeta)^{w-1}(h_n(\zeta)-h(\zeta))d\zeta\Big|\\
&+ \frac{1}{\Gamma(w)}\Big|\sum_{i=1}^{m}\int_{\theta_{i-1}}^{\theta_i}(\theta_i-\zeta)^{w-1}(h_n(\zeta)-h(\zeta)d\zeta\Big| + |\phi(p_n)-\phi(p)|,
\end{aligned}\tag{4.15}$$

where $h_n \in \Pi$, such that $h_n = \mathcal{L}(\theta, p_n, h_n)$ then for $\theta \in \mathfrak{I}_k$ one can get

$$\begin{aligned}
|h_n - h| &= |\mathcal{L}(\theta,p_n,h_n) - \mathcal{L}(\theta,p,h)|\\
&\leq \mathscr{K}|p_n - p| + \mathscr{G}|h_n - h|\\
&\leq \frac{\mathscr{K}|p_n-p|}{1-\mathscr{G}}.
\end{aligned}\tag{4.16}$$

Taking the norm of $\mathcal{PC}(\mathfrak{I}_T,\mathbb{R})$, in (4.16) then

$$||h_n - h||_{\mathcal{PC}} \leq \frac{\mathscr{K}}{1-\mathscr{G}}||p_n - p||_{\mathcal{PC}}. \tag{4.17}$$

Using (4.17) in (4.15), then we obtain

$$||\mathfrak{X}p_n - \mathfrak{X}q||_{\mathcal{PC}} \leq ||p_n-p||_{\mathcal{PC}}\Big(\sum_{i=1}^{m}\mathscr{L}_i + \frac{\mathscr{K}T^w(m+1)}{(1-\mathscr{G})\Gamma(w+1)} + \mathscr{H}\Big).$$

If $p_n \to p$ as $n \to \infty$, then $||\mathfrak{X}p_n - \mathfrak{X}q||_{\mathcal{PC}} \to 0$. Hence the operator is continuous.
For $\theta \in \mathfrak{I}_0$, the proof is similiar.
Step 2: The operator $\mathfrak{X}$ map Π to equicontinous set of $\mathcal{PC}(\mathfrak{I}_T,\mathbb{R})$

Let $x_1, x_2 \in \mathfrak{J}_k, k = 1, 2, 3, \cdots, m$, such that $x_1 < x_2$ then

$$\begin{aligned}
&||\mathfrak{X}p(x_2) - \mathfrak{X}p(x_1)||_{\mathcal{PC}} \\
&= \sup_{\theta \in \mathfrak{J}_k} \big|\mathfrak{X}p(x_2) - \mathfrak{X}q(x_1)\big| \\
&\leq \Big|\frac{1}{\Gamma(w)} \int_{\theta_k}^{x_1} \big((x_2 - \rho(\zeta))^{w-1} - (x_1 - \rho(\zeta)^{w-1})\big)h(\zeta)\nabla\zeta\Big| \\
&+ \Big|\frac{1}{\Gamma(w)} \int_{x_1}^{x_2} (x_2 - \rho(\zeta))^{w-1} h(\zeta)\nabla\zeta\Big| + \sum_{0<\theta_k<x_2-x_1} \big|\mathscr{I}_{\theta_k}(\theta_k, p(\theta_k^-))\big| \\
< &\Big|\frac{1}{\Gamma(w)} \int_{\theta_k}^{x_1} \big((x_2 - \zeta)^{w-1} - (x_1 - \zeta)^{w-1}\big)h(\zeta)d\zeta\Big| + \Big|\frac{1}{\Gamma(w)} \int_{x_1}^{x_2} (x_2 - \zeta)^{w-1} h(\zeta)d\zeta\Big| \\
&+ \sum_{0<\theta_k<x_2-x_1} \big|\mathscr{I}_{\theta_k}(\theta_k, p(\theta_k^-))\big| \\
&\leq \frac{\mathscr{A} + \mathscr{F}\sigma}{(1-\mathscr{E})\Gamma(w)} \Big(\Big| \int_{\theta_k}^{x_1} \big((x_2 - \zeta)^{w-1} - (x_1 - \zeta)^{w-1}\big)d\zeta\Big| + \Big|\frac{1}{\Gamma(w)} \int_{x_1}^{x_2} (x_2 - \zeta)^{w-1} d\zeta\Big|\Big) \\
&+ \sum_{0<\theta_k<x_2-x_1} \big|\mathscr{I}_{\theta_k}(\theta_k, p(\theta_k^-))\big|
\end{aligned}$$

