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New Spaces Over Modulus Function

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ABSTRACT: Our main aim of this paper is to introduce some new techniques of spaces using modulus function. Some of basic inclusion properties will be taken care of.

Key Words: Modulus function, Paranormed sequence, Infinite matrices.

Contents

1. Introduction

We represent the set of all sequences with complex terms by $\mho$. By a *sequence space* we define a linear subspace of $\mho$ i.e., the sequence space is the set of scalar sequences(real or complex) which is closed under co-ordinate wise addition and scalar multiplication. Here we symbolize $\mathbf{N}$ and $\mathbf{C}$ to represent the set of non-negative integers and the set of complex numbers, respectively. By ℓ_∞, c and c_0, respectively, we shall mean the set of all bounded sequences, the set of all convergent sequences and those sequences having limit as zero. Note that ℓ_1, $\ell(p)$, cs and bs will specify the spaces of all absolutely, p-absolutely convergent, convergent and bounded series, respectively [10], [14]- [17], [32].

For an infinite matrix $\Lambda = (w_{i,j})$ and $\eta = (\eta_k) \in \Psi$, the Λ-transform of η is $\Lambda\eta = \{(\Lambda\eta)_i\}$ provided it exists $\forall\ i \in \mathbb{N}$, where $(\Lambda\eta)_i = \sum_{j=0}^{\infty} w_{i,j}\eta_j$.

For the matrix $\Lambda = (w_{i,j})$, the set G_Λ, where

$$G_\Lambda = \{\eta = (\eta_j) \in \Psi \ : \ \Lambda\eta \in G\}, \tag{1.1}$$

is known as the matrix domain of Λ in G (see, [18], [21], [27]- [29]).

Consider the sequence of positive numbers (q_k) and write $S_n = \sum\limits_{k=0}^{n} q_k$ for $n \in \mathbf{N}$.

Then the matrix $S^q = (s^q_{nk})$ of the Riesz mean (S, q_n) is given by

$$s^q_{nk} = \begin{cases} \frac{q_k}{S_n}, & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n \end{cases}$$

The Riesz mean (S, q_n) is regular if and only if $S_n \to \infty$ as $n \to \infty$ (see, [24], [30]).

In [22], the author introduced the concept of modulus function. We call a function $\mathcal{G} : [0,\infty) \to [0,\infty)$ to be modulus function if

(i) $\mathcal{G}(\zeta) = 0$ if and only if $\zeta = 0$,

(ii) $\mathcal{G}(\zeta + \eta) \le \{(\zeta) + \{(\eta)\ \forall\ \zeta \ge 0, \eta \ge 0$

(iii) $\mathcal{G}$ is increasing, *and*

(iv) $\mathcal{G}$ is continuous from the right at 0.

One can easily see that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are modulus functions then so is $\mathcal{G}_1 + \mathcal{G}_2$; and the function $\mathcal{G}^j$ $(j \in \mathbf{N})$, the composition of a modulus function $\mathcal{G}$ with itself j times is also modulus function. It has also been studied in [11].

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Recently, in [25] the new space was introduced by using notion of modulus function as follows:

$$L(\mathcal{F}) = \left\{ \zeta = (\zeta_r) : \sum_r |\mathcal{F}(\zeta_r|)| < \infty \right\}$$

In this direction of forming a new sequence space by virtue of matrix domain method has been given by several authors viz., $(\ell_p)_{R^t} = r_p^t$ (see, [2]), $[c_0(u,p)]_{A^r} = a_0^r(u,p)$ and $[c(u,p)]_{A^r} = a_c^r(u,p)$ (see, [3], $(\ell_\infty)_{R^t} = r_\infty^t$, $(c)_{R^t} = r_c^t$ and $(c_o)_{R^t} = r_0^t$ (see, [19]), $(\ell_p)_{C_1} = X_p$ and $(\ell_\infty)_{C_1} = X_\infty$(see, [23]), $r^q(u,p) = \{l(p)\}_{R_u^q}$ (see, [27]), $(\ell_\infty)_{N_q}$ and c_{N_q}(see, [31]), and etc.

2. The space $s_\infty^q(\mathcal{F},p)$, $s_c^q(\mathcal{F},p)$ and $s_0^q(\mathcal{F},p)$

In this section, we shall introduce the space $s_\infty^q(\mathcal{F},p)$, $s_c^q(\mathcal{F},p)$ and $s_0^q(\mathcal{F},p)$ of Riesz type and show that they are complete.

