



## Some Common Fixed Point Results on $(\psi, \phi)$ -contraction

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ABSTRACT: The aim of the paper is to obtain common fixed point theorems for  $(\psi, \phi)$ -contraction under the generalized rational type condition in a complete metric space. Moreover, these theorems generalize recent well known results in the literature.

Key Words: Weak compatibility,  $(\psi, \phi)$ -contraction, common fixed point.

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### 1. Introduction

In 1976, Jungck [9] initiated the notion of commutativity of mappings and established a common fixed point theorem on a complete metric space. In 1982, Sessa [18] also introduced the concept of weak commutativity by weakening the commutativity and obtained some interesting results on the existence of common fixed points. Further, Jungck [10] generalized the weak commutativity by a new notion of compatible mappings. However, in 1996, Jungck [11] again introduced a more generalized concept known as weakly compatibility, and defined as follows.

**Definition 1.1** ([11]). *Let  $f$  and  $g$  be self mappings of a set  $X$ . Then the pair  $\{f, g\}$  is said to be weakly compatible if they commute on the set of coincidence points, i.e.,  $f gx = g f x$  whenever  $f x = g x$  for some  $x \in X$ .*

On the other hand, generalizing Banach contraction condition, Boyd and Wong [5] defined a new class of contractive condition which is generally known as  $\phi$ -contraction. Further Alber et al. [2] generalized this concept by introducing weak  $\phi$ -contraction and established a fixed point theorem for the mapping satisfying such type of contractive condition. By the way, a self mapping  $T$  on a metric space  $(X, d)$  is said to be *weak  $\phi$ -contractive* if there exists a function  $\phi : [0, +\infty) \rightarrow [0, +\infty)$  with  $\phi(0) = 0$  and  $\phi(t) > 0$  for all  $t > 0$ , such that  $d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$  for each  $x, y \in X$ . Thereafter, Rhoades [15] again generalized the result of Alber et al. [2] and obtained the following interesting theorem.

**Theorem 1.2** ([15]). *Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  such that, for every  $x, y \in X$ ,*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \tag{1.1}$$

*where  $\phi : [0, +\infty) \rightarrow [0, +\infty)$  is a continuous and non-decreasing function with  $\phi(0) = 0$  and  $\phi(t) > 0$  for all  $t > 0$ . Then  $T$  has a unique fixed point.*

Now, for further discussions, we consider following classes of functions:

$(C_1)$   $\Phi = \{\phi \mid \phi : [0, +\infty) \rightarrow [0, +\infty)$  is lower semi continuous with  $\phi(t) > 0$  for all  $t > 0$  and  $\phi(0) = 0\}$ .

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(C<sub>2</sub>)  $\Psi = \{\psi \mid \psi : [0, +\infty) \rightarrow [0, +\infty) \text{ is continuous and nondecreasing with } \psi(t) = 0 \text{ if and only if } t = 0\}$ .

Moreover, in 2008, Dutta et al. [17] generalized the  $\phi$ -contraction by a new extended class contractive mappings known as  $(\psi, \phi)$ -contraction and established the following result.

**Theorem 1.3** ([17]). *Let  $X$  be a complete metric space and  $T : X \rightarrow X$  such that, for every  $x, y \in X$ ,*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad (1.2)$$

where  $\phi \in \Phi$ ,  $\psi \in \Psi$ . Then  $T$  has a unique fixed point in  $X$ .

Furthermore, in 2015, Murty et al. [14] also obtained the following common fixed point theorem for  $(\psi, \phi)$ -contraction which generalizes various results in the literature.

**Theorem 1.4** ([14]). *Suppose that  $A, B, S$  and  $T$  are self mappings of a complete metric space  $(X, d)$ ,  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ , and the pairs  $\{A, S\}$  and  $\{B, T\}$  are weakly compatible. If, for every  $x, y \in X$  with  $x \neq y$ ,*

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \phi(N(x, y)), \quad (1.3)$$

where  $\psi \in \Psi$ ,  $\phi \in \Phi$  such that  $\phi$  is discontinuous at  $t = 0$ , and

$$M(x, y) = \max \left\{ d(Sx, Ty), \frac{d(Ax, Sx) + d(By, Ty)}{2}, \frac{d(Sx, By) + d(Ax, Ty)}{2} \right\}$$

and

$$N(x, y) = \min \left\{ d(Sx, Ty), \frac{d(Ax, Sx) + d(By, Ty)}{2}, \frac{d(Sx, By) + d(Ax, Ty)}{2} \right\}.$$

Then  $A, B, S$ , and  $T$  have a unique common fixed point in  $X$ .

