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A Generalization of the Regular Function Modulo n

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ABSTRACT: A new generalization for von Neumann regular elements modulo n (regular elements modulo n) will be defined and studied. Also we survey general properties of the multiplicative function $V(n,m)$ which counts the number of n-regular elements in the ring $\mathbb{Z}_m$.

Key Words: n-regular elements, n-regular elements modulo m, Von Neumann regular function modulo n, summatory function.

Contents

1. Introduction

Throughout this article, all rings will be assumed to be commutative and have unity 1. An element $a \in R$ is called von Neumann regular element if there exists $x \in R$ such that $a^2x = a$. A ring R is called von Neumann regular ring if all its elements are von Neumann regular elements. Von Neumann regular rings (elements) were studied extensively in the literature, see [2], [3] and [5]. In [2], Alkam et al. constructed a number theoretic function $V(n)$ that counts the number of von Neumann regular elements in the ring of integers modulo n, $\mathbb{Z}_n$, and studied it with algebraic tools. Other articles have looked at the function $V(n)$ using number-theoretic techniques, see [4].
Anderson et al. expanded the concept of a von Neumann regular element of a ring R to the concept of (m,n)-von Neumann regular element in [6]. The n-regular elements of the ring R are defined in this article, followed by the function $V(n,m)$, which counts the number of n-regular elements in the ring $\mathbb{Z}_m$. Several properties of the function $V(n,m)$ are discussed throughout the article. Finally, we introduce the generalization $F_n(m)$ of the function $F(m)$, then we relate it with the divisor function $\sigma(m)$.

2. n-regular elements of a ring

The n-regular elements of a ring R are defined in this section. Anderson et al. introduced the concept of an (m,n)-von Neumann regular element of a ring R in [6].

Definition 2.1. *Let R be a ring and let m,n be two positive integers. An element $a \in R$ is said to be (m,n)-von Neumann regular element (in short $(m,n)-vnr$) if there exists $b \in R$ such that $a^m b = a^n$.*

This article will focus on a special case of this definition, namely when $m = n+1$, and we will call to the $(n+1,n)-vnr$ element as the n-regular element. Each von Neumann regular element is clearly an n-regular element. Note: In [6], the term "n-regular" has been defined in a way that differs from ours.

Example 2.2. *In $\mathbb{Z}_4$, 2 is not a regular element, while 2 is a 2-regular element.*

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3. n-regular elements of the ring $\mathbb{Z}_m$

The arithmetic function $V(m)$, which counts the number of von Neumann regular elements in the ring $\mathbb{Z}_m$, was examined by Alkam et al. in [2] and To'th in [4]. In this section, we will generalize the function $V(m)$ to the function $V(n,m)$ which counts the number of n-regular elements in the ring $\mathbb{Z}_m$. Since each von Neumann regular element is an n-regular element, $V(m) \leq V(n,m)$.
The following definition separates the nilpotent elements of R, $Nil(R)$, into subsets each of certain nilpotincy.

Definition 3.1. *Let R be a ring and let n be a positive integer. Then $Nil_n(R) = \{x \in R : x^n = 0\}$. It is clear that $Nil(R) = \bigcup_{n\in\mathbb{N}} Nil_n(R)$.*

Lemma 3.2. *Let R be a local ring with maximal ideal M. Then the set of n-regular elements of R is $U(R) \bigcup Nil_n(R)$.*

Proof. Suppose that a is an n-regular element of the local ring R. Then there exists $b \in R$ such that $a^{n+1}b = a^n$. Hence either $a^n \in M$ or $a^n \notin M$. If $a^n \in M$, then $a \in M$ and a must be in $Nil_n(R)$. If $a^n \notin M$, then $a \in U(R)$. It is clear that $U(R)$ and $Nil_n(R)$ are subsets of the n-regular elements. □

Note that, if $x \in Nil_n(R)$, then $x^k = 0$ for any $k \geq n$.

The proof of the following theorem is straightforward by using Lemma [3.2] although we include the proof for the sake of completeness.