Since $(x - \zeta)^{w-1}$ is continuous and if $x_1 \to x_2$, then $||\mathfrak{X}p(x_2) - \mathfrak{X}p(x_1)||_{\mathcal{PC}} \to 0$. Thus the operator $\mathfrak{X}$ is equicontinuous in $\mathfrak{J}_k$. Since the result at $x_1, x_2 \in \mathfrak{J}_0$ is similar, so the proof is omitted.

Step 3: The operators $\mathfrak{X}$ map Π to bounded set of $\mathcal{PC}(\mathfrak{J}_T, \mathbb{R})$.

From (4.7) it is clear that $||\mathfrak{X}(p)|| \leq \sigma$ for $\sigma \in \mathbb{R}$. As a consequences of the Step 1 to Step 3 together with the theorem of Arzela-Ascoli, we arrived that the mapping $\mathfrak{X}$ is continuous completely.

Step 4: For any $\lambda \in (0, 1)$, the set $\mathcal{K} = \{p \in \mathcal{PC}(\mathfrak{J}_k, \mathbb{R}) : p = \lambda\mathfrak{X}(p), 0 < \lambda < 1\}$ is bounded, for $\theta \in \mathfrak{J}_k, k = 1, 2, 3, \cdots, m$, we have

$$\begin{aligned}
|p(\theta)| &= |\lambda\mathfrak{X}(p)\theta| \\
&= \Big|\lambda\Big(\phi(p) + \frac{1}{\Gamma(w)} \sum_{i=1}^{k} \int_{\theta_{i-1}}^{\theta_i} \big(\theta_i - \rho(\zeta)\big)^{w-1} h(\zeta)\nabla\zeta \\
&+ \frac{1}{\Gamma(w)} \int_{\theta_k}^{\theta} \big(\theta - \rho(\zeta)\big)^{w-1} h(\zeta)\nabla\zeta + \sum_{i=1}^{k} \mathscr{I}_i(\theta_i, p(\theta_i^-))\Big)\Big| \\
&\leq \mu||p||_{\mathcal{PC}} + \sum_{i=1}^{n} \mathscr{M}_i + \frac{(\mathscr{A} + \mathscr{F}||p||_{\mathcal{PC}})T^w(m+1)}{\Gamma(w+1)(1-\mathscr{E})}.
\end{aligned}$$

Thus,

$$\frac{||p||_{\mathcal{PC}}}{\mu||p||_{\mathcal{PC}} + \sum_{i=1}^{n} \mathscr{M}_i + \frac{(\mathscr{A}+\mathscr{F}||p||_{\mathcal{PC}})T^w(m+1)}{(1-\mathscr{E})\Gamma(w+1)}} \leq 1.$$

From the equation (4.14), we get a positive constant β such that $||p||_{\mathcal{PC}} \neq \beta$. Consider a set $\Psi = \{p \in \mathcal{PC}(\mathfrak{J}_T, \mathbb{R}) : ||p||_{\mathcal{PC}} < \beta\}$. Then the operator $\mathfrak{X} : \overline{\Psi} \to \mathcal{PC}(\mathfrak{J}_T, \mathbb{R})$ is continuous and completely continuous. So there is no $p \in \partial(\Psi)$ such that $p = \lambda\mathfrak{X}(p),\ \lambda \in (0, 1)$. Hence, the nonlinear alternatives of Leray-Schauder's type fixed point theorem gives that the operator $\mathfrak{X}$ has a fixed point, which is the solution of the Equation (1.1).