Moreover, during last three decades, a number of researchers have extended and weakened  $(\psi, \phi)$ -contractive condition in different settings and obtained several common fixed point theorems for pairs of mappings (see, [1,3,4,6,7,8,12,13,14,16,17,19] and references therein).

Now, we are in a position to state and prove our results which have been obtained for mappings satisfying a generalized rational type condition under the weak compatibility and  $(\psi, \phi)$ -contraction in complete metric spaces as follow.

## 2. Main Results

**Theorem 2.1.** *Suppose that  $A, B, S$  and  $T$  are self mappings of a complete metric space  $(X, d)$ ,  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ , and the pairs  $\{A, S\}$  and  $\{B, T\}$  are weakly compatible. If, for every  $x, y \in X$ ,*

$$\psi(d(Ax, By)) \leq \psi(M_1(x, y)) - \phi(N_1(x, y)), \quad (2.1)$$

where  $\psi \in \Psi$ ,  $\phi \in \Phi$  such that  $\phi$  is discontinuous at  $t = 0$ , and

$$\begin{aligned} M_1(x, y) = & \max \left\{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), \right. \\ & \left( \frac{d(By, Sx) + d(Ax, Ty)}{2} \right), \left( \frac{d(Sx, Ax) + d(Ty, By)}{2} \right), \\ & \left. d(By, Ty) \left( \frac{1 + d(Ax, Sx)}{1 + d(Sx, Ty)} \right), d(Ax, Sx) \left( \frac{1 + d(By, Ty)}{1 + d(Sx, Ty)} \right) \right\}. \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} N_1(x, y) = & \min \left\{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), \right. \\ & \left( \frac{d(By, Sx) + d(Ax, Ty)}{2} \right), \left( \frac{d(Sx, Ax) + d(Ty, By)}{2} \right), \\ & \left. d(By, Ty) \left( \frac{1 + d(Ax, Sx)}{1 + d(Sx, Ty)} \right), d(Ax, Sx) \left( \frac{1 + d(By, Ty)}{1 + d(Sx, Ty)} \right) \right\}. \end{aligned} \quad (2.3)$$

Then  $A, B, S$ , and  $T$  have a unique common fixed point in  $X$ , whenever one of the range  $A(X)$ ,  $B(X)$ ,  $S(X)$ ,  $T(X)$  is closed in  $X$ .

*Proof.* Let  $x_0$  be an arbitrary point in  $X$ . Since  $A(X) \subseteq T(X)$ , we can choose an  $x_1 \in X$  such that  $y_0 = Ax_0 = Tx_1$ . Similarly, since  $B(X) \subseteq S(X)$ , there exists an  $x_2 \in X$  such that  $y_1 = Bx_1 = Sx_2$ . Continuing in this way, we construct a sequence  $\{y_n\}_{n \in \mathbb{N}_0}$  in  $X$ , where  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  and  $\mathbb{N}$  is a set of natural numbers, such that  $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$  and  $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ .

We shall now show that  $\{y_n\}_{n \in \mathbb{N}_0}$  is a Cauchy sequence in  $X$ . If  $y_{2n} = y_{2n+1}$  for some  $n \in \mathbb{N}_0$ , it is obvious to say that  $\{y_n\}_{n \in \mathbb{N}_0}$  is a Cauchy sequence. So, we assume the case when  $y_{2n} \neq y_{2n+1}$  for every  $n \in \mathbb{N}_0$ . Then, by taking  $x = x_{2n}, y = x_{2n+1}$  in (2.2) and (2.3), we have

$$\begin{aligned} M_1(x_{2n}, x_{2n+1}) &= \max \left\{ d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \right. \\ &\quad \left( \frac{d(Bx_{2n+1}, Sx_{2n}) + d(Ax_{2n}, Tx_{2n+1})}{2} \right), \left( \frac{d(Sx_{2n}, Ax_{2n}) + d(Tx_{2n+1}, Bx_{2n+1})}{2} \right), \\ &\quad \left. d(Bx_{2n+1}, Tx_{2n+1}) \left( \frac{1 + d(Ax_{2n}, Sx_{2n})}{1 + d(Sx_{2n}, Tx_{2n+1})} \right), d(Ax_{2n}, Sx_{2n}) \left( \frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})} \right) \right\} \\ &= \max \left\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \right. \\ &\quad \left( \frac{d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})}{2} \right), \left( \frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2} \right), \\ &\quad \left. d(y_{2n+2}, y_{2n+1}) \left( \frac{1 + d(y_{2n+1}, y_{2n})}{1 + d(y_{2n}, y_{2n+1})} \right), d(y_{2n+1}, y_{2n}) \left( \frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})} \right) \right\} \\ &= \max \left\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+1}), \left( \frac{d(y_{2n+2}, y_{2n})}{2} \right), \right. \\ &\quad \left. \left( \frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2} \right), d(y_{2n+1}, y_{2n}) \left( \frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})} \right) \right\} \end{aligned}$$