Theorem 3.3. *Let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of commutative local rings and let n be a positive integer. Then the element $(x_\alpha)_{\alpha\in\Lambda} \in \{R_\alpha\}_{\alpha\in\Lambda}$ is an n-regular element if and only if for each $\alpha \in \Lambda$ either $x_\alpha \in U(R_\alpha)$ or $x_\alpha \in Nil_n(R_\alpha)$.*

Proof. $(x_\alpha)_{\alpha\in\Lambda}$ is n-regular element iff
there is $(y_\alpha)_{\alpha\in\Lambda}$ such that $(x_\alpha)^{n+1}_{\alpha\in\Lambda}(y_\alpha)_{\alpha\in\Lambda} = (x_\alpha)^n_{\alpha\in\Lambda}$ iff
$(x_\alpha^{n+1})_{\alpha\in\Lambda}(y_\alpha)_{\alpha\in\Lambda} = (x_\alpha^n)_{\alpha\in\Lambda}$ iff
$(x_\alpha^{n+1}y_\alpha)_{\alpha\in\Lambda} = (x_\alpha^n)_{\alpha\in\Lambda}$ iff
$x_\alpha^{n+1}y_\alpha = x_\alpha^n$ for each $\alpha \in \Lambda$ iff
x_α is n-regular element in R_α for each $\alpha \in \Lambda$ iff
for each $\alpha \in \Lambda$ either $x_\alpha \in U(R_\alpha)$ or $x_\alpha \in Nil_n(R_\alpha)$. □

It is known that for any prime number p and any positive integer α, $Nil(\mathbb{Z}_{p^\alpha}) = \langle\overline{p}\rangle = \overline{p}\mathbb{Z}_{p^\alpha} \cong \mathbb{Z}_{p^{\alpha-1}}$. Similarly, $\langle\overline{p}^2\rangle = \overline{p}\langle\overline{p}\rangle \cong \overline{p}\mathbb{Z}_{p^{\alpha-1}} \cong \mathbb{Z}_{p^{\alpha-2}}$. In general, for any integer k such that $1 \leq k \leq \alpha$, $\langle\overline{p}^k\rangle \cong \mathbb{Z}_{p^{\alpha-k}}$.

Now, We can use Lemma [3.2] to prove the following lemma

Lemma 3.4. *Let n be a positive integer. Then for any prime number p and any positive integer α, $V(n,p^\alpha) = \phi(p^\alpha) + p^{\alpha-\lceil\frac{\alpha}{n}\rceil}$.*

Proof. Since $\mathbb{Z}_{p^\alpha}$ is a local ring with maximal ideal $Nil(\mathbb{Z}_{p^\alpha}) = \langle\overline{p}\rangle$, $V(n,p^\alpha) = |U(\mathbb{Z}_{p^\alpha})| + |Nil_n(\mathbb{Z}_{p^\alpha})| = \phi(p^\alpha) + |\langle\overline{p}^{\lceil\frac{\alpha}{n}\rceil}\rangle| = \phi(p^\alpha) + p^{\alpha-\lceil\frac{\alpha}{n}\rceil}$. □

The following result is based on Lemma [3.4], where it highlights some special cases for n.

Corollary 3.5. *Let n be a positive integer. Then for any prime number p and any positive integer $\alpha \leq n$, $V(n,p^\alpha) = p^\alpha$.*

It is a well known fact that if the standard prime factorization of the positive integer m is $m = \prod_{i=1}^t p_i^{\alpha_i}$, then $\mathbb{Z}_m \cong \prod_{i=1}^t \mathbb{Z}_{p_i^{\alpha_i}}$. Hence by Theorem [3.3], we deduce that $V(n,m) = \prod_{i=1}^t V(n,p_i^{\alpha_i})$. Thus, $V(n,m)$ is multiplicative function with respect to m(that is, if m_1 and m_2 are relatively prime, then $V(n,m_1m_2) = V(n,m_1)V(n,m_2)$).
Some properties of the multiplicative function $V(n,m)$ are presented here. To prove the following theorem, we use some of the conclusions from [2] and [4]. Recall that, if $a|b$ and $gcd(a,\frac{b}{a}) = 1$, then a is said to be a unitary divisor of b, denoted by $a||b$.