The result for $\theta \in \mathfrak{J}_0$ is almost same, so it is omitted. □

5. Example

Example 5.1. *Consider a impulsive fractional dynamic equation involving a non-local initial condition on the time scale* $\mathbb{T} = [0, \frac{1}{3}] \cup [\frac{1}{2}, 1]$.

$$\begin{cases} {}^{C}\mathfrak{D}^{\frac{1}{2}} p(\theta) = \frac{e^{-3\theta}\left(2+|p(\theta)|+|{}^{C}\mathfrak{D}^{w} p(\theta)|\right)}{35e^{2\theta}\left(1+|p(\theta)|\right)}, & \theta = [0,1] \cap T, \theta \neq \frac{1}{3} \\ p(\frac{1}{3}^{+}) - p(\frac{1}{3}^{-}) = \frac{1+p(\frac{1}{3})}{10}, & \theta_1 = \frac{1}{3} \\ p(0) = \frac{p}{5}. \end{cases} \tag{5.1}$$

We set

$$\mathcal{L}(\theta, p, q) = \frac{e^{-3\theta}\big(2 + |p(\theta)| + |q(\theta)|\big)}{35e^{2\theta}\big(1 + |p(\theta)|\big)} \tag{5.2}$$

Clearly, the right side of (5.2) is continuous for $p, q \in \mathbb{R}$ *in the context of time scale. Again for all* $\theta \in [0,1] \cap \mathbb{T}$ *and* $h, g \in \mathbb{R}$, *we get*

$$\begin{aligned} |\mathcal{L}(\theta, p, q)| \leq & \frac{2 + |p(\theta)| + |q(\theta)|}{35e^2} \\ & \leq \frac{2}{35e^2} + \frac{1}{35e^2}|p(\theta)| + \frac{1}{35e^2}|q(\theta)|. \end{aligned}$$

Then we get, $\mathscr{A} = \frac{2}{35e^2}$, $\mathscr{F} = \frac{1}{35e^2}$, $\mathscr{E} = \frac{1}{35e^2}$. *Next*

$$|\mathcal{L}(\theta, p, q) - \mathcal{L}(\theta, h, g)| \leq \frac{1}{35e^2}|p - h| + \frac{1}{35e^2}|q - g|,$$

$$|\mathscr{I}_1(\theta, p) - \mathscr{I}_1(\theta, q)| \leq \frac{1}{10}|p - h|, \ |\phi(p) - \phi(h)| \leq \frac{1}{5}|p - h|, \ |\phi(p)| \leq \frac{1}{5}.$$

From here we get $\mathscr{K} = \frac{1}{35e^2}$, $\mathscr{G} = \frac{1}{35e^2}$, $\mathscr{L}_1 = \frac{1}{10}$, $\mathscr{H} = \frac{1}{5}$. *Thus, from the above data we can say that the equation (5.1) satisfies all the conditions of* $\mathcal{A}_1 - \mathcal{A}_4$.

Again, for $m = 1$ *we get*

$$\begin{aligned} \mathscr{L}_1 + \frac{\mathscr{K} T^w (p+1)}{(1 - \mathscr{L})\Gamma(w+1)} + \mathscr{H} \leq & \frac{1}{10} + \frac{1}{5} + \frac{2\frac{1}{35e^2}}{(1 - \frac{1}{35e^2})\Gamma(\frac{1}{2} + 1)} \\ \leq & \frac{3}{10} + \frac{2\frac{1}{35e^2}}{(1 - \frac{1}{35e^2})\Gamma(\frac{1}{2} + 1)} \\ < & 1. \end{aligned}$$

Thus, the conditions of the Theorem 4.3 are satisfied. Therefore we arrived in the conclusion that the equation (5.1) has a unique solution.

6. Conclusion

In this manuscript we discussed the fractional dynamic equation by the Caputo nabla derivative involving instantaneous impulses with non local initial condition, and the comparision of Caputo nabla derivative and Riemann-Liouville nabla derivative have been discussed in the context of time scale. Later we have given an example on the basis of all theoritical results on existence and uniqueness of the solution in the time scale interval $[0, 1] \cap \mathbb{T}$, where $\mathbb{T} = [0, \frac{1}{3}] \cup [\frac{1}{2}, 1]$. The stability analysis of solution of the equation (1.1) with different types of initial and boundarey conditions are our future work. The theory of impulsive fractional dynamic equation has a potential application on the filed of mathematical analysis, moreover it has a wide application in physics, field of engineering, and economics.

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