and

$$\begin{aligned} N_1(x_{2n}, x_{2n+1}) &= \min \left\{ d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \right. \\ &\quad \left( \frac{d(Bx_{2n+1}, Sx_{2n}) + d(Ax_{2n}, Tx_{2n+1})}{2} \right), \left( \frac{d(Sx_{2n}, Ax_{2n}) + d(Tx_{2n+1}, Bx_{2n+1})}{2} \right), \\ &\quad \left. d(Bx_{2n+1}, Tx_{2n+1}) \left( \frac{1 + d(Ax_{2n}, Sx_{2n})}{1 + d(Sx_{2n}, Tx_{2n+1})} \right), d(Ax_{2n}, Sx_{2n}) \left( \frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})} \right) \right\} \\ &= \min \left\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \right. \\ &\quad \left( \frac{d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})}{2} \right), \left( \frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2} \right), \\ &\quad \left. d(y_{2n+2}, y_{2n+1}) \left( \frac{1 + d(y_{2n+1}, y_{2n})}{1 + d(y_{2n}, y_{2n+1})} \right), d(y_{2n+1}, y_{2n}) \left( \frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})} \right) \right\} \\ &= \min \left\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+1}), \left( \frac{d(y_{2n+2}, y_{2n})}{2} \right), \right. \\ &\quad \left. \left( \frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2} \right), d(y_{2n+1}, y_{2n}) \left( \frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})} \right) \right\}. \end{aligned}$$

Now, if  $M_1(x_{2n}, x_{2n+1}) = d(y_{2n+1}, y_{2n+2})$  then

$$\begin{aligned} \psi(d(y_{2n+1}, y_{2n+2})) &= \psi(d(Ax_{2n}, Bx_{2n+1})) \\ &\leq \psi(M_1(x_{2n}, x_{2n+1})) - \phi(N_1(x_{2n}, x_{2n+1})) \\ &= \psi(d(y_{2n+1}, y_{2n+2})) - \phi(N_1(x_{2n}, x_{2n+1})) \\ &< \psi(d(y_{2n+1}, y_{2n+2})), \end{aligned}$$

which is a contradiction. Therefore  $d(y_{2n+2}, y_{2n+1}) \leq d(y_{2n}, y_{2n+1})$ , and

$$d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1}) \left( \frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})} \right),$$

which implies

$$d(y_{2n}, y_{2n+1}) \left( \frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})} \right) \leq d(y_{2n}, y_{2n+1}).$$

Hence,  $M_1(x_{2n}, x_{2n+1}) = d(y_{2n}, y_{2n+1})$  and  $N_1(x_{2n}, x_{2n+1}) = \frac{d(y_{2n}, y_{2n+2})}{2}$ . Using (2.1), we have

$$\begin{aligned} \psi(d(y_{2n+1}, y_{2n+2})) &= \psi(d(Ax_{2n}, Bx_{2n+1})) \\ &\leq \psi(M_1(x_{2n}, x_{2n+1})) - \phi(N_1(x_{2n}, x_{2n+1})) \\ &\leq \psi(d(y_{2n}, y_{2n+1})) - \phi(N_1(x_{2n}, x_{2n+1})). \end{aligned} \quad (2.4)$$

This implies  $\psi(d(y_{2n}, y_{2n+1}))$  for all  $n \in \mathbb{N}_0$ , and the sequence is monotonically decreasing of non-negative real numbers. Hence, there exists  $r > 0$  such that  $\lim_{n \rightarrow +\infty} d(y_{2n}, y_{2n+1}) = r$ . Moreover,  $\lim_{n \rightarrow +\infty} \psi(M_1(x_{2n}, x_{2n+1})) = \psi(r)$ . Now, taking upper limits on each side of (2.4) to obtain the following inequality

$$\limsup_{n \rightarrow +\infty} \psi(d(y_{2n+1}, y_{2n+2})) \leq \limsup_{n \rightarrow +\infty} \psi(d(y_{2n+1}, y_{2n})) - \limsup_{n \rightarrow +\infty} \phi(N_1(x_{2n}, x_{2n+1})).$$

Thus, the lower semi continuity of  $\phi$  gives

$$\psi(r) \leq \psi(r) - \limsup_{n \rightarrow +\infty} \phi(N_1(x_{2n}, x_{2n+1})).$$

