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Stability Analysis of Nonlinear Riemann-Liouville Fractional Differential Equations

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ABSTRACT: In this paper, we give sufficient conditions to guarantee the asymptotic stability of the zero solution to a kind of nonlinear fractional differential equations with the Riemann Liouville fractional derivative of order $\alpha \in (n-1,n)$ by using Krasnoselskii's fixed point theorem and the Banach contraction mapping principle in a weighted Banach space. The results obtained here extend the work of Li and Kou [6].

Key Words: Fractional differential equations, Riemann-Liouville fractional derivatives, stability, fixed point theorem.

Contents

1. Introduction

Study of fractional differential equations appears from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]- [8], [10], [11] and the references therein.

In [6], the authors have used the Schauder fixed point theorem and the Banach contraction mapping principle for study the stability of solutions for nonlinear FDEs with the Riemann-Liouville fractional derivative of order $\alpha \in (n-1,n)$

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f(t, x(t)), \ t \geq 0, \\ D_{0^+}^{\alpha-k} x(0^+) = b_k, \ k = 1, 2, ..., n. \end{cases}$$

Inspired and motivated by the above work, we concentrate on the stability of solutions for nonlinear FDEs with the Riemann-Liouville fractional derivative of order $\alpha \in (n-1,n)$

$$\begin{cases} D_{0^+}^{\alpha} (x(t) - g(t, x(t))) = f(t, x(t)), \ t \geq 0, \\ D_{0^+}^{\alpha-k} (x(t) - g(t, x(t)))|_{0^+} = b_k, \ k = 1, 2, ..., n, \end{cases} \tag{1.1}$$

where $f, g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $f(t,0) = g(t,0) = 0$. We first establish the equivalence between the IVPs for nonlinear FDEs and the Volterra integral equation on an infinite interval. Then, we introduce a special weighted Banach space and utilize the Krasnoselskii fixed point theorem and the Banach contraction mapping principle to investigate the stability to solutions of IVPs (1.1). The results obtained here extend the work of Li and Kou [6].

This paper is organized as follows. In Section 2, we present some preliminaries needed in later sections. In Section 3, we give and prove our main results on stability. In Section 4, we give an example to illustrate our main results.

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2. Preliminaries

In this section, we introduce several elementary definitions and lemmas to be applied throughout the paper. Let $\mathbb{R}^+ = [0, +\infty)$ and $C_{n-\alpha}(\mathbb{R}^+) = \{x : \mathbb{R}^+ \to \mathbb{R},\ t^{n-\alpha}x \in C(\mathbb{R}^+)\}$. For any $b > 0$, we denote $C_{n-\alpha}([0,b]) = \{x : [0,b] \to \mathbb{R},\ t^{n-\alpha}x \in C([0,b])\}$, $C_{n-\alpha}([0,b])$ is a Banach space equipped with the norm

$$\|x\|_{C_{n-\alpha}([0,b])} = \left\|t^{n-\alpha}x\right\|_{C([0,b])} = \max_{t\in[0,b]} \left|t^{n-\alpha}x(t)\right|.$$

Definition 2.1 ([3]). *The left-side fractional integral of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$I_{0^+}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\,ds,$$

provided the right-side is pointwise defined on $\mathbb{R}^+$.

Definition 2.2 ([3]). *The left-side Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$D_{0^+}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)\,x(s)\,ds,\ \ n-1 < \alpha < n,$$

provided the right-side is pointwise defined on $\mathbb{R}^+$.

In [3] and [4], the equivalence between the IVPs for FDEs (1.1) and the Volterra integral equation on a finite interval has been proven in detail. Similarly, we have the following lemma.

Lemma 2.3. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(.,x) \in C_{n-\alpha}(\mathbb{R}^+)$ for any $x \in C_{n-\alpha}(\mathbb{R}^+)$. If $x \in C_{n-\alpha}(\mathbb{R}^+)$, then x is a solution of (1.1) if and only if it satisfies the Volterra integral equation*

$$\begin{aligned} x(t) &= \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)}t^{\alpha-k} + g(t, x(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s))\,ds,\ \ t \geq 0. \end{aligned} \tag{2.1}$$

Lemma 2.4 ([6]). *Let*

$$E = \left\{x \in C_{n-\alpha}\left(\mathbb{R}^+\right),\ \lim_{t\to\infty}\frac{t^{n-\alpha}x(t)}{1+t^{n+1}} = 0\right\},$$

with the norm

$$\|x\| = \sup_{t\geq 0}\frac{t^{n-\alpha}|x(t)|}{1+t^{n+1}}.$$

Then, $(E, \|.\|)$ is a Banach space.