Theorem 3.6. *Let n be a positive integer and let m be a positive integer with the standard prime factorization $m = \prod_{i=1}^{t} p_i^{\alpha_i}$, also let $k = \prod_{i=1}^{t} p_i^{\lceil \frac{\alpha_i}{n} \rceil}$. Then*

1. *If $n \geq \max\{\alpha_i\}_{i=1}^{t}$, then $V(n,m) = m$.*
2. *If $n \leq \min\{\alpha_i\}_{i=1}^{t}$, then $V(n,m) = \prod_{i=1}^{t}[\phi(p_i^{\alpha_i}) + p_i^{\alpha_i - \lceil \frac{\alpha_i}{n} \rceil}]$. As a special case, $V(m) = V(1,m) = \prod_{i=1}^{t}[\phi(p_i^{\alpha_i}) + 1]$.*
3. *$V(n,m) = \frac{m}{k} V(k)$.*
4. *$V(n,m) = \sum\limits_{d||k} \frac{m}{k}\phi(d)$.*
5. *$V(n,m)$ is increasing with respect to both n and m (that is, if $n_1 \leq n_2$, then $V(n_1,m) \leq V(n_2,m)$, and if $m_1 \leq m_2$, then $V(n,m_1) \leq V(n,m_2)$).*
6. *$\frac{V(n,m)}{\phi(m)} = \sum\limits_{d||k} \frac{1}{\phi(d)}$.*

Proof. Since $V(n,m)$ is multiplicative in m, we can deduce (1) and (2) using Lemma [3.4]. To prove (3),

$$\begin{aligned} V(n,m) &= \prod_{i=1}^{t}[\phi(p_i^{\alpha_i}) + p_i^{\alpha_i - \lceil \frac{\alpha_i}{n} \rceil}] \\ &= \prod_{i=1}^{t}[(p_i^{\alpha_i} - p_i^{\alpha_i - 1}) + p_i^{\alpha_i - \lceil \frac{\alpha_i}{n} \rceil}] \\ &= \prod_{i=1}^{t} p_i^{\alpha_i - \lceil \frac{\alpha_i}{n} \rceil}[(p_i^{\lceil \frac{\alpha_i}{n} \rceil} - p_i^{\lceil \frac{\alpha_i}{n} \rceil - 1}) + 1] \\ &= \frac{m}{k} V(k). \end{aligned}$$

To prove (4), you can use (3) and the property $V(k) = \sum\limits_{d||k} \phi(d)$ that is found in [2] and [4].

The proof of (5) is straightforward.

To prove (6), combine (3) and the fact, if $m = \prod_{i=1}^{t} p_i^{\alpha_i}$ and $n = \prod_{i=1}^{t} p_i^{\beta_i}$ are positive integers, then $\frac{\phi(m)}{\phi(n)} = \frac{m}{n}$, as well as the property $\frac{V(k)}{\phi(k)} = \sum\limits_{d||k} \frac{1}{\phi(d)}$ that is found in [2] and [4].

□

Example 3.7. *Take $m = 2^4.3^3$ and $n = 3$, then $k = 2^2.3$ and $\phi(m) = 144$. Thus, $V(k) = 9$ and $V(3,m) = 324 = \frac{2^4.3^3}{2^2.3}.9$.*
Also, the values of d such that $d||k$ are $1, 3, 4$ and 12. So, $\sum\limits_{d||k} \frac{1}{\phi(d)} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{9}{4} = \frac{324}{144}$.

4. Sum of n-regular elements modulo m

In this section, we will use the number-theoretic consideration to prove some results concerning the n-regular elements modulo m. Firstly, consider $Reg(n,m) = \{a \in \mathbb{Z} : 1 \leq a \leq m, a\, is\, n-regular\,(mod\, m)\}$ denotes the set of all n-regular elements modulo m. Then $V(n,m) = |Reg(n,m)|$.