Therefore, by the property of  $\phi$ , we get a contradiction. Hence, we have

$$\lim_{n \rightarrow +\infty} d(y_{2n}, y_{2n+1}) = 0.$$

Similarly, taking  $x = x_{2n+1}$  and  $y = x_{2n+2}$  in (2.1) and arguing as above, we have

$$\lim_{n \rightarrow +\infty} d(y_{2n+1}, y_{2n+2}) = 0.$$

Therefore, for all  $n \in \mathbb{N}_0$ , we have

$$\lim_{n \rightarrow +\infty} d(y_{2n}, y_{2n+1}) = 0. \quad (2.5)$$

Next, we prove that  $\{y_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $X$ . For this, it is sufficient to show  $\{y_{2n}\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $X$ . To the contrary, suppose  $\{y_{2n}\}_{n \in \mathbb{N}}$  is not a Cauchy sequence, then there exists an  $\epsilon > 0$  and the sequence of natural numbers  $\{2m_k\}$ ,  $\{2n_k\}$  with  $2m_k > 2n_k > k$  such that

$$d(y_{2m_k}, y_{2n_k}) \geq \epsilon \text{ and } d(y_{2m_k-2}, y_{2n_k}) < \epsilon.$$

Using (2.5) and the inequality

$$\epsilon \leq d(y_{2m_k}, y_{2n_k}) \leq d(y_{2n_k}, y_{2m_k-2}) + d(y_{2m_k-1}, y_{2m_k-2}) + d(y_{2m_k-1}, y_{2m_k})$$

we get

$$\lim_{k \rightarrow +\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon. \quad (2.6)$$

Also (2.5), (2.6) and the inequality,  $d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2m_k+1}) + d(y_{2m_k+1}, y_{2n_k})$ , yield

$$\epsilon \leq \lim_{k \rightarrow +\infty} d(y_{2m_k+1}, y_{2n_k}),$$

and (2.5), (2.6) and the inequality,  $d(y_{2m_k+1}, y_{2n_k}) \leq d(y_{2m_k+1}, y_{m_k}) + d(y_{2m_k}, y_{2n_k})$ , yield that

$$\lim_{k \rightarrow \infty} d(y_{2m_k+1}, y_{2n_k}) \leq \epsilon.$$

Hence, we obtain

$$\lim_{k \rightarrow +\infty} d(y_{2m_k+1}, y_{2n_k}) = \epsilon. \quad (2.7)$$

In similar manner, it can be shown that

$$\lim_{k \rightarrow +\infty} d(y_{2m_k}, y_{2n_k-1}) = \lim_{k \rightarrow +\infty} d(y_{2n_k-1}, y_{2n_k+1}) = \epsilon. \quad (2.8)$$

Now, we find

$$\begin{aligned} M_1(x_{2m_k-1}, x_{2n_k-1}) &= \max \left\{ d(Sx_{2m_k-1}, Tx_{2n_k-1}), d(Ax_{2m_k-1}, Sx_{2m_k-1}), d(Bx_{2n_k-1}, Tx_{2n_k-1}), \right. \\ &\quad \left( \frac{d(Bx_{2n_k-1}, Sx_{2m_k-1}) + d(Ax_{2m_k-1}, Tx_{2n_k-1})}{2} \right), \\ &\quad \left( \frac{d(Sx_{2m_k-1}, Ax_{2m_k-1}) + d(Tx_{2n_k-1}, Bx_{2n_k-1})}{2} \right), \\ &\quad d(Bx_{2n_k-1}, Tx_{2n_k-1}) \left( \frac{1 + d(Ax_{2m_k-1}, Sx_{2m_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})} \right), \\ &\quad \left. d(Ax_{2m_k-1}, Sx_{2m_k-1}) \left( \frac{1 + d(Bx_{2n_k-1}, Tx_{2n_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})} \right) \right\} \\ &= \max \left\{ d(y_{2m_k+1}, y_{2n_k}), d(Ax_{2m_k-1}, Sx_{2m_k-1}), d(Bx_{2n_k-1}, Tx_{2n_k-1}), \right. \\ &\quad \left( \frac{d(Bx_{2n_k-1}, Sx_{2m_k-1}) + d(Ax_{2m_k-1}, Tx_{2n_k-1})}{2} \right), \\ &\quad \left( \frac{d(Sx_{2m_k-1}, Ax_{2m_k-1}) + d(Tx_{2n_k-1}, Bx_{2n_k-1})}{2} \right), \\ &\quad d(Bx_{2n_k-1}, Tx_{2n_k-1}) \left( \frac{1 + d(Ax_{2m_k-1}, Sx_{2m_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})} \right), \\ &\quad \left. d(Ax_{2m_k-1}, Sx_{2m_k-1}) \left( \frac{1 + d(Bx_{2n_k-1}, Tx_{2n_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})} \right) \right\} \end{aligned}$$