Lemma 2.5 ([4]). *Let Ω be a subset of a Banach space E. Then, Ω is is relatively compact in E if the following conditions are satisfied*

(i) $\left\{t^{n-\alpha}x(t)/\left(1+t^{n+1}\right) : x \in \Omega\right\}$ is uniformly bounded;

(ii) $\left\{t^{n-\alpha}x(t)/\left(1+t^{n+1}\right) : x \in \Omega\right\}$ is equicontinuous in $\mathbb{R}^+$;

(iii) $\left\{t^{n-\alpha}x(t)/\left(1+t^{n+1}\right) : x \in \Omega\right\}$ equiconverges to 0 as $t \to \infty$, i.e., for any given $\varepsilon > 0$, there exists $T > 0$, such that for all $x \in \Omega$ and $t > T$, it holds

$$\left|t^{n-\alpha}x(t)/\left(1+t^{n+1}\right)\right| < \varepsilon.$$

Theorem 2.6 (Banach's fixed point theorem [9]). *Let Ω be a non-empty closed convex subset of a Banach space $(E, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 2.7 (Krasnoselskii [9]). *Let Ω be a non-empty bounded closed convex subset of a Banach space $(E, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map Ω into E such that*

(i) $\mathcal{A}x + \mathcal{B}y \in \Omega$ for all $x, y \in \Omega$,

(ii) $\mathcal{A}$ is continuous and compact,

(iii) $\mathcal{B}$ is a contraction.

Then there is a $x \in \Omega$ with $\mathcal{A}x + \mathcal{B}x = x$.

Definition 2.8. *The trivial solution $x = 0$ of (1.1) is said to be stable in a Banach space E, if for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $t \geq 0$, $\sum_{k=1}^{n} |b_k| \leq \delta$ implies that the solution $x(t) = x(t, b_1, ..., b_k)$ satisfies $\|x\| \leq \varepsilon$.*

3. Main results

This section is devoted to proving the stability of solutions to IVPs for nonlinear FDEs with the Riemann-Liouville fractional derivative.

Theorem 3.1. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(., x), g(., x) \in C_{n-\alpha}(\mathbb{R}^+)$ for any $x \in C_{n-\alpha}(\mathbb{R}^+)$, $f(t, 0) = g(t, 0) = 0$. Assume that*

(H1) There exist two nonnegative continuous functions p and r defined on $\mathbb{R}^+$, such that

$$|f(t, x(t))| \leq p(t)\, r\left(\frac{|x(t)|}{1 + t^{n+1}}\right), \tag{3.1}$$

where p is bounded on $\mathbb{R}^+$ and $r(t) \leq t$.

(H2) There exists nonnegative continuous function q defined on $\mathbb{R}^+$, such that $\|q\|_\infty = \sup_{t\geq 0} |q(t)| < 1$, and

$$|g(t, x(t)) - g(t, y(t))| \leq q(t)\, |x(t) - y(t)|. \tag{3.2}$$

If

$$\|q\|_\infty + \frac{M_1 \|p\|_\infty B(\alpha - n + 1, \alpha)}{\Gamma(\alpha)} < 1,$$

where $M_1 \geq \sup_{t\geq 0} \left|\frac{t^\beta}{1+t^{n+1}}\right|$ when $0 \leq \beta \leq n$ and $\|p\|_\infty = \sup_{t\geq 0} |p(t)|$. Then, the trivial solution $x = 0$ of (1.1) is stable in the Banach space E.

Proof. For any given $\varepsilon > 0$, our aim is to prove the existence of $\delta > 0$ such that $\sum_{k=1}^{n} |b_k| \leq \delta$ implies that $\|x\| \leq \varepsilon$. Let

$$l \geq \frac{1}{\Lambda}\left(\sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha - k + 1)} M_1\right),$$

where

$$\Lambda = 1 - \|q\|_\infty - \frac{M_1 \|p\|_\infty B(\alpha - n + 1, \alpha)}{\Gamma(\alpha)}.$$

Consider the non-empty bounded closed convex subset $\Omega \subset E$ where $\Omega = \{x \in E,\ \|x\| \leq l\}$. We define two mappings $\mathcal{A}$ and $\mathcal{B}$ on Ω as follows

$$(\mathcal{A}x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, x(s))\, ds, \tag{3.3}$$

and

$$(\mathcal{B}x)(t) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha - k + 1)} t^{\alpha - k} + g(t, x(t)). \tag{3.4}$$

Note that

$$\begin{aligned} |g(t, x(t))| &= |g(t, x(t)) - g(t, 0) + g(t, 0)| \\ &\leq |g(t, x(t)) - g(t, 0)| + |g(t, 0)| \\ &\leq q(t)\, |x(t)| \leq \|q\|_\infty |x(t)|. \end{aligned}$$

Therefore

$$\left|\frac{t^{n-\alpha}}{1+t^{n+1}}g\left(t,x\left(t\right)\right)\right|\leq\left\Vert q\right\Vert_{\infty}\left|\frac{t^{n-\alpha}}{1+t^{n+1}}x\left(t\right)\right|. \tag{3.5}$$

Thus

$$\left|\frac{t^{n-\alpha}}{1+t^{n+1}}g\left(t,x\left(t\right)\right)\right|\leq\left\Vert q\right\Vert_{\infty}\left\Vert x\right\Vert. \tag{3.6}$$

Here, we divide the proof into four steps.

Step 1. We first prove that $\mathcal{A}$ and $\mathcal{B}$ maps Ω into E.