Theorem 4.1. $V(n,m) = \sum\limits_{\substack{d||m \\ t||k \\ d\, and\, t\, have \\ common\, prime \\ divisors}} \frac{d}{t}\phi(\frac{m}{d})$

Proof. Let $y_i = \frac{p_i^{\alpha_i}}{p_i^{\lceil \frac{\alpha_i}{n} \rceil}\phi(p_i^{\alpha_i})}$, $1 \leq i \leq t$, and $y = \prod_{i=1}^{t} y_i = \frac{m}{k\phi(m)}$. Then

$$\begin{aligned} \phi(m)[y + \sum_{1\leq i\leq t} \frac{y}{y_i} + \sum_{1\leq i<j\leq t} \frac{y}{y_i y_j} + \cdots + \frac{y}{\prod_{i=1}^{t} y_i}] &= \prod_{i=1}^{t} \phi(p_i^{\alpha})(y_i + 1) \\ &= \prod_{i=1}^{t} V(n, p_i^{\alpha}) \\ &= V(n, m). \end{aligned}$$

□

Tóth in [4] gave a formula for the sum of regular elements $(mod\, m)$, $S(m) = \frac{m(V(m)+1)}{2}$, Tóth's formula was analogous to the formula $\sum_{\substack{1\leq a\leq m \\ gcd(a,m)=1}} a = \frac{m\phi(m)}{2}$.

The following theorem gives a formula for $S(n, m)$, which is analogous to $S(m) = \frac{m(V(m)+1)}{2}$.

Theorem 4.2. *For the positive integers n and m, $S(n, m) = \frac{m(V(n,m)+1)}{2}$.*

Proof. $S(n, m) = \sum_{a\in Reg(n,m)} a$

$$= \sum_{\substack{d||m \\ t||k}} \sum_{\substack{(a,m)=d \\ (a,k)=t}} a$$

$$= \sum_{\substack{d||m \\ t||k}} \sum_{\substack{(\frac{a}{d},\frac{m}{d})=1 \\ (a,k)=t}} a$$

$$= \sum_{\substack{d||m \\ t||k}} \sum_{\substack{j=1 \\ (j,\frac{m}{d})=1 \\ (a,k)=t}}^{\frac{m}{d}} jd$$

$$= \sum_{\substack{d||m \\ t||k}} t \sum_{i=1}^{\frac{d}{t}} i \left(\sum_{\substack{j=1 \\ (j,\frac{m}{d})=1 \\ (a,k)=t}}^{\frac{m}{d}} j \right)$$

$$= k(1 + 2 + 3 + \ldots + \tfrac{m}{k}) + \sum_{\substack{d||m \\ t||k \\ d<m \\ t<k}} t \sum_{i=1}^{\frac{d}{t}} i \left(\tfrac{m}{2d}\phi(\tfrac{m}{d})\right)$$

$$= k(\tfrac{m}{k} + 1)\tfrac{m}{2k} + \sum_{\substack{d||m \\ t||k \\ d<m \\ t<k}} \sum_{i=1}^{\frac{d}{t}} ti \left(\tfrac{m}{2d}\phi(\tfrac{m}{d})\right)$$

$$= \tfrac{m}{2}(\tfrac{m}{k} + 1) + \sum_{\substack{d||m \\ t||k \\ d<m \\ t<k}} \sum_{i=1}^{\frac{d}{t}} d \left(\tfrac{m}{2d}\phi(\tfrac{m}{d})\right)$$

$$= \tfrac{m}{2} \left((\tfrac{m}{k} + 1) + \sum_{\substack{d||m \\ t||k \\ d<m \\ t<k}} \sum_{i=1}^{\frac{d}{t}} \phi(\tfrac{m}{d}) \right)$$

$$= \frac{m}{2}\left(1 + \sum_{\substack{d||m \\ t||k}} \phi(\tfrac{m}{d})\tfrac{d}{t}\right)$$
$$= \frac{m}{2}\left(1 + V(n,m)\right)$$
□

Example 4.3. *For* $m = 16$, $n = 3$, $V(n,m) = 12$ *and* $R(n,m) = \{1,3,4,5,7,8,9,11,12,13,15,16\}$. *Then* $\sum_{a\in Reg(n,m)} a = 104 = \frac{16}{2}(1+12)$.