and

$$\begin{aligned} N_1(x_{2m_k-1}, x_{2n_k-1}) &= \min \left\{ d(Sx_{2m_k-1}, Tx_{2n_k-1}), d(Ax_{2m_k-1}, Sx_{2m_k-1}), d(Bx_{2n_k-1}, Tx_{2n_k-1}), \right. \\ &\quad \left( \frac{d(Bx_{2n_k-1}, Sx_{2m_k-1}) + d(Ax_{2m_k-1}, Tx_{2n_k-1})}{2} \right), \\ &\quad \left( \frac{d(Sx_{2m_k-1}, Ax_{2m_k-1}) + d(Tx_{2n_k-1}, Bx_{2n_k-1})}{2} \right), \\ &\quad d(Bx_{2n_k-1}, Tx_{2n_k-1}) \left( \frac{1 + d(Ax_{2m_k-1}, Sx_{2m_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})} \right), \\ &\quad \left. d(Ax_{2m_k-1}, Sx_{2m_k-1}) \left( \frac{1 + d(Bx_{2n_k-1}, Tx_{2n_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})} \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \min \left\{ d(y_{2m_k+1}, y_{2n_k}), d(Ax_{2m_k-1}, Sx_{2m_k-1}), d(Bx_{2n_k-1}, Tx_{2n_k-1}), \right. \\
&\quad \left( \frac{d(Bx_{2n_k-1}, Sx_{2m_k-1}) + d(Ax_{2m_k-1}, Tx_{2n_k-1})}{2} \right), \\
&\quad \left( \frac{d(Sx_{2m_k-1}, Ax_{2m_k-1}) + d(Tx_{2n_k-1}, Bx_{2n_k-1})}{2} \right), \\
&\quad d(Bx_{2n_k-1}, Tx_{2n_k-1}) \left( \frac{1 + d(Ax_{2m_k-1}, Sx_{2m_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})} \right), \\
&\quad \left. d(Ax_{2m_k-1}, Sx_{2m_k-1}) \left( \frac{1 + d(Bx_{2n_k-1}, Tx_{2n_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})} \right) \right\}.
\end{aligned}$$

Thus, using (2.2), (2.5), (2.6), (2.7) and (2.8), we have  $\lim_{k \rightarrow +\infty} M_1(x_{2m_k-1}, x_{2n_k-1}) = \epsilon$  and  $\lim_{k \rightarrow +\infty} N_1(x_{2m_k-1}, x_{2n_k-1}) = 0$ . Moreover, by taking  $x = x_{2m_k-1}$  and  $y = x_{2n_k-1}$  in (2.1), we get

$$\begin{aligned}
\psi(d(y_{2m_k}, y_{2n_k+1})) &= \psi(d(Ax_{2m_k-1}, Bx_{2n_k-1})) \\
&\leq \psi(M_1(x_{2m_k-1}, x_{2n_k-1}) - \phi(N_1(x_{2m_k-1}, x_{2n_k-1})).
\end{aligned}$$

Therefore, taking the limit as  $k \rightarrow +\infty$ , we get  $\psi(\epsilon) \leq \psi(\epsilon) - \phi(N_1(x_{2m_k-1}, x_{2n_k-1}))$ , which is a contradiction for  $\epsilon > 0$  (due to discontinuity of  $\phi$  at  $t = 0$ ). Hence  $\{y_{2n}\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $X$ .

Thus, in both cases, it has been shown that  $\{y_n\}_{n \in \mathbb{N}_0}$  is a Cauchy sequence in  $X$ . Since  $X$  is complete, it has a limit in  $X$ , say  $z$ . We shall now show that  $z$  is a common fixed point for mappings  $A$  and  $S$ . It is clear that

$$\lim_{n \rightarrow +\infty} y_{2n+1} = \lim_{n \rightarrow +\infty} Ax_{2n} = \lim_{n \rightarrow +\infty} Tx_{2n+1} = z,$$

and

$$\lim_{n \rightarrow +\infty} y_{2n+2} = \lim_{n \rightarrow +\infty} Bx_{2n+1} = \lim_{n \rightarrow +\infty} Sx_{2n+2} = z.$$