What we need to verify $\mathcal{A}x,\mathcal{B}x\in C_{n-\alpha}\left(\mathbb{R}^{+}\right)$ and

$$\lim_{t\to\infty}\frac{t^{n-\alpha}\left(\mathcal{A}x\right)\left(t\right)}{1+t^{n+1}}=\lim_{t\to\infty}\frac{t^{n-\alpha}\left(\mathcal{B}x\right)\left(t\right)}{1+t^{n+1}}=0.$$

Note that

$$\frac{t^{n-\alpha}\left(\mathcal{A}x\right)\left(t\right)}{1+t^{n+1}}=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\frac{t^{n-\alpha}}{1+t^{n+1}}\left(t-s\right)^{\alpha-1}f\left(s,x\left(s\right)\right)ds,$$

and

$$\frac{t^{n-\alpha}\left(\mathcal{B}x\right)\left(t\right)}{1+t^{n+1}}=\sum_{k=1}^{n}\frac{b_{k}}{\Gamma\left(\alpha-k+1\right)}\frac{t^{n-k}}{1+t^{n+1}}+\frac{t^{n-\alpha}g\left(t,x\left(t\right)\right)}{1+t^{n+1}}.$$

First, it is obvious that each term of $\frac{t^{n-\alpha}g(t,x(t))}{1+t^{n+1}}$ and the following formula

$$\sum_{k=1}^{n}\frac{b_{k}}{\Gamma\left(\alpha-k+1\right)}\frac{t^{n-k}}{1+t^{n+1}},$$

are continuous on $\mathbb{R}^{+}$, we only prove

$$\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\frac{t^{n-\alpha}}{1+t^{n+1}}\left(t-s\right)^{\alpha-1}f\left(s,x\left(s\right)\right)ds\in C\left(\mathbb{R}^{+}\right).$$

Note that $t^{n-\alpha}\left(\mathcal{A}x\right)\left(t\right)$ is continuous at $t_{0}=0$. We consider the right-side continuity of

$$\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\frac{t^{n-\alpha}}{1+t^{n+1}}\left(t-s\right)^{\alpha-1}f\left(s,x\left(s\right)\right)ds,$$

on the interval $(0,\infty)$.

Let $t>t_{0}$, for given ε, there exists $\delta_{1}=\min\left\{\delta_{2},2\delta_{3},\delta_{4}\right\}>0$, such that $0<t-t_{0}<\delta_{1}$, our aim is to prove

$$\left|\int_{0}^{t}\frac{t^{n-\alpha}}{1+t^{n+1}}\left(t-s\right)^{\alpha-1}f\left(s,x\left(s\right)\right)ds-\int_{0}^{t_{0}}\frac{t_{0}^{n-\alpha}}{1+t_{0}^{n+1}}\left(t_{0}-s\right)^{\alpha-1}f\left(s,x\left(s\right)\right)ds\right|\leq\varepsilon.$$

We know that

$$\begin{aligned}
&\left|\int_{0}^{t}\frac{t^{n-\alpha}}{1+t^{n+1}}\left(t-s\right)^{\alpha-1}f\left(s,x\left(s\right)\right)ds-\int_{0}^{t_{0}}\frac{t_{0}^{n-\alpha}}{1+t_{0}^{n+1}}\left(t_{0}-s\right)^{\alpha-1}f\left(s,x\left(s\right)\right)ds\right|\\
&\leq\left|\int_{0}^{t_{0}}\left(\frac{t^{n-\alpha}}{1+t^{n+1}}\left(t-s\right)^{\alpha-1}-\frac{t_{0}^{n-\alpha}}{1+t_{0}^{n+1}}\left(t_{0}-s\right)^{\alpha-1}\right)f\left(s,x\left(s\right)\right)ds\right|\\
&+\left|\int_{t_{0}}^{t}\frac{t^{n-\alpha}}{1+t^{n+1}}\left(t-s\right)^{\alpha-1}f\left(s,x\left(s\right)\right)ds\right|\\
&\leq\left\Vert f\left(s,x\left(s\right)\right)\right\Vert_{C_{n-\alpha}([0,t_{0}+\delta_{1}])}\left[\int_{0}^{t_{0}}\left(\frac{t^{n-\alpha}}{1+t^{n+1}}\left(t-s\right)^{\alpha-1}-\frac{t_{0}^{n-\alpha}}{1+t_{0}^{n+1}}\left(t_{0}-s\right)^{\alpha-1}\right)s^{\alpha-n}ds\right.\\
&\left.+\int_{t_{0}}^{t}\frac{t^{n-\alpha}}{1+t^{n+1}}\left(t-s\right)^{\alpha-1}s^{\alpha-n}ds\right].
\end{aligned}$$

We analyze the two terms in brackets, respectively.

Clearly, we have

$$\int_{t_0}^{t} \frac{t^{n-\alpha}}{1+t^{n+1}} (t-s)^{\alpha-1} s^{\alpha-n} ds \le t_0^{\alpha-n} \frac{t^{n-\alpha}(t-t_0)^{\alpha}}{\alpha(1+t^{n+1})}$$

by part integral. Thus, for the given ε, there exists $\delta_2 > 0$, such that for $0 < t - t_0 < \delta_2$, we have

$$\int_{t_0}^{t} \frac{t^{n-\alpha}}{1+t^{n+1}} (t-s)^{\alpha-1} s^{\alpha-n} ds \le \frac{\varepsilon}{2}. \tag{3.7}$$

Now we analyze the rest. It is easy to show

$$\int_{0}^{t} \frac{t^{n-\alpha}}{1+t^{n+1}} (t-s)^{\alpha-1} s^{\alpha-n} ds = \frac{t^{\alpha}}{1+t^{n+1}} B(\alpha-n+1,\alpha). \tag{3.8}$$