5. The summatory function $F_n(m)$

The summatory function $F(m) = \sum_{d|m} V(d)$ is calculated in [2]. In this section, we will calculate $F_n(m)$, the generalized form of the summatory function $F(m)$. Let $F_n(m) = \sum_{d|m} V(n,d)$. Since $V(n,m)$ is multiplicative concerning m, $F_n(m)$ is also multiplicative concerning m, hence $F_n(m)$ is completely characterized by its values on powers of primes. Recall that the functions $\sigma(n)$ and $\sigma_k(n)$ are defined as the sum, or the sum of k-th powers, of the divisors of n respictively.

Theorem 5.1. *Let p be a prime number and let α and n be positive integers. Then*

1. *If $\alpha \le n$, then $F_n(p^\alpha) = \sigma(p^\alpha)$.*
2. *If $\alpha > n$, then $F_n(p^\alpha) = p^\alpha + 1 + \sigma(p^{n-1})\sigma_{n-1}(p^{q-1}) + p^{q(n-1)}\sigma(p^{r-1})$, where q, r are the quotient and remainder when we divide α by n.*

Proof. 1. In Corollary [3.5] if $\alpha \le n$, then $V(n,p^\alpha) = p^\alpha$. Hence, the result is clear.

2. If $\alpha > n$, then $F_n(p^\alpha) = \sum_{d|p^\alpha} V(n,d)$

$$= \sum_{k=0}^{\alpha} V(n,p^k)$$
$$= \sum_{k=0}^{\alpha} \phi(p^k) + p^{k-\lceil \frac{k}{n}\rceil}$$
$$= p^\alpha + 1 + \sum_{k=1}^{\alpha} p^{k-\lceil \frac{k}{n}\rceil}$$
$$= p^\alpha + 1 + \sum_{k=1}^{n} p^{k-\lceil \frac{k}{n}\rceil} + \sum_{k=n+1}^{2n} p^{k-\lceil \frac{k}{n}\rceil} + \sum_{k=2n+1}^{3n} p^{k-\lceil \frac{k}{n}\rceil} + \cdots + \sum_{k=(q-1)n+1}^{qn} p^{k-\lceil \frac{k}{n}\rceil} + \sum_{k=qn+1}^{\alpha} p^{k-\lceil \frac{k}{n}\rceil}$$
$$= p^\alpha + 1 + \sum_{k=1}^{n} p^{k-1} + \sum_{k=n+1}^{2n} p^{k-2} + \sum_{k=2n+1}^{3n} p^{k-3} + \cdots + \sum_{k=(q-1)n+1}^{qn} p^{k-q} + \sum_{k=qn+1}^{\alpha} p^{k-(q+1)}$$
$$= p^\alpha + 1 + \left(\sum_{k=0}^{n-1} p^k\right)\left[1 + p^{n-1} + p^{2(n-1)} + \cdots + p^{(q-1)(n-1)}\right] + p^{q(n-1)}\sum_{k=0}^{r-1} p^k$$
$$= p^\alpha + 1 + \sigma(p^{n-1})\sigma_{n-1}(p^{q-1}) + p^{q(n-1)}\sigma(p^{r-1}).$$
□

.

6. Conclusions

This article defines the n-regular elements of the ring R, as well as the function $V(n,m)$, which counts the number of n-regular elements in the ring $\mathbb{Z}_m$. Throughout the article, we have established that the set of n-regular elements of the ring R is $U(R)\bigcup Nil_n(R)$. Also, we show that the function $V(n,m)$ is multiplicative. The sum of the n-regular elements modulo m is calculated using number theoretic considerations. Also, we introduce the summatory function $F_n(m) = \sum\limits_{d|m} V(n,d)$ and we find the relationship between $F_n(m)$ and the divisor function $\sigma(m)$.

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