Assuming that  $S(X)$  is closed, there exists a  $u \in X$  such that  $z = Su$ . We claim that  $Au = z$ . If not, then

$$\begin{aligned}
M_1(u, x_{2n+1}) &= \max \left\{ d(Su, Tx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}), \right. \\
&\quad \left( \frac{d(Bx_{2n+1}, Su) + d(Au, Tx_{2n+1})}{2} \right), \left( \frac{d(Su, Au) + d(Tx_{2n+1}, Bx_{2n+1})}{2} \right), \\
&\quad \left. d(Bx_{2n+1}, Tx_{2n+1}) \left( \frac{1 + d(Au, Su)}{1 + d(Su, Tx_{2n+1})} \right), d(Au, Su) \left( \frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Su, Tx_{2n+1})} \right) \right\}
\end{aligned}$$

and

$$\begin{aligned}
N_1(u, x_{2n+1}) &= \min \left\{ d(Su, Tx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}), \right. \\
&\quad \left( \frac{d(Bx_{2n+1}, Su) + d(Au, Tx_{2n+1})}{2} \right), \left( \frac{d(Su, Au) + d(Tx_{2n+1}, Bx_{2n+1})}{2} \right), \\
&\quad \left. d(Bx_{2n+1}, Tx_{2n+1}) \left( \frac{1 + d(Au, Su)}{1 + d(Su, Tx_{2n+1})} \right), d(Au, Su) \left( \frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Su, Tx_{2n+1})} \right) \right\}.
\end{aligned}$$

Taking the limit as  $n \rightarrow +\infty$ , we get

$$\lim_{n \rightarrow +\infty} M_1(u, x_{2n+1}) = d(Au, z) \text{ and } \lim_{n \rightarrow +\infty} N_1(u, x_{2n+1}) = 0.$$

Therefore, by (2.1), we have

$$\psi(d(Au, Bx_{2n+1})) \leq \psi(M_1(u, x_{2n+1})) - \phi(N_1(u, x_{2n+1})),$$

which, taking the limit as  $n \rightarrow +\infty$ , implies that

$$\psi(d(Au, z)) \leq \psi(d(Au, z)) - \phi(N_1(u, x_{2n+1})),$$

a contradiction for  $d(Au, z) > 0$ . Hence  $Au = z$ , and  $Au = Su = z$ . Since the mappings  $A$  and  $S$  are weakly compatible,  $Az = ASu = SAu = Sz$ .

Next we claim that  $Az = z$ . If not, we find

$$\begin{aligned} M_1(z, x_{2n+1}) &= \max \left\{ d(Sz, Tx_{2n+1}), d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1}), \right. \\ &\quad \left( \frac{d(Bx_{2n+1}, Sz) + d(Az, Tx_{2n+1})}{2} \right), \left( \frac{d(Sz, Az) + d(Tx_{2n+1}, Bx_{2n+1})}{2} \right), \\ &\quad \left. d(Bx_{2n+1}, Tx_{2n+1}) \left( \frac{1 + d(Az, Sz)}{1 + d(Sz, Tx_{2n+1})} \right), d(Az, Sz) \left( \frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sz, Tx_{2n+1})} \right) \right\} \end{aligned}$$

and

$$\begin{aligned} N_1(z, x_{2n+1}) &= \min \left\{ d(Sz, Tx_{2n+1}), d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1}), \right. \\ &\quad \left( \frac{d(Bx_{2n+1}, Sz) + d(Az, Tx_{2n+1})}{2} \right), \left( \frac{d(Sz, Az) + d(Tx_{2n+1}, Bx_{2n+1})}{2} \right), \\ &\quad \left. d(Bx_{2n+1}, Tx_{2n+1}) \left( \frac{1 + d(Az, Sz)}{1 + d(Sz, Tx_{2n+1})} \right), d(Az, Sz) \left( \frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sz, Tx_{2n+1})} \right) \right\}. \end{aligned}$$

Taking the limit as  $n \rightarrow +\infty$ , we get

$$\lim_{n \rightarrow +\infty} M_1(z, x_{2n+1}) = d(Sz, z) = d(Az, z).$$

Using (2.1), we have

$$\psi(d(Az, Bx_{2n+1})) \leq \psi(M_1(z, x_{2n+1})) - \phi(N_1(z, x_{2n+1}))$$

which, on taking limit as  $n \rightarrow +\infty$ , gives

$$\psi(d(Az, z)) \leq \psi(d(Az, z)) - \phi(d(Az, z)),$$

a contradiction for  $d(Az, z) > 0$ . Therefore  $Az = z$ .

Moreover, we show that  $z$  is a fixed point for mappings  $B$  and  $T$ . Since  $A(X) \subseteq T(X)$ , there is some  $v \in X$  such that  $Az = Tv$ . Then  $Az = Tv = Sz = z$ . We claim that  $Bv = z$ . If not then by (2.1), we have

$$\begin{aligned} \psi(d(z, Bv)) &= \psi(d(Az, Bv)) \\ &\leq \psi(M_1(z, v)) - \phi(N_1(z, v)) \\ &= \psi(d(Bv, z)), \end{aligned}$$

a contradiction for  $d(Bv, z) > 0$ , hence  $Bv = z$ . Thus  $Bv = Tv = z$ , and by the weak compatibility of mappings  $B$  and  $T$ , we get  $Bz = BTv = TBv = Tz$ . If  $Bz \neq z$  then by (2.1), we have