Thus, for the given ε, there exists a small enough $\delta_3 > 0$ such that for $0 < t \le \delta_3$, we have

$$\left| \int_{0}^{t} \frac{t^{n-\alpha}}{1+t^{n+1}} (t-s)^{\alpha-1} s^{\alpha-n} ds \right| \le \frac{\varepsilon}{8}. \tag{3.9}$$

Let $h(t) = \frac{t^{n-\alpha}}{1+t^{n+1}} (t-s)^{\alpha-1}$, $t \ge s$, we have

$$\begin{aligned}
h'(t) = & \frac{1}{(1+t^{n+1})^2} \Big\{ \Big[(n-\alpha) t^{n-\alpha-1} (t-s)^{\alpha-1} + (\alpha-1)(t-s)^{\alpha-2} t^{n-\alpha} \Big] (1+t^{n+1}) \\
& - (n+1) t^{n} \Big[t^{n-\alpha} (t-s)^{\alpha-1} \Big] \Big\} \\
& = \frac{t^{n-\alpha-1}(t-s)^{\alpha-2}}{(1+t^{n+1})^2} \big[(n-1)t - (n-\alpha)s + (n-1)t^{n+2} - (n-\alpha)st^{n+1} \\
& - (n+1)t^{n+2} + (n+1)st^{n+1} \big] \\
& = \frac{t^{n-\alpha-1}(t-s)^{\alpha-2}}{(1+t^{n+1})^2} \big\{ \big[(\alpha+1)t^{n+1} - (n-\alpha) \big] s + \big[(n-1) - 2t^{n+1} \big] t \big\} \\
& \ge \frac{t^{n-\alpha-1}(t-s)^{\alpha-2}}{(1+t^{n+1})^2} \big[(\alpha-1)t^{n+1} + (\alpha-1) \big] s \\
& \ge 0,
\end{aligned}$$

through calculation, thus h is a monotonous increasing function on $\mathbb{R}^+$. We divide the interval $[0, t_0]$ of s into the small enough interval $[0, \delta_3]$ and the other interval $[\delta_3, t_0]$, together with (3.9), we get that

$$\begin{aligned}
& \left| \int_{0}^{\delta_3} \left(\frac{t^{n-\alpha}}{1+t^{n+1}} (t-s)^{\alpha-1} - \frac{t_0^{n-\alpha}}{1+t_0^{n+1}} (t_0-s)^{\alpha-1} \right) s^{n-\alpha} ds \right| \\
& \le \left| \int_{0}^{\delta_3} \frac{t^{n-\alpha}}{1+t^{n+1}} (t-s)^{\alpha-1} s^{n-\alpha} ds \right| + \left| \int_{0}^{\delta_3} \frac{t_0^{n-\alpha}}{1+t_0^{n+1}} (t_0-s)^{\alpha-1} s^{n-\alpha} ds \right| \\
& \le 2 \left| \int_{0}^{\delta_3} \frac{\delta_3^{n-\alpha} (\delta_3-s)^{\alpha-1}}{1+\delta^{n+1}} s^{n-\alpha} ds \right| \\
& \le \frac{\varepsilon}{4}.
\end{aligned} \tag{3.10}$$

Meanwhile, we have

$$\begin{aligned}
&\left|\int_{\delta_3}^{t_0}\left(\frac{t^{n-\alpha}}{1+t^{n+1}}(t-s)^{\alpha-1}-\frac{t_0^{n-\alpha}}{1+t_0^{n+1}}(t_0-s)^{\alpha-1}\right)s^{n-\alpha}ds\right| \\
&\le \delta_3^{n-\alpha}\left|\int_{\delta_3}^{t_0}\left(\frac{t^{n-\alpha}}{1+t^{n+1}}(t-s)^{\alpha-1}-\frac{t_0^{n-\alpha}}{1+t_0^{n+1}}(t_0-s)^{\alpha-1}\right)ds\right| \\
&\le \frac{\delta_3^{n-\alpha}}{\alpha}\left(\int_{\delta_3}^{t_0}\left|\frac{t^{n-\alpha}}{1+t^{n+1}}(t-\delta_3)^{\alpha-1}-\frac{t_0^{n-\alpha}}{1+t_0^{n+1}}(t_0-\delta_3)^{\alpha-1}\right|+\left|\frac{t^{n-\alpha}(t-t_0)^{\alpha}}{1+t^{n+1}}\right|\right),
\end{aligned}$$

which implies that there exists δ_4 such that for $0<t-t_0<\delta_4$, we have

$$\int_{\delta_3}^{t_0}\left(\frac{t^{n-\alpha}}{1+t^{n+1}}(t-s)^{\alpha-1}-\frac{t_0^{n-\alpha}}{1+t_0^{n+1}}(t_0-s)^{\alpha-1}\right)s^{n-\alpha}ds\le\frac{\varepsilon}{4}. \tag{3.11}$$

Then, (3.10) together with (3.11) leads to

$$\int_{0}^{t_0}\left(\frac{t^{n-\alpha}}{1+t^{n+1}}(t-s)^{\alpha-1}-\frac{t_0^{n-\alpha}}{1+t_0^{n+1}}(t_0-s)^{\alpha-1}\right)s^{n-\alpha}ds\le\frac{\varepsilon}{2}. \tag{3.12}$$