Let Λ be a real or complex linear space, define the function $\tau : \Lambda \to \mathbb{R}$ with $\mathbb{R}$ as set of real numbers. Then, the paranormed space is a pair $(\Lambda;\tau)$ and τ is a paranorm for Λ, if the following axioms are satisfied for all $\zeta,\ \eta \in \Lambda$ and for all scalars β:

(i) $\tau(\theta) = 0$,

(ii) $\tau(-\zeta) = \tau(\zeta)$,

(iii) $\tau(\zeta+\eta) \le \tau(\zeta) + \tau(\eta)$, *and*

(iv) scalar multiplication is continuous, that is,

$|\beta_n - \beta| \to 0$ and $h(\zeta_n - \zeta) \to 0$ imply $\tau(\beta_n\zeta_n - \beta\zeta) \to 0$ for all $\beta's$ in $\mathbb{R}$ and $\zeta's$ in Λ , where θ is a zero vector in the linear space Λ. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$and $M = max\{1, H\}$. Then, the linear space $\ell_\infty(p)$ was defined by Maddox [18] as follows :

$$\ell_\infty(p) = \{\zeta = (\zeta_k) : \sup_k |\zeta_k|^{p_k} < \infty\}$$

which is complete space paranormed by

$$\tau_1(\zeta) = \left[\sup_k |\zeta_k|^{p_k}\right]^{1/M}.$$

We shall assume throughout that $p_k^{-1} + \{p_k^{'}\}^{-1}$ provided $1 < inf p_k \le H < \infty$, and we denote the collection of all finite subsets of N by F, where N=\{0, 1, 2, ... \}.

Following Altay (see, [2]), Başarir and Öztürk (see, [6]), Choudhary and Mishra (see, [5]), Ganie et al. (see, [6]-[13], Mursaleen (see, [20]), Ruckle [25], Sengönül [26], we define the spaces $s_\infty^q(\mathcal{F},p)$, $s_c^q(\mathcal{F},p)$ and $s_0^q(\mathcal{F},p)$ as the set of all sequences whose $R_\mathcal{F}^q$-transform are in the spaces $c(p)$ and $c_0(p)$, respectively i.e.,

$$s_\infty^q(\mathcal{F},p) = \left\{ x \in \omega : \sup_k \left| \mathcal{F}\left(\frac{1}{Q_k} \sum_{j=0}^{k} q_j x_j \right) \right|^{p_k} < \infty \right\},$$

$$s_c^q(\mathcal{F},p) = \left\{ x \in \omega : \lim_k \left| \mathcal{F}\left(\frac{1}{Q_k} \sum_{j=0}^{k} q_j x_j - l \right) \right|^{p_k} = 0 \text{ for some } l \in \mathbf{R} \right\},$$

$$s_0^q(\mathcal{F},p) = \left\{ x \in \omega : \lim_k \left| \mathcal{F}\left(\frac{1}{Q_k} \sum_{j=0}^{k} q_j x_j \right) \right|^{p_k} = 0 \right\}.$$

These spaces can be written with the help of (2) as follows:

$$s^q_\infty(\mathcal{F},p)=\{l_\infty(p)\}_{S^q_{\mathcal{F}}},\ \ s^q_c(\mathcal{F},p)=\{c(p)\}_{S^q_{\mathcal{F}}} \text{ and } s^q_0(\mathcal{F},p)=\{c_0(p)\}_{S^q_{\mathcal{F}}},$$

where, $0<p_k\le H<\infty$.

Define the sequence $y=(y_k)$, which will be used, by the $S^q_{\mathcal{F}}$-transform of a sequence $x=(x_k)$, i.e.,

$$y_k=\mathcal{F}\ \frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j \text{ for all } k\in\mathbb{N}. \tag{2.1}$$

Theorem 2.1. *The spaces $s^q_\infty(\mathcal{F},p)$, $s^q_c(\mathcal{F},p)$ and $s^q_0(\mathcal{F},p)$ are complete linear metric space paranormed by $\mathcal{G}$ defined*

$$\mathcal{G}(x)=\sup_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right|^{\frac{p_k}{M}}.$$

Proof. We only prove the theorem for the space $s^q_\infty(\mathcal{F},p)$. The linearity of $s^q_\infty(\mathcal{F},p)$ with respect to the co-ordinate wise addition and scalar multiplication follows from from the inequalities which are satisfied for $z,x\in s^q_\infty(\mathcal{F},p)$(see [18], p.30])

$$\sup_k\left|\mathcal{F}\left(\tfrac{1}{Q_k}\textstyle\sum_{j=0}^{k} q_j(z_j+x_j)\right)\right|^{\frac{p_k}{M}}$$
$$\le\sup_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j z_j\right)\right|^{\frac{p_k}{M}}+\sup_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right|^{\frac{p_k}{M}} \tag{2.2}$$

and for any $\alpha\in\mathbb{R}$ (see, [17])

$$|\alpha|^{p_k}\le max\{1,|\alpha|^M\}. \tag{2.3}$$

It is clear that, $\mathcal{G}(\theta)$=0 and $\mathcal{G}(x)=\mathcal{G}(-x)$ for all $x\in s^q_0(\mathcal{F},p)$. Again the inequality (4) and (5), yield the subadditivity of g and

$$\mathcal{G}(\alpha x)\le max\{1,|\alpha|\}\mathcal{G}(x).$$

Let $\{x^n\}$ be any sequence of points of the space $s^q_0(\mathcal{F},p)$ such that $g(x^n-x)\to 0$ and (α_n) is a sequence of scalars such that $\alpha_n\to\alpha$. Then, since the inequality,

$$\mathcal{G}(x^n)\le\mathcal{G}(x)+\mathcal{G}(x^n-x)$$

holds by subadditivity of $\mathcal{G}$, $\{\mathcal{G}(x^n)\}$ is bounded and we thus have

$$g(\alpha_n x^n-\alpha x)=\sup_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j(\alpha_n x^n_j-\alpha x_j)\right)\right|^{\frac{p_k}{M}}$$
$$\le|\alpha_n-\alpha|\,g(x^n)+|\alpha|\,g(x^n-x)$$

which tends to zero as $n\to\infty$. That is to say that the scalar multiplication is continuous. Hence, g is paranorm on the space $s^q_\infty(\mathcal{F},p)$.