$$\begin{aligned} \psi(d(z, Bz)) &= \psi(d(Az, Bz)) \\ &\leq \psi(M_1(z, z)) - \phi(N_1(z, z)) \\ &= \psi(d(z, Tz)) - \phi(N_1(z, z)) = \psi(d(z, Bz)) - \phi(N_1(z, z)), \end{aligned}$$

a contradiction for  $d(z, Bz) > 0$ . Hence  $Az = Bz = Sz = Tz = z$ . A similar analysis is also valid for the case in which  $T(X)$  is closed as well as for the cases in which  $A(X)$  or  $B(X)$  is closed. Also, the uniqueness of the common fixed point  $z$  follows from (2.1).  $\square$

**Remark 2.2.** Our Theorem 2.1 also generalizes the results in [3], [4], [7], [13], ([14], Theorem 1.4), ([17], Theorem 1.3), ([15], Theorem 1.2) and many others.

Now, we obtain some special cases of our Theorem 2.1 in the form of corollaries as follow.

**Corollary 2.3.** Suppose that  $A, B, S$  and  $T$  are self mappings of a complete metric space  $(X, d)$ ,  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ , and the pairs  $\{A, S\}$  and  $\{B, T\}$  are weakly compatible. If, for every  $x, y \in X$ ,

$$\begin{aligned} \psi(d(Ax, By)) \leq & \psi \left( \max \left\{ d(By, Ty) \left( \frac{1+d(Ax, Sx)}{1+d(Sx, Ty)} \right), d(Ax, Sx) \left( \frac{1+d(By, Ty)}{1+d(Sx, Ty)} \right), d(Sx, Ty) \right\} \right) \\ & - \phi \left( \min \left\{ d(By, Ty) \left( \frac{1+d(Ax, Sx)}{1+d(Sx, Ty)} \right), d(Ax, Sx) \left( \frac{1+d(By, Ty)}{1+d(Sx, Ty)} \right), d(Sx, Ty) \right\} \right), \end{aligned}$$

where  $\phi \in \Phi$ ,  $\psi \in \Psi$ . Then  $A, B, S$ , and  $T$  have a unique common fixed point in  $X$ , whenever one of the range  $A(X), B(X), S(X), T(X)$  is closed in  $X$ .

**Corollary 2.4.** Suppose  $A, B, S$  and  $T$  are self mappings of a complete metric space  $(X, d)$ ,  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ , and the pairs  $\{A, S\}$  and  $\{B, T\}$  are weakly compatible. If, for every  $x, y \in X$ ,

$$\psi(d(Ax, By)) \leq \psi \left( \max \left\{ d(By, Ty) \left( \frac{1+d(Ax, Sx)}{1+d(Sx, Ty)} \right) \right\} \right) - \phi \left( \min \left\{ d(By, Ty) \left( \frac{1+d(Ax, Sx)}{1+d(Sx, Ty)} \right) \right\} \right)$$

or

$$\psi(d(Ax, By)) \leq \psi \left( \max \left\{ d(Ax, Sx) \left( \frac{1+d(By, Ty)}{1+d(Sx, Ty)} \right) \right\} \right) - \phi \left( \min \left\{ d(Ax, Sx) \left( \frac{1+d(By, Ty)}{1+d(Sx, Ty)} \right) \right\} \right),$$

where  $\phi \in \Phi$  and  $\psi \in \Psi$ . Then  $A, B, S$ , and  $T$  have a unique common fixed point in  $X$ , whenever one of the range  $A(X), B(X), S(X), T(X)$  is closed in  $X$ .

Also, by taking  $S = T = I$  (identity mapping) in Theorem 2.1, we obtain the following.

**Corollary 2.5.** Let  $(X, d)$  be a complete metric space and let  $A, B : X \rightarrow X$  be two mappings such that, for every  $x, y \in X$ ,

$$\begin{aligned} \psi(d(Ax, By)) \leq & \psi \left( \max \left\{ d(y, By) \left( \frac{1+d(x, Ax)}{1+d(x, y)} \right), d(x, y) \right\} \right) \\ & - \phi \left( \min \left\{ d(y, By) \left( \frac{1+d(x, Ax)}{1+d(x, y)} \right), d(x, y) \right\} \right) \end{aligned}$$

or

$$\begin{aligned} \psi(d(Ax, By)) \leq & \psi \left( \max \left\{ d(x, Ax) \left( \frac{1+d(y, By)}{1+d(x, y)} \right), d(x, y) \right\} \right) \\ & - \phi \left( \min \left\{ d(x, Ax) \left( \frac{1+d(y, By)}{1+d(x, y)} \right), d(x, y) \right\} \right), \end{aligned}$$

where  $\phi \in \Phi$  and  $\psi \in \Psi$ . Then  $A$  and  $B$  have a unique common fixed point in  $X$ .