Therefore, by (3.12) and (3.7), we complete the proof of the right-side continuity of

$$\frac{1}{\Gamma(\alpha)}\int_0^t\frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}f(s,x(s))\,ds,$$

i.e., $t^{n-\alpha}(\mathcal{A}x)(t)$ is right-continuous on $(0,+\infty)$. Similar procedure can be applied on left-continuous discussion. Next, we only prove

$$\lim_{t\to\infty}\frac{t^{n-\alpha}(\mathcal{A}x)(t)}{1+t^{n+1}}=0 \text{ and } \lim_{t\to\infty}\frac{t^{n-\alpha}(\mathcal{B}x)(t)}{1+t^{n+1}}=0.$$

By (3.1), we have

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}f(s,x(s))\,ds\right| \\
&\le\left|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}\left(p(s)\,r\left(\frac{|x(s)|}{1+s^{n+1}}\right)\right)ds\right| \\
&\le\frac{\|p\|_\infty}{\Gamma(\alpha)}\int_0^t\frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}s^{\alpha-n}\frac{s^{n-\alpha}x(s)}{1+s^{n+1}}ds \\
&\le\frac{\|x\|\,\|p\|_\infty}{\Gamma(\alpha)}\int_0^t\frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}s^{\alpha-n}ds.
\end{aligned}$$

By (3.8), we obtain the result

$$\begin{aligned}
\left|\frac{t^{n-\alpha}(\mathcal{A}x)(t)}{1+t^{n+1}}\right| &\le\frac{\|x\|\,\|p\|_\infty B(\alpha-n+1,\alpha)}{\Gamma(\alpha)}\left|\frac{t^\alpha}{1+t^{n+1}}\right| \\
&\le\frac{l\,\|p\|_\infty B(\alpha-n+1,\alpha)}{\Gamma(\alpha)}\left|\frac{t^\alpha}{1+t^{n+1}}\right|\ \to\ 0,
\end{aligned} \tag{3.13}$$

as $t\to\infty$.

By (3.2) and (3.5), we get

$$\begin{aligned}
\left|\frac{t^{n-\alpha}(\mathcal{B}x)(t)}{1+t^{n+1}}\right| &\le \left|\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}\frac{t^{n-k}}{1+t^{n+1}}\right|+\left|\frac{t^{n-\alpha}g(t,x(t))}{1+t^{n+1}}\right| \\
&\le \left|\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}\frac{t^{n-k}}{1+t^{n+1}}\right|+\|q\|_\infty\left|\frac{t^{n-\alpha}}{1+t^{n+1}}x(t)\right|\ \to\ 0,
\end{aligned} \tag{3.14}$$

as $t \to \infty$.

Step 2. We prove that $\mathcal{A}x + \mathcal{B}y \in \Omega$ for all $x, y \in \Omega$.

It is easy to get $\sup_{t\geq 0}\left|\frac{t^{\beta}}{1+t^{n+1}}\right| \leq M_1$ when $0 \leq \beta \leq n$, and with (3.13) and (3.14) we have

$$\begin{aligned}
&\left|\frac{t^{n-\alpha}(\mathcal{A}x)(t)}{1+t^{n+1}} + \frac{t^{n-\alpha}(\mathcal{B}y)(t)}{1+t^{n+1}}\right| \\
&\leq M_1 \frac{\|x\|\,\|p\|_{\infty}}{\Gamma(\alpha)} B(\alpha - n + 1, \alpha) + \sup_{t\geq 0}\left|\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}\frac{t^{n-k}}{1+t^{n+1}}\right| + \|q\|_{\infty}\|x\| \\
&\leq \frac{M_1 l\,\|p\|_{\infty} B(\alpha-n+1,\alpha)}{\Gamma(\alpha)} + \sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}M_1 + \|q\|_{\infty}\,l.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\mathcal{A}x + \mathcal{B}y\| \\
&\leq \sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}M_1 + \left(\|q\|_{\infty} + \frac{M_1\|p\|_{\infty}B(\alpha-n+1,\alpha)}{\Gamma(\alpha)}\right) l \\
&\leq l.
\end{aligned}$$

Hence $\mathcal{A}x + \mathcal{B}y \in \Omega$ for all $x, y \in \Omega$.

Step 3. We show that $\mathcal{A}$ is continuous.

According to the definition of continuity for a mapping, let $\{x_k\} \subset E$, for the given ε, there exists $N > 0$ such that for any $k \geq N$ implies $\|x_k - x\| \leq \varepsilon$, our aim is to prove $\|\mathcal{A}x_k - \mathcal{A}x\| \leq \varepsilon$.