It remains to prove the completeness of the space $s^q_\infty(\mathcal{F},p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $s^q_\infty(\mathcal{F},p)$, where $x^i=\{x^i_0,x^i_2,...\}$. Then, for a given $\epsilon>0$ there exists a positive integer $n_0(\epsilon)$ such that

$$\mathcal{G}(x^i-x^j)<\epsilon \tag{2.4}$$

for all $i, j \geq n_0(\epsilon)$. Using definition of $\mathcal{G}$ and for each fixed $k \in \mathbb{N}$ that

$$\left|(S^q_{\mathcal{F}} x^i)_k - (S^q_{\mathcal{F}} x^j)_k\right| \leq \sup_k \left|(S^q_{\mathcal{F}} x^i)_k - (S^q_{\mathcal{F}} x^j)_k\right|^{\frac{p_k}{M}} < \epsilon$$

for $i, j \geq n_0(\epsilon)$, which leads us to the fact that $\{(S^q_{\mathcal{F}} x^0)_k, (S^q_{\mathcal{F}} x^1)_k, \dots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbf{N}$. Since $\mathbf{R}$ is complete, it converges,say, $(S^q_{\mathcal{F}} x^i)_k \to ((S^q_{\mathcal{F}} x)_k$ as $i \to \infty$. Using these infinitely many limits $(S^q_{\mathcal{F}} x)_0, (S^q_{\mathcal{F}} x)_1, \dots$, we define the sequence $\{(S^q_{\mathcal{F}} x)_0, (S^q_{\mathcal{F}} x)_1, \dots\}$. From (6) for with $j \to \infty$ we have

$$\left|(S^q_{\mathcal{F}} x^i)_k - (S^q_{\mathcal{F}} x)_k\right| \leq \epsilon, \tag{2.5}$$

for all k, i.e.,

$$\mathcal{G}(x^i - x) \leq \epsilon \ (i \geq n_0(\epsilon)).$$

Finally, taking $\epsilon = 1$ in (7) and letting $i \geq n_0(1)$. we have by Minkowski's inequality for each $m \in \mathbf{N}$ that

$$\left|(S^q_{\mathcal{F}} x)_k\right|^{\frac{p_k}{M}} \leq \mathcal{G}(x^i - x) + \mathcal{G}(x^i) \leq 1 + \mathcal{G}(x^i)$$

which implies that $x \in s^q_\infty(\mathcal{F}, p)$. Since $\mathcal{G}(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$, hence we have shown that $s^q_\infty(\mathcal{F}, p)$ is complete, hence the proof.
□

Remark 2.2. *One can easily see the absolute property does not hold on the spaces $s^q_\infty(\mathcal{F}, p)$, $s^q_c(\mathcal{F}, p)$ and $s^q_0(\mathcal{F}, p)$, that is $\mathcal{G}(x) \neq \mathcal{G}(|x|)$ for atleast one sequence in the spaces $s^q_\infty(\mathcal{F}, p)$,, $s^q_c(\mathcal{F}, p)$ and $s^q_0(\mathcal{F}, p)$ and this says that the spaces $s^q_\infty(\mathcal{F}, p)$, $s^q_c(\mathcal{F}, p)$ and $s^q_0(\mathcal{F}, p)$ are sequence spaces of non-absolute type.*

Theorem 2.3. *If p_k and t_k are bounded sequences of positive real numbers with $0 < p_k \leq t_k < \infty$ for each $k \in \mathbf{N}$, then for any modulus function $\mathcal{F}$, we have*
(i) $s^q_c(\mathcal{F}, p) \subseteq s^q_c(\mathcal{F}, t)$.
(ii) $s^q_c(\mathcal{F}, p) \subseteq s^q_c(\mathcal{F}, t)$.
(iii) $s^q_{c_0}(\mathcal{F}, p) \subseteq s^q_{c_0}(\mathcal{F}, t)$.

Proof. We only prove (i) and the rest can be proven similarly. For $\zeta \in s^q_\infty(\mathcal{F}, p)$ it is obvious that

$$\sup_k \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right| < \infty.$$

Consequently, for sufficiently large values of k say $k \geq k_0$ for some fixed $k_0 \in \mathbf{N}$, we have

$$\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right| < \infty.$$

But $\mathcal{F}$ being increasing and $p_k \leq t_k$, we have

$$\sup_{k \geq k_0} \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right|^{t_k} \leq \sup_{k \geq k_0} \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right|^{p_k} < \infty.$$

From this, it is clear that $\zeta \in s^q_\infty(\mathcal{F}, t)$ and the result follows. □

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