However, if we assume  $A = B$  and  $S = T = I$  (identity mapping) in Theorem 2.1, we have the following.

**Corollary 2.6.** Let  $(X, d)$  be a complete metric space and let  $A : X \rightarrow X$  be a mapping such that, for every  $x, y \in X$ ,

$$\begin{aligned} \psi(d(Ax, Ay)) \leq & \psi \left( \max \left\{ d(y, Ay) \left( \frac{1+d(x, Ax)}{1+d(x, y)} \right), d(x, Ax) \left( \frac{1+d(y, Ay)}{1+d(x, y)} \right), d(x, y) \right\} \right) \\ & - \phi \left( \min \left\{ d(y, Ay) \left( \frac{1+d(x, Ax)}{1+d(x, y)} \right), d(x, Ax) \left( \frac{1+d(y, Ay)}{1+d(x, y)} \right), d(x, y) \right\} \right), \end{aligned}$$

where  $\phi \in \Phi$  and  $\psi \in \Psi$ . Then  $A$  has a unique fixed point in  $X$ .



Hereto, we have obtained another result for mappings satisfying a generalized rational type condition under the weak compatibility and  $(\psi, \phi)$ -weak contraction in complete metric spaces. This result also generalizes many other results in the literature.

**Theorem 2.7.** *Suppose that  $A, B, S$  and  $T$  are self mappings of a complete metric space  $(X, d)$ ,  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ , and the pairs  $\{A, S\}$  and  $\{B, T\}$  are weakly compatible. If, for every  $x, y \in X$ ,*

$$\psi(d(Ax, By)) \leq \psi(M_1(x, y)) - \phi(M_1(x, y)), \quad (2.9)$$

where  $\psi \in \Psi$ ,  $\phi \in \Phi$  such that  $\phi$  is discontinuous at  $t = 0$ , and  $M_1(x, y)$  is defined by (2.2). Then  $A, B, S$ , and  $T$  have a unique common fixed point in  $X$ , whenever one of the range  $A(X), B(X), S(X), T(X)$  is closed in  $X$ .

*Proof.* By property of function  $\psi$ , we have  $\phi(N_1(x, y)) \leq \phi(M_1(x, y))$ , and therefore

$$\begin{aligned} \psi(d(Ax, By)) &\leq \psi(M_1(x, y)) - \phi(M_1(x, y)) \\ &\leq \psi(M_1(x, y)) - \phi(N_1(x, y)). \end{aligned}$$

Hence, Theorem 2.1 completes the proof.  $\square$

Here, we give the following example for the vindication of our result (Theorem 2.1) on  $(\psi, \phi)$ -contraction.

**Example 2.8.** *Let  $X = \{(1, 1), (1, 4), (4, 1), (4, 5), (5, 4)\}$  be endowed with metric  $d$  defined by*

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

*Suppose  $A, B, S, T : X \rightarrow X$  are such that*

$$\begin{aligned} A(x_1, x_2) &= B(x_1, x_2) = \begin{cases} (x_1, 1) & \text{if } x_1 \leq x_2 \\ (1, x_2) & \text{if } x_1 > x_2. \end{cases} \\ S(x_1, x_2) &= T(x_1, x_2) = (x_1, x_2). \end{aligned}$$

*Choose  $\psi(t) = t$  and  $\phi(t) = \frac{t}{6}$ . Clearly, mappings  $A, B, S, T$  do not satisfy the condition (1.3) of Theorem 1.4. To see this, at  $x = (4, 5)$  and  $y = (5, 4)$ , we have  $d(Ax, By) = 6$ ,  $M(x, y) = 4$ ,  $N(x, y) = 2$ ,  $M_1(x, y) = \frac{20}{3}$  and  $N_1(x, y) = 2$ . Then,  $\psi(d(Ax, By)) \leq \psi(M(x, y)) - \phi(N(x, y))$  implies  $6 \leq 4 - \frac{1}{3}$ , which is not possible. Hence the condition (1.3) is not satisfied. However, the condition (2.1) of our Theorem 2.1 is satisfied for all  $x, y \in X$  and  $(1, 1)$  is the only common fixed point. Moreover, it is clear that  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ , and the pairs  $\{A, S\}, \{B, T\}$  are weakly compatible.*

### 3. Compliance with Ethical Standards

#### 3.1. Conflict of interest

The authors declare that they have no conflict of interest.

#### 3.2. Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

#### 3.3. Informed consent

Informed consent was obtained from all individual participants included in the study.

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