In view of $\lim_{k\to\infty}\|x_k - x\| = 0$, there exists $K > 0$ such that $\|x_k\| \leq K$ $(k = 1, 2, \ldots)$ and $\|x\| \leq K$. By virtue of (3.8) and (3.1), for any k, we have

$$\begin{aligned}
&\|\mathcal{A}x_k - \mathcal{A}x\| \\
&= \sup_{t\geq 0}\left|\frac{t^{n-\alpha}(\mathcal{A}x_k)(t)}{1+t^{n+1}} - \frac{t^{n-\alpha}(\mathcal{A}x)(t)}{1+t^{n+1}}\right| \\
&= \sup_{t\geq 0}\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}\left[f(s, x_k(s)) - f(s, x(s))\right] ds\right| \\
&\leq \frac{2K\|p\|_{\infty}}{\Gamma(\alpha)} B(\alpha-n+1,\alpha)\left|\frac{t^{\alpha}}{1+t^{n+1}}\right|. \qquad (3.15)
\end{aligned}$$

From the above formula, because the fact (3.15) tends to zero at infinity and continuous dependence at $t = 0$. Thus, for the given ε, there exists $T > 0$ and $\delta_5 > 0$, such that

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}\left[f(s, x_k(s)) - f(s, x(s))\right] ds\right| < \varepsilon,\ \ t \geq T, \qquad (3.16)$$

and

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}\left[f(s, x_k(s)) - f(s, x(s))\right] ds\right| < \varepsilon,\ \ 0 \leq t \leq \delta_5. \qquad (3.17)$$

Now we analyze the case when $t \in [\delta_5, T]$. For the above $k \geq N$,

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}\left[f(s, x_k(s)) - f(s, x(s))\right] ds\right| \\
&\leq \sup_{s\in[0,T]}\left|f(s, x_k(s)) - f(s, x(s))\right| \frac{1}{\Gamma(\alpha)} \sup_{t\in[\delta_5,T]}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}} ds \\
&\leq \frac{1}{\alpha\Gamma(\alpha)} \sup_{s\in[0,T]}\left|f(s, x_k(s)) - f(s, x(s))\right|.
\end{aligned}$$

In view of $\lim_{k\to\infty}\|x_k - x\| = 0$, we know $\lim_{k\to\infty}\sup_{s\in[0,T]}|x_k(s) - x(s)| = 0$ and f is uniformly continuous on any compact subsets. Thus, we have

$$\sup_{s\in[0,T]}|f(s,x_k(s)) - f(s,x(s))| \to 0,\ k\to\infty.$$

Therefore

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}}\left[f(s,x_k(s)) - f(s,x(s))\right]ds\right| < \varepsilon,\ \delta_5 \le t \le T,$$

together with (3.16) and (3.17), we obtain $\|\mathcal{A}x_k - \mathcal{A}x\| \le \varepsilon$, i.e., mapping $\mathcal{A}$ is continuous.

Step 3. We prove mapping $\mathcal{A}$ is compact.

Let $\Omega \subset E$ be a bounded set. For any $x\in\Omega$, there exists $l>0$ such that $\|x\|\le l$. Now we only need to utilize Lemma 2.5 to prove that $\mathcal{A}(\Omega)$ is a relatively compact set in E. In fact, $\sup_{t\ge 0}\left|\frac{t^\beta}{1+t^{n+1}}\right| \le M_1$ when $0\le\beta\le n$, together with (3.13), we know $\left\{t^{n-\alpha}(\mathcal{A}x)(t)/\left(1+t^{n+1}\right)\right\}$ is uniformly bounded in E and equiconvergent at infinity.

Finally, we shall verify $\left\{t^{n-\alpha}(\mathcal{A}x)(t)/\left(1+t^{n+1}\right)\right\}$ is equicontinuous on $\mathbb{R}^+$. For any $x\in\Omega$ and $t_1,t_2\in\mathbb{R}^+$, $t_1<t_2$,

$$\begin{aligned}
&\left|\frac{t_2^{n-\alpha}(\mathcal{A}x)(t_2)}{1+t_2^{n+1}} - \frac{t_1^{n-\alpha}(\mathcal{A}x)(t_1)}{1+t_1^{n+1}}\right| \\
&\le \sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-n-k)}\left|\frac{t_2^{n-\alpha}}{1+t_2^{n+1}} - \frac{t_1^{n-\alpha}}{1+t_1^{n+1}}\right| \\
&+ \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}\frac{t_2^{n-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^{n+1}}f(s,x(s))\,ds\right| \\
&+ \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}\frac{t_1^{n-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^{n+1}}f(s,x(s))\,ds\right| \\
&\le \sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-n-k)}\left|\frac{t_2^{n-\alpha}}{1+t_2^{n+1}} - \frac{t_1^{n-\alpha}}{1+t_1^{n+1}}\right| \\
&+ \|f(s,x(s))\|_{C_{n-\alpha}([0,t_2])}\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}\frac{t_2^{n-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^{n+1}}s^{\alpha-n}ds\right| \\
&+ \|f(s,x(s))\|_{C_{n-\alpha}([0,t_1])}\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{t_1^{n-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^{n+1}}s^{\alpha-n}ds\right| \\
&\le \sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-n-k)}\left|\frac{t_2^{n-\alpha}}{1+t_2^{n+1}} - \frac{t_1^{n-\alpha}}{1+t_1^{n+1}}\right| \\
&+ \|f(s,x(s))\|_{C_{n-\alpha}[0,t_2]}\frac{B(\alpha-n+1,\alpha)}{\Gamma(\alpha)}\left|\frac{t_2^{\alpha}}{1+t_2^{n+1}} - \frac{t_1^{\alpha}}{1+t_1^{n+1}}\right| \ \to 0,
\end{aligned}$$

as $t_1\to t_2$. That is to say that $\mathcal{A}(\Omega)$ is equicontinuous. Hence, mapping $\mathcal{A}$ is compact.

Step 4. We prove mapping $\mathcal{B}$ is contraction.

For all $x\in\Omega$, from (H2), we have

$$\begin{aligned}
\left|\frac{t^{n-\alpha}(\mathcal{B}x)(t)}{1+t^{n+1}} - \frac{t^{n-\alpha}(\mathcal{B}y)(t)}{1+t^{n+1}}\right| &= \left|\frac{t^{n-\alpha}}{1+t^{n+1}}\right||g(t,x(t)) - g(t,y(t))| \\
&\le \|q\|_\infty\left|\frac{t^{n-\alpha}}{1+t^{n+1}}\right||x(t)-y(t)|.
\end{aligned}$$

Therefore

$$\left|\frac{t^{n-\alpha}(\mathcal{B}x)(t)}{1+t^{n+1}} - \frac{t^{n-\alpha}(\mathcal{B}y)(t)}{1+t^{n+1}}\right| \le \|q\|_\infty\|x-y\|.$$

Thus

$$\|\mathcal{B}x - \mathcal{B}y\| \leq \|q\|_\infty \|x - y\|.$$

Hence, the operator $\mathcal{B}$ is a contraction.

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus the operator $\mathcal{A} + \mathcal{B}$ has a fixed point in Ω.

Now, we consider the stability of (1.1). For any given $\varepsilon > 0$, together with (3.1) and (3.6), there exists

$$0 \leq \delta \leq \frac{\Lambda}{\sum\limits_{k=1}^{n} \frac{M_1}{\Gamma(\alpha-k+1)}} \varepsilon,$$

such that $\sum_{k=1}^{n} |b_k| < \delta$ implies

$$\begin{aligned}
\|x\| &= \sup_{t\geq 0} \left| \frac{t^{n-\alpha} (\mathcal{A}x)(t)}{1+t^{n+1}} + \frac{t^{n-\alpha} (\mathcal{B}y)(t)}{1+t^{n+1}} \right| \\
&\leq \sup_{t\geq 0} \left| \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} \frac{t^{n-k}}{1+t^{n+1}} \right| + \sup_{t\geq 0} \left| \frac{t^{n-\alpha} g(t, x(t))}{1+t^{n+1}} \right| \\
&+ \|x\| \sup_{t\geq 0} \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{t^{n-\alpha} (t-s)^{\alpha-1} s^{\alpha-n}}{1+t^{n+1}} p(s)\, ds \right| \\
&\leq \sum_{k=1}^{n} \frac{1}{\Gamma(\alpha-k+1)} M_1 \delta + \|q\|_\infty \|x\| + M_1 \frac{\|x\| \|p\|_\infty}{\Gamma(\alpha)} B(\alpha-n+1, \alpha) \\
&\leq \epsilon.
\end{aligned}$$

Then, $x = 0$ is stable in Banach space E. □

Remark 3.1. *The implication of the expression*

$$\lim_{t\to\infty} \frac{t^{n-\alpha} x(t)}{1+t^{n+1}} = 0,$$

in the definition of the Banach space E means that $x = 0$ is weighted asymptotically stable in the Banach space E.

Moreover, we obtain the following theorem using the Banach contraction mapping principle.

Theorem 3.2. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(., x), g(., x) \in C_{n-\alpha}(\mathbb{R}^+)$ for any $x \in C_{n-\alpha}(\mathbb{R}^+)$, $f(t, 0) = g(t, 0) = 0$. Assume (H2) and the following hypotheses*

(H3) $|f(t, x(t)) - f(t, y(t))| \leq \dfrac{p(t)}{1+t^{n+1}} |x(t) - y(t)|$, $t \geq 0$, $x, y \in \mathbb{R}$,

(H4) $\dfrac{1}{\Gamma(\alpha)} \sup_{t\geq 0} \int_0^t \dfrac{t^{n-\alpha} (t-s)^{\alpha-1} s^{\alpha-n}}{1+t^{n+1}} p(s)\, ds + \|q\|_\infty < 1$ *and*

$$\lim_{t\to\infty} \sup_{t\geq 0} \int_0^t \frac{t^{n-\alpha} (t-s)^{\alpha-1} s^{\alpha-n}}{1+t^{n+1}} p(s)\, ds = 0.$$

Then the trivial solution $x = 0$ of (1.1) is stable in the Banach space E.

Proof. Like the procedure proven in Theorem 3.1, we can get that $\Phi x \in C_{n-\alpha}(\mathbb{R}^+)$ where

$$(\Phi x)(t) = (\mathcal{A}x)(t) + (\mathcal{B}x)(t).$$

In accordance with the requirements, we have

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}} f(s,x(s))\,ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\sup_{t\geq 0}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}} \left|f(s,x(s))-f(t,0)\right| ds \\
&\leq \frac{\|x\|}{\Gamma(\alpha)}\sup_{t\geq 0}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}s^{\alpha-n}}{1+t^{n+1}} p(s)\,ds \\
&\to 0 \text{ as } t\to\infty.
\end{aligned}$$

Thus, we know that Φ maps Ω into E. Then, we will use the contraction mapping principle to discuss the stability of (1.1). In fact

$$\begin{aligned}
&\left|\frac{t^{n-\alpha}(\Phi x)(t)}{1+t^{n+1}} - \frac{t^{n-\alpha}(\Phi y)(t)}{1+t^{n+1}}\right| \\
&\leq \left|\frac{t^{n-\alpha}}{1+t^{n+1}}\right| \left|g(t,x(t))-g(t,y(t))\right| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}}{1+t^{n+1}} \left|f(s,x(s))-f(s,y(s))\right| ds \\
&\leq \|q\|_\infty \|x-y\| + \frac{\|x-y\|}{\Gamma(\alpha)}\sup_{t\geq 0}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}s^{\alpha-n}}{1+t^{n+1}} p(s)\,ds \\
&\lambda\|x-y\|.
\end{aligned}$$

Thus

$$\|\Phi x - \Phi y\| \leq \lambda\|x-y\|,$$

where

$$\lambda = \frac{1}{\Gamma(\alpha)}\sup_{t\geq 0}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}s^{\alpha-n}}{1+t^{n+1}} p(s)\,ds + \|q\|_\infty < 1.$$

Thus, Φ is a contraction mapping and Φ has a fixed point x in Ω.

Note that for any given $\epsilon > 0$, there exists $0 \leq \delta \leq \frac{(1-\lambda)}{\sum_{k=1}^{n}\frac{M_1}{\Gamma(\alpha-k+1)}}\varepsilon$ such that $\sum_{k=1}^{n}|b_k| < \sigma$ implies

$$\begin{aligned}
\|x\| &= \sup_{t\geq 0}\left|\frac{t^{n-\alpha}(\mathcal{A}x)(t)}{1+t^{n+1}} + \frac{t^{n-\alpha}(\mathcal{B}y)(t)}{1+t^{n+1}}\right| \\
&\leq \sup_{t\geq 0}\left|\sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}\frac{t^{n-k}}{1+t^{n+1}}\right| + \sup_{t\geq 0}\left|\frac{t^{n-\alpha}g(t,x(t))}{1+t^{n+1}}\right| \\
&+ \|x\|\sup_{t\geq 0}\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}s^{\alpha-n}}{1+t^{n+1}} p(s)\,ds\right| \\
&\leq \sum_{k=1}^{n}\frac{1}{\Gamma(\alpha-k+1)}M_1\delta \\
&+ \left(\|q\|_\infty + \sup_{t\geq 0}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{t^{n-\alpha}(t-s)^{\alpha-1}s^{\alpha-n}}{1+t^{n+1}} p(s)\,ds\right)\|x\| \\
&\leq \epsilon.
\end{aligned}$$

thus

$$\begin{aligned}\|x\| &\leq \sum_{k=1}^{n} \frac{1}{\Gamma(\alpha-k+1)} M_1 \delta + \lambda \|x\| \\ &\leq \epsilon.\end{aligned}$$

Therefore, the trivial solution $x = 0$ of (1.1) is stable in the Banach space E. □

4. An example

Example 4.1. *Consider the following initial value problem for the nonlinear fractional differential equation*

$$\begin{cases} D_{0^+}^{\frac{3}{2}} \left(x(t) - \frac{1}{5} \exp(-t) \sin(x(t)) \right) \\ = \frac{1}{10} \arctan \left(\frac{\exp(t)+1}{t^2+3} \right) \left(\frac{x(t)}{1+t^3} \right)^{\frac{2}{3}} \sin \left(\frac{x(t)}{1+t^3} \right)^{\frac{1}{4}}, \ t \geq 0, \\ D_{0^+}^{\frac{3}{2}-1} \left(x(0^+) - g(0^+, x(0^+)) \right) = D_{0^+}^{\frac{3}{2}-2} \left(x(0^+) - g(0^+, x(0^+)) \right) = 1, \end{cases} \tag{4.1}$$

where $\alpha = \frac{3}{2}$. *Assuming that* $p(t) = \frac{1}{10} \arctan \left(\frac{\exp(t)+1}{t^2+3} \right) \left(\frac{1+t^2}{1+t^3} \right)^{\frac{2}{3}}$ *and* $r(t) = t^{\frac{3}{4}} \sin t^{\frac{1}{4}}$, *then*

$$|f(t, x(t))| \leq p(t)\, r \left(\frac{|x(t)|}{1+t^3} \right). \tag{4.2}$$

On the other hand $g(t, x(t)) = \frac{1}{5} \exp(-t) \sin(x(t))$ *satisfied*

$$|g(t, x(t)) - g(t, y(t))| \leq q(t) |x(t) - y(t)|, \tag{4.3}$$

where $q(t) = \frac{1}{5} \exp(-t)$. *The condition*

$$\begin{aligned}\|q\|_\infty + \frac{M_1 \|p\|_\infty B(\alpha-n+1, \alpha)}{\Gamma(\alpha)} &= \frac{1}{5} + \frac{\pi}{20} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} \\ &\simeq 0.48 < 1. \end{aligned} \tag{4.4}$$

Note that (4.2), (4.3) and (4.4) satisfy Theorem 3.1, then the trivial solution of (4.1) is stable in the Banach space